A note on coupling for CBI processes

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Coupling

- A coupling (X_t, Y_t): if both X_t and Y_t are Markov processes associated with the same transition probability P_t (possibly with different initial distributions), where X_t and Y_t are called the marginal processes of the coupling.

- A coupling (X_t, Y_t) is called successful if the coupling time

$$T := \inf\{t \geq 0 : X_t = Y_t\} < \infty, \ \ a.s.$$

- A (strong) Markov process with P_t is said to have a coupling property: for any μ_1 and μ_2, there exists a successful coupling with marginal processes starting from μ_1 and μ_2, respectively.

  ⋄ Due to strong Markov property setting $X_t = Y_t$ for $t \geq T$.

$$\|\mu_1 P_t - \mu_2 P_t\|_{var} \leq 2\mathbb{P}(T > t), t \geq 0,$$

where $\|\cdot\|$ is the total variational norm.

The coupling property of a strong Markov process with semigroup P_t is equivalent to each of the following:

- For any μ_1, μ_2, $\lim_{t\to\infty} \|\mu_1 P_t - \mu_2 P_t\|_{var} = 0$.
- All bounded time-space harmonic functions (i.e. $u(t,\cdot) = P_s u(t+s,\cdot)$) are constant.
- The tail σ-algebra of the process is trivial.

We refer the reader to Chen (2004) and Lindvall (1992) for the systematical study.

Motivation

- A growing literature on coupling for jump processes.
  Chen (2004): Q processes; Schilling and Wang (2011): Levy processes; Wang (2011): Ornstein-Uhlenbeck processes...

- Nice applications of coupling to the study of ergodicity, Liouville theorem, convergence rate, gradient estimate for the processes.

Motivation

- To consider the coupling for CB(I) processes.

  ⋄ Dawson and Li (2006), Fu and Li (2010), Li and Mytnik (2011): a class of stochastic equation with jumps, e.g.

$$\begin{aligned} X_t(x) &= x + \int_0^t (a + bX_s)ds + \sigma \int_0^t \sqrt{X_s}dB_s \\ &\quad + \int_0^t \int_{\mathbb{R}_+} \int_0^{X_{s-}} z\tilde{N}_1(ds, dz, du). \end{aligned}$$

  The diffusion and jump terms with degenerate and non-Lipschitz coefficients

  ⋄ α-CIR model for interest rate; See Jiao *et al* (2017); Li and Ma (2015).

$$r_t = r_0 + \int_0^t \beta(\gamma - r_s)\,ds + \sigma \int_0^t \sqrt{r_s}dB_s + \sigma_Z \int_0^t r_{s-}^{1/\alpha} dZ_s$$

  Ergodicity property by using coupling method (key for statistical inference)

CBI processes

CBI (Kawazu and Watanabe (1971)) of branching mechanism $\Psi(\cdot)$ and immigration mechanism $\Phi(\cdot)$: Markov process X on state space $\mathbb{R}_+$ with transition semigroup given by

$$\mathbb{E}_x\left[e^{-pX_t}\right] = \exp\left[-xv(t,p) - \int_0^t \Phi\big(v(s,p)\big)ds\right],$$

where $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p$$

and Ψ and Φ are functions on $\mathbb{R}_+$ given by

$$\Psi(q) = bq + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(du),$$
$$\Phi(q) = aq + \int_0^\infty (1 - e^{-qu})\nu(du),$$

with $\sigma, a \geq 0$, $b \in \mathbb{R}$ and π, ν being two Lévy measures such that $\int_0^\infty (u \wedge u^2)\pi(du) < \infty$ and $\int_0^\infty (1 \wedge u)\nu(du) < \infty$.

CB processes

- X_t is called CB process if $\Phi \equiv 0$. The extinction time of CB is defined by $\tau_0 = \inf\{t \geq 0 : X_t = 0\}$. and

$$\mathbb{P}_x(\tau_0 \leq t) = \mathbb{P}_x(X_t = 0) = \exp\{-x\bar{v}_t\}.$$

where the limit $\bar{v}_t = \uparrow \lim_{\lambda\to\infty} v(t,\lambda)$ exists in $(0,\infty]$.

- Grey (1974): $\bar{v}_t < \infty$ for all $t > 0$ if and only if there is some constant $\theta > 0$ such that $\Psi(q) > 0$ for $z > \theta$ and Grey condition satisfied:

$$\int_\theta^\infty \Psi(q)^{-1} dq < \infty.$$

- For (sub)critical CB processes,

$$\text{Grey condition holds} \iff \mathbb{P}(\tau_0 < \infty) = 1.$$

Coupling for CBI processes when Grey condition holds

- Li and Ma (2015).

  Assume Grey condition holds. the (sub)critical CBI process with the transition semigroup $(P_t)_{t\geq 0}$ has the strong Feller property. Moreover, for any $t > 0$ and $x, y \in \mathbb{R}_+$, we have

  $$\left\| P_t(x,\cdot) - P_t(y,\cdot) \right\|_{var} \leq 2(1 - e^{-\bar{v}_t |x-y|}),$$

  which goes to 0 as $t \to \infty$. In this case, the CBI processes have successful coupling.

Exponential ergodicity

- Assume Grey condition holds and the Levy measure ν in immigration mechanism satisfies $\int_1^\infty y^\delta \nu(dy) < \infty$ for some $\delta > 0$. Then

  (i) the subcritical CBI process is exponentially ergodic, i.e.

  $$\|P_t(x,\cdot) - \mu(\cdot)\|_{var} \leq 2(x\bar{v}_1 + M_\gamma)e^{-\gamma b(t-1)},$$

  where μ is the stationary measure, $\gamma = \delta \wedge 1$ and

  $$M_\gamma = \begin{cases} \gamma \bar{v}_1^\gamma \int_0^\infty (1 - L_\mu(\lambda))\lambda^{-(1+\gamma)} d\lambda & \text{if } \gamma < 1, \\ \bar{v}_1 b^{-1}(a + \int_0^\infty u\,\nu(du)) & \text{if } \gamma = 1. \end{cases}$$

  (ii) the critical CBI process is ergodic.

Sketch of proof (coupling)

- Construct the flow $\{X_t(x) : t \geq 0, x \geq 0\}$ by

$$\begin{aligned} Y_t(x) &= x + \int_0^t (a - bY_s(x))ds + \sigma \int_0^t \int_0^{Y_{s-}(x)} W(ds, du) \\ &\quad + \int_0^t \int_0^\infty \int_0^{Y_{s-}(x)} z\tilde{N}_1(ds, dz, du) + \int_0^\infty zN_0(ds, dz). \end{aligned}$$

  For fixed x, the solution $\{Y_t(x), t \geq 0\}$ is a CBI process with branching mechanism Ψ and immigration mechanism Φ.

- Dawson and Li (2012):
  For any $x \geq y \geq 0$ we have $\mathbb{P}(Y_t(x) \geq Y_t(y)$ for all $t \geq 0) = 1$ and $(Y_t(x) - Y_t(y))_{t \geq 0}$ is a CB process with branching mechanism Ψ.

- The coupling time is the extinction time of the above CB process.

Grey condition is necessary for coupling?

- **Theorem** The (sub)critical CB processes have successful coupling if and only if Grey condition holds.
- when Grey condition fails, $X_t \to 0$ as $t \to \infty$ but never hit 0 a.s. The processes belong to one of the following classes:
  - Class I (Ψ is of finite variation): $\sigma = 0$ and $\int_0^1 um(du) < \infty$.
  - Case II (Ψ is of infinite variation): $\sigma = 0$, $\int_0^1 um(du) = \infty$ and $\int_\theta^\infty \Psi(q)^{-1} dq = \infty$ for $\theta > 0$.
- we need to consider their asymptotic behavior to 0.

Asymptotic behavior of (sub)critical CB process

- Foucart and Ma (2016+)
  (i) Suppose that the process is of Class I. For any fixed $\lambda > 0$ and $x > 0$,

$$\eta_t(\lambda) X_t(x) \to W_x^\lambda, \ a.s.$$

as $t \to \infty$ and the process $\{W_x^\lambda : x \geq 0\}$ is a subordinator with $P(W_x^\lambda = 0) = 0$.

(ii) Suppose that the process is of Class II. Fix some $\lambda_0 > 0$ and set $G(y) = \exp\left(-\int_{\lambda_0}^{y} \frac{du}{\Psi(u)}\right)$ on $y \in (0, \infty)$. Then the map G is slowly varying at ∞ and for all $x \geq 0$,

$$e^t G\left(\frac{1}{X_t(x)}\right) \xrightarrow[t\to\infty]{} Z_x \text{ a.s.}$$

Here $(Z_x, x \geq 0)$ is a extremal process (i.e. $Z_{x+y} = Z_x \vee Z'_y$, Z'_y is independent of $(Z_u, 0 \leq u \leq x)$ and $Z'_y \overset{d}{=} Z_y$)

Grey condition is necessary for coupling of CB

- **Theorem** The (sub)critical CB processes have successful coupling if and only if Grey condition holds.

- sketch of proof: Choose some $c \in \Gamma$ such that $0 < G^{-1}(c) < \infty$. Since (Z_x) is an extremal process, for any $x > y$,

$$P(Z_x > c > Z_y) = e^{-yG^{-1}(c)}\big(1 - e^{-(x-y)G^{-1}(c)}\big).$$

Let $B_t = \{u > 0 : e^t G(1/u) > c\}$. Then

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{var}} \ \geq\ P(X_t(x) \in B_t) - P(X_t(y) \in B_t)$$

which implies that
$\liminf_{t\to\infty} \|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{var}} \geq e^{-yG^{-1}(c)}\big(1 - e^{-(x-y)G^{-1}(c)}\big)$.

Coupling for CBI processes when Grey condition fails

- $D[0,\infty)$: the space of càdlàg paths $t \mapsto w_t$ from $[0,\infty)$ to $\mathbb{R}_+$.
- $\mathbb{Q}_x(dw)$ denote the distribution on $D[0,\infty)$ of the CB process $(X_t(x) : t \geq 0)$ with $X_0(x) = x$.
- $\mathbb{Q}_\nu(dw)$ on $D[0,\infty)$ by

$$\mathbb{Q}_\nu(dw) = \int_0^\infty \nu(dx)\mathbb{Q}_x(dw).$$

- $M(ds, dw)$ be a Poisson random measure on $(0,\infty) \times D[0,\infty)$ with intensity measure $ds\mathbb{Q}_n(dw)$.

Suppose that M and $X_t(x)$ are independent of each other. We define the process $Y_t(x)$ by

$$Y_t(x) = X_t(x) + \int_0^t \int_{D[0,\infty)} w_{t-s} M(ds, dw), \quad t \geq 0.$$

The Lévy measure in immigration mechanism should have a local non-trivial absolutely continuous part to make the process active.

Coupling for CBI processes when Grey condition fails

- Based on idea of Wang (2011) dealing with coupling for OU type process.
- Assumption A: there exists some $z_0 \in \mathbb{R}_+$ and some $\varepsilon > 0$ with $B(z_0, \varepsilon) \subset (0, \infty)$ such that the Lévy measure $\nu(dz)$ has an absolutely continuous part in $B(z_0, \varepsilon)$, i.e.,

$$\nu(dz) \geq \rho(z) dz$$

  for some non-negative function ρ and

$$\int_{B(z_0,\varepsilon)} \rho(z)^{-1} dz < \infty.$$

- Assumption A holds. Then for the subcritical CBI process

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{var} \leq \frac{K(1 + |x - y|)}{\sqrt{t}}, \quad x, y \in \mathbb{R}_+, \ t > 0,$$

  for some constant $K > 0$.

Idea of proof

- Add a "random jump" in immigration part which generate a independent CB process.
- By Palm formula

$$\begin{aligned}
&\mathbb{E}\left[f\left(X_T(x) + \int_0^T \int_{D[0,\infty)} w_{T-s} \left(M + \delta_{(\tau,\xi)} \right)(ds, dw) \right) \right] \\
= \quad & \frac{1}{T n_\rho(B_{\varepsilon/2})} \mathbb{E}\left[f(Y_T^1(x)) \sum_{i=1}^{J_T} 1_{B_{\varepsilon/2}}(\eta_i) \right].
\end{aligned}$$

- Similarly

$$
\begin{aligned}
& \mathbb{E}\left[f\left(X_T(x) + \int_0^T \int_{D[0,\infty)} w_{T-s}\left(M + \delta_{(\tau,\xi)}\right)(ds, dw)\right)\right] \\
= \; & \mathbb{E}\left[f\left(X_T(y) + \tilde{X}_T + \int_0^T \int_{D[0,\infty)} w_{T-s}\left(M + \delta_{(\tau,\xi)}\right)(ds, dw)\right)\right] \\
= \; & \frac{1}{T n_\rho(B_{\varepsilon/2})} \mathbb{E}\left[f(Y_T(y)) \sum_{i=1}^{J_T} \ldots\right] + I
\end{aligned}
$$

- The bound of I is given by

$$
|I| \leq \frac{1}{T} \int_0^T \mathbb{P}(X_t(x-y) > \varepsilon/2) dt
$$

Absolute continuity

- Schilling and Wang (2011) :

  The Lévy process $(Z_t)_{t\geq 0}$ has the coupling property if and only if there exists $t_0 > 0$ such that for any $t \geq t_0$, the transition probability $P_t(x, \cdot)$ has (with respect to Lebesgue measure) an absolutely continuous component.

- However it might be interesting to note that for the (general) jump processes (e.g. O-U type processes), absolutely continuous property is incomparable with the coupling property.

- A natural problem is to consider the pure jump CB process with $b > 0$:

$$X_t = x - \int_0^t bX_s ds + \int_0^t \int_0^{X_{s-}} \int_0^\infty z\tilde{N}(ds, du, dz),$$

- Grey condition ($\Rightarrow \int_0^1 u\Pi(du) = \infty$) if and only if the process has successful coupling.
- The law of X_t has a density on $\mathbb{R}_+ \setminus \{0\}$ for every $t > 0$ $\iff$ $\int_0^1 zm(dz) = \infty$??
- The above condition is inspired by Bertoin and Legall (2000; Appendix) and they proved that

the process does not weight points for every $t > 0$

$$\iff \int_0^1 zm(dz) = \infty.$$

Thanks for your attention !