A continuous-state polynomial branching process

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Classical branching process

The transition function $(P_t)_{t\geq 0}$ of a classical continuous-time branching process satisfies the following so-called branching property:

$$P_t(x,\cdot) * P_t(y,\cdot) = P_t(x+y,\cdot), \qquad x, y \in E, \tag{1}$$

where "$*$" denotes the convolution operation.

In the continuous-time Markov chain case, the branching property implies that the Q-matrix (q_{ij}) of the process must be of the form:

$$q_{ij} = \begin{cases} \alpha i b_{j-i+1}, & j \geq i+1, i \geq 1, \\ -\alpha i, & j = i \geq 1, \\ \alpha i b_0, & j = i-1, i \geq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $\alpha > 0$ and $\{b_i : i \in \mathbb{N}\}$ is a probability measure satisfying $b_1 = 0$.

Here the jumping rate is given by the linear function αi, so we also call it a linear branching process.

Discrete-state nonlinear branching process

In realistic situations, the particles may interact with each other. Then the branching property may does not hold and the jumping rate could be a nonlinear function.

Let $\{r_i : i \geq 0\}$ be a sequence of positive constants. Chen (1997) considered following Q-matrix $(q_{ij})_{i,j\in\mathbb{N}}$:

$$q_{ij} = \begin{cases} r_i b_{j-i+1}, & j \geq i+1, i \geq 1, \\ -r_i, & j = i \geq 1, \\ r_i b_0, & j = i-1, i \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The corresponding Markov chain is called a extended branching process or a nonlinear branching process.

See also Chen (2002a, b), Chen et al. (2002, 2006, 2008), Pakes (2007) and Zhang (2001).

Special case: polynomial branching process: $r_i = \alpha i^\theta$ ($\theta > 0, \alpha > 0$). When $\theta = 1$, it reduces to the linear branching process.

The Q-matrix of discrete-state polynomial branching process (q_{ij}) has the representation $q_{ij} = i^\theta \rho_{ij}$, where (ρ_{ij}) is a Q-matrix of a random walk with jumps larger than -1:

$$\rho_{ij} = \begin{cases} \alpha b_{j-i+1}, & j \geq i+1, i \geq 1, \\ -\alpha, & j = i \geq 1, \\ \alpha b_0, & j = i-1, i \geq 1. \end{cases} \tag{4}$$

random walk with jumps larger than -1 (discrete-state)

$\Updownarrow$

spectrally positive Lévy process (continuous-state)

Relation between polynomial branching process and Lévy process

The generator of a spectrally positive Lévy process is given by :

$$Af(x) = -bf'(x) + cf''(x) + \int_{(0,\infty)} (f(x+z) - f(x) - zf'(x)1_{\{z\leq 1\}})m(dz),$$

where $b \in \mathbb{R}$, $c \geq 0$ be constants and $m(du)$ a σ-finite measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1 \wedge u^2)m(du) < \infty$.

The generator L of continuous-state polynomial branching process is defined by:

$$Lf(x) = x^{\theta}Af(x), \qquad x \in [0,\infty).$$

Continuous-state polynomial branching process

Let $(B_t : t \geq 0)$ be a Brownian motion and $M(ds, dz, du)$ a Poisson random measure with intensity $dsm(dz)du$. Consider positive solutions of the equation

$$X_t = X_0 + \sqrt{2c}\int_0^t \sqrt{X_s^\theta} dB_s + \int_0^t \int_{(0,1]} \int_0^{X_{s-}^\theta} z\tilde{M}(ds, dz, du) \\ - b\int_0^t X_s^\theta ds + \int_0^t \int_{(1,\infty)} \int_0^{X_{s-}^\theta} zM(ds, dz, du), \tag{5}$$

where $\tilde{M}(ds, dz, du) = M(ds, dz, du) - dsm(dz)du$. The case $\theta = 1$ was studied by Dawson and Li (2006, 2012) and others. Lambert (2005) and Pardoux (2016) considered another nonlinear structure.

By saying $(X_t : t \geq 0)$ is a solution to (5), we mean it satisfies (5) before hitting the state $\mathbf{0}$ or ∞ and is trapped by those states.

Proposition

For any $X_0 = x \in [0, \infty)$, there exists a pathwise unique positive solution to (5).

Continuous-state polynomial branching process

Research topics:

- Explicit representation of mean hitting times;
- Sufficient and necessary conditions for extinction and explosion;
- The process coming down from infinity.

Main difficulty: The process doesn't satisfy the branching property, so the traditional methods based on Laplace transform are not available.

Main methods:

- Stochastic integral equation;
- Equation for resolvent;
- Random time change.

Equation for resolvent

From the transition semigroup $(P_t)_{t\geq 0}$ of the process we define its resolvent $(U^\lambda)_{\lambda>0}$ by

$$U^\lambda(x,dy) = \int_0^\infty e^{-\lambda t} P_t(x,dy)dt, \qquad x \in [0,\infty). \tag{6}$$

Let $e_\lambda(x) = e^{-\lambda x}$. We have $Le_\lambda(x) = x^\theta \psi(\lambda) e_\lambda(x)$, where

$$\psi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} (e^{-\lambda z} - 1 + \lambda z 1_{\{z\leq 1\}}) m(dz).$$

Kolmogorov forward equation

$$\frac{d}{dt} P_t e_\lambda(x) = \psi(\lambda) \int_{[0,\infty)} y^\theta e^{-\lambda y} P_t(x,dy).$$

Equation for resolvent

The following proposition gives a characterization of the resolvent and plays a important role in the study of continuous-state polynomial branching process.

Proposition

For any $\eta > 0$, $\lambda \geq 0$ and $x \geq 0$ we have

$$\eta U^\eta e_\lambda(x) - e^{-\lambda x} = \psi(\lambda) \int_{[0,\infty)} y^\theta e^{-\lambda y} U^\eta(x, dy)$$

and

$$\int_\lambda^\infty l_x(\eta, z)(z-\lambda)^{\theta-1} dz = \Gamma(\theta) \int_0^\infty e^{-\eta t} dt \int_{(0,\infty)} e^{-\lambda y} P_t(x, dy),$$

where $l_x(\eta, z) = \psi(z)^{-1}\big[\eta U^\eta e_z(x) - e^{-zx}\big]$.

Mean extinction and explosion times

Let $(X_t^x : t \geq 0)$ be the solution to (5) with $X_0^x = x \geq 0$.

Let $\tau_y^x = \inf\{t \geq 0 : X_t^x = y\}$ and $q = \inf\{\lambda > 0 : \psi(\lambda) > 0\}$.

Theorem 1

For any $y \leq x \in (0, \infty)$ we have

$$\mathbf{E}(\tau_0^x 1_{\{X_\infty^x = 0\}}) = \frac{1}{\Gamma(\theta)} \int_0^\infty \frac{e^{-qx} - e^{-(\lambda+q)x}}{\psi(\lambda+q)} \lambda^{\theta-1} d\lambda,$$

$$\mathbf{E}(\tau_\infty^x 1_{\{X_\infty^x = \infty\}}) = \frac{1}{\Gamma(\theta)} \int_0^\infty \Big[\frac{e^{-qx} - e^{-\lambda x}}{\psi(\lambda)} - \frac{e^{-qx} - e^{-(\lambda+q)x}}{\psi(\lambda+q)}\Big] \lambda^{\theta-1} d\lambda,$$

$$\mathbf{E}(\tau_\infty^x \wedge \tau_y^x) = \frac{e^{-qx}}{\Gamma(\theta)} \int_0^\infty \frac{e^{-(\lambda-q)y} - e^{-(\lambda-q)x}}{\psi(\lambda)} \lambda^{\theta-1} d\lambda.$$

Those formulas seem new also for linear branching. Duhalde et al. (2014) gave $\mathbf{E}(e^{-\lambda \tau_y^x})$ in linear case.

Necessary and sufficient condition for extinction

The proofs of the following theorems are based on the equation for the resolvent.

Theorem 2

For any $x \in (0, \infty)$ we have $\mathbf{P}(\tau_0^x < \infty) > 0$ if and only if

$$\int^{\infty-} \frac{\lambda^{\theta-1}}{\psi(\lambda)} d\lambda < \infty. \tag{7}$$

In this case, we have $\mathbf{P}(\tau_0^x < \infty) = \mathbf{P}(\lim_{t\to\infty} X_t^x = 0) = e^{-qx}$.

When $\theta = 1$, (7) is called Grey's condition.

Necessary and sufficient condition for explosion

Theorem 3

For any $x \in (0, \infty)$ we have $\mathbf{P}(\tau_\infty^x < \infty) = 0$ if and only if one of the following two conditions is satisfied:

(i) $\psi'(0) \geq 0$;

(ii) $\psi'(0) < 0$ and

$$\int_{0+} \frac{\lambda^{\theta-1}}{-\psi(\lambda)} d\lambda = \infty. \tag{8}$$

In the case of $\mathbf{P}(\tau_\infty^x < \infty) > 0$, we have $\mathbf{P}(\tau_\infty^x < \infty) = \mathbf{P}(\lim_{t\to\infty} X_t^x = \infty) = 1 - e^{-qx}$.

When $\theta = 1$, this theorem was given by Kawazu and Watanabe (1971).

Some limits of hitting times

Recall $(X_t^x : t \geq 0)$ is the solution to (5) with $X_0^x = x \geq 0$.

We have comparison property : $x_1 \leq x_2 \Rightarrow X_t^{x_1} \leq X_t^{x_2}$.

Recall $\tau_y^x = \inf\{t \geq 0 : X_t^x = y\}$ and let $\tau_y^\infty = \uparrow\lim_{x\to\infty} \tau_y^x$.

Proposition

Suppose that $\psi'(0) \geq 0$ (critical or subcritical). Then the following four statements are equivalent:

(i) $\mathbf{P}(\tau_y^\infty < \infty) > 0$ for each $y \in (0, \infty)$;

(ii) $\mathbf{P}(\tau_y^\infty < \infty) = 1$ for each $y \in (0, \infty)$;

(iii) $\mathbf{E}(\tau_y^\infty) < \infty$ for each $y \in (0, \infty)$;

(iv) we have ($\theta > 1$ is necessary)

$$\int_{0+} \frac{\lambda^{\theta-1}}{\psi(\lambda)} d\lambda < \infty. \tag{9}$$

Some limits of hitting times

Proposition

Suppose $\psi'(0) \geq 0$. Then for each $y \in [0, \infty)$ we have

$$\mathbf{E}(\tau_y^\infty) = \lim_{x\to\infty} \mathbf{E}(\tau_y^x) = \frac{1}{\Gamma(\theta)} \int_0^\infty \frac{e^{-\lambda y}}{\psi(\lambda)} \lambda^{\theta-1} d\lambda.$$

Corollary

If $\psi'(0) \geq 0$ and (9) is satisfied, then $\lim_{y\to\infty} \mathbf{E}(\tau_y^\infty) = 0$. Therefore a.s. $\lim_{y\to\infty} \tau_y^\infty = 0$.

In particular, we have:

$$(0, \infty) = \bigcup_{n=1}^\infty [\tau_n^\infty, \infty).$$

We shall use this representation to construct a process $(X_t^\infty : t > 0)$.

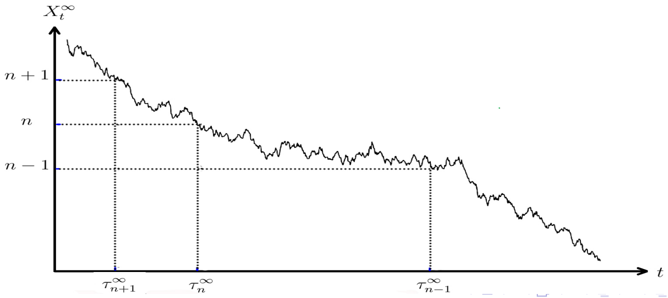
Some limits of hitting times

Proposition

Suppose $\psi'(0) \geq 0$ and (9) holds. Then $(X^k_{\tau_n^\infty + t})_{t\geq 0} \uparrow (X_t^{(n)})_{t\geq 0}$ as $k \uparrow \infty$. The limit $(X_t^{(n)})_{t\geq 0}$ is a weak solution to (5) with $X_0^{(n)} = n$.

Observe that $X^{(n+1)}_{t-\tau^\infty_{n+1}} = X^{(n)}_{t-\tau^\infty_n}$ for $t \in [\tau_n^\infty, \infty)$.

We can define a process $(X_t^\infty)_{t>0}$ by letting $X_t^\infty = X^{(n)}_{t-\tau_n^\infty}$ for $t \in [\tau_n^\infty, \infty)$.



Coming down from ∞

By saying $(X_t)_{t\geq 0}$ is a solution to (5) with initial state ∞, we mean $\lim_{t\downarrow 0} X_t = \infty$ and $(0 < r < t)$

$$X_t = X_r + \sqrt{2c}\int_r^t \sqrt{X_s^\theta}dB_s + \int_r^t \int_{(0,1]} \int_0^{X_{s-}^\theta} z\tilde{M}(ds, dz, du)$$
$$- b\int_r^t X_s^\theta ds + \int_r^t \int_{(1,\infty)} \int_0^{X_{s-}^\theta} zM(ds, dz, du).$$

Theorem 4

Suppose that $\psi'(0) \geq 0$ and (9) holds. Then $(X_t^\infty)_{t\geq 0} := \lim_{x\to\infty}(X_t^x)_{t\geq 0}$ is the pathwise unique solution to (5) with $X_0^\infty = \infty$.

Convergence of discrete-state process

Recall Q-matrix of discrete-state polynomial branching process

$$q_{i,j} = \begin{cases} i^\theta b_{j-i+1}, & i+1 \leq j < \infty, i \geq 1, \\ -i^\theta, & 1 \leq i = j < \infty, \\ i^\theta b_0, & 0 \leq i-1 = j < \infty, \\ 0, & \text{otherwise}, \end{cases}$$

where $\{b_i\}_{i\in\mathbb{N}}$ is a probability measure satisfies $b_1 = 0$.

Proposition

There exists a sequence of discrete-state polynomial branching processes $\xi_n = (\xi_n(t) : t \geq 0)$ and a sequence of positive number γ_n, $n = 1, 2, \ldots$ such that $(n^{-1}\xi_n(\gamma_n t) : t \geq 0)$ converges to $(X_t : t \geq 0)$ weakly in (D, d_∞), where d_∞ is the Skorokhod distance.

Problems to be considered in the future

- The speed of coming down from infinity;
- Immigration structures;
- Quasi-stationary distribution;
- General nonlinear branching;

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Thank you!