Alpha-CIR Model with Branching Processes in Sovereign Interest Rate Modelling

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Motivation

- Current sovereign bond markets in the Euro zone:
  - persistency of low interest rates
  - significant fluctuations at local extent.

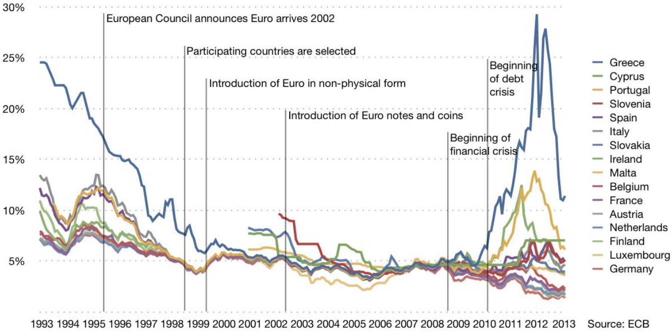


Figure: 10-years interest rates of Euro area countries.

Modelling approaches

- Large fluctuations in financial data motivate the introduction of jumps in the interest rate dynamics: Eberlein & Raible (1999), Filipović, Tappe & Teichmann (2010)...
- Hawkes process to model the "self-exciting" and the "clustering" feature: Aït-Sahalia & Jacod (2009), Errais, Giesecke & Goldberg (2010), Dassios & Zhao (2011), Rambaldi, Pennesi & Lillo (2014), and Jaisson & Rosenbaum (2015)...
- Difficulty: jump presence v.s. trend of low interest rate

Plan of our work

- Objective: a simple model of interest rate for these seemingly puzzling phenomena in a unified and parsimonious framework.
- Jump model as natural extension of the Cox-Ingersoll-Ross (CIR) model, using the α-stable branching processes
  - CIR model is the particular case with continuous path
- Integral representation to highlight the branching property
  - CBI process approach based on Dawson and Li (2006, 2012), Li and Ma (2015)
  - link with affine models: exponential affine structure for bond price, Duffie, Filipović & Schachermayer (2001)
  - limit of Hawkes processes: clustering and self-exciting properties
  - MLE estimator properties studied by Barczy, Ben Alaya, Kebaier and Pap (2016)
- The so-called α-CIR model provides nice properties in terms of trajectory behaviors and bond pricing

The α-CIR model setup

We consider α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ model for the short interest rate

$$r_t = r_0 + \int_0^t a\,(b - r_s)\,ds + \sigma \int_0^t \sqrt{r_s} dB_s + \sigma_Z \int_0^t r_{s-}^{1/\alpha} dZ_s \qquad (1)$$

- $B = (B_t, t \geq 0)$ a Browinan motion
- $Z = (Z_t, t \geq 0)$ a spectrally positive α-stable compensate Lévy process with parameter $\alpha \in (1, 2]$ with

$$\mathbb{E}\left[e^{-qZ_t}\right] = \exp\left\{-\frac{tq^\alpha}{\cos(\pi\alpha/2)}\right\}, \quad q \geq 0.$$

- B and Z are independent

Z_t follows the α-stable distribution $S_\alpha(t^{1/\alpha}, 1, 0)$ with scale parameter $t^{1/\alpha}$, skewness parameter 1 and zero drift.

A natural extension of the CIR model

- Existence of the unique strong solution by Fu and Li (2010).
- When $\sigma_Z = 0$, we recover the CIR model.
- When $\alpha = 2$, it also reduces to a CIR model but with volatility parameter $(\sigma^2 + 2\sigma_Z^2)^{1/2}$.
- The difference of Z from a Brownian motion is controlled by the tail index α:
  $\diamond$ $\alpha = 2$: Z is a Brownian motion scaled by $\sqrt{2}$;
  $\diamond$ $\alpha < 2$: Z is a pure jump process with heavy tails. More as α close to 1, more likely Z_t takes values far from median;
  $\diamond$ comparison with Poisson process: Z has an infinite number of (small) jumps over any time interval, allowing it to capture the extreme activity.

Simulation of processes Z and r with different α

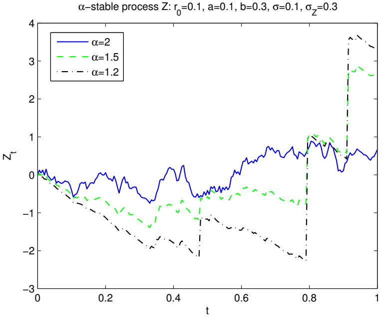


Figure: Three parameters of α: 2 (blue), 1.5 (green) and 1.2 (black)

Several advantages from the financial point of view

- By combining heavy-tailed jump distribution with infinite activity, the model describes in a unified way both the large fluctuations in recent sovereign crisis and the usual small oscillations.
- The interest rate can be split into different components in a branching process framework by Dawson and Li (2012), which can eventually be interpreted as spreads.
- By the link with the CBI processes, the bond prices are given in an explicit way by using the joint Laplace transform of the affine model in Filipović (2001)
- The most interesting feature is that the bond prices decrease w.r.t. α, which inversely related to the tail fatness. This allows to interpret the low interest rate phenomenon from the viewpoint of bond pricing.

Similar properties with CIR model

Boundary condition:

The point 0 is an inaccessible boundary if and only if $2ab \geq \sigma^2$. In particular, a pure jump α-CIR process with $ab > 0$ never reaches 0 since $\sigma = 0$.

Branching property (Dawson and Li 2012):

r can be decomposed as $r = r^{(1)} + r^{(2)}$ where for $i = 1, 2$, $r^{(i)}$ is an α-CIR$(a, b^{(i)}, \sigma, \sigma_Z, \alpha)$ process such that $r_0 = r_0^{(1)} + r_0^{(2)}$ and $b = b^{(1)} + b^{(2)}$.

Integral representation

Integral form by using the random fields

$$r_t = r_0 + \int_0^t a\,(b - r_s)\,ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) \\ + \sigma_Z \int_0^t \int_0^{r_{s-}} \int_{\mathbb{R}^+} \zeta \widetilde{N}(ds, du, d\zeta), \tag{2}$$

- $W(ds, du)$: white noise on $\mathbb{R}_+^2$ with intensity $dsdu$,
- $\widetilde{N}(ds, du, d\zeta)$: compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $dsdu\mu(d\zeta)$,
- $\mu(d\zeta)$ is a Lévy measure satisfying $\int_0^\infty (\zeta \wedge \zeta^2)\mu(d\zeta) < \infty$. Besides, W and N are independent of each other.
- It follows from of Dawson and Li (2012) that this equation has a unique strong solution.

Random fields for interest rate modelling: Kennedy (1994), Albeverio, Lytvynov & Mahnig (2004).

Equivalence of two representations

We choose the Lévy measure to be

$$\mu(d\zeta) = -\frac{1_{\{\zeta>0\}}d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1 < \alpha < 2, \tag{3}$$

Then the root representation (1) and the integral representation (2) are equivalent in the following sense by Li (2011):

- The solutions of the two equations have the same probability law.
- On an extended probability space, they are equal almost surely.

Link to Hawkes process

- When $\sigma = 0$ and $\mu(d\zeta) = \delta_1(dz)$, then r is given by

$$r_t = r_0 + abt - \int_0^t (a + \sigma_Z) r_s ds + \sigma_Z \int_0^t \int_0^{r_{s-}} N(ds, du) \quad (4)$$

which is the intensity of Hawkes process $\int_0^t \int_0^{r_{s-}} N(ds, du)$, N being the Poisson random measure with intensity $dsdu$.

- Consider a sequence $\left\{ r_t^{(n)}, t \geq 0 \right\}$ defined by (4) with parameters $(a/n, nb, \sigma_Z)$. Then

$$r_{nt}^{(n)}/n \xrightarrow{\mathcal{L}} Y_t \quad \text{in } D(\mathbb{R}_+),$$

where $D(\mathbb{R}_+)$ is the Skorokhod space of càdlàg processes and

$$Y_t = \int_0^t a(b - Y_s) ds + \sigma_Z \int_0^t \int_0^{Y_s} W(ds, du).$$

- Jaisson and Rosenbaum (2015): nearly unstable Hawkes process converges, after suitable scaling, to a CIR process.

Locally equivalent Lévy-Ornstein-Uhlenbeck process

- Consider the α-CIR process with initial value r_0 and introduce

$$\lambda_t = r_0 + \int_0^t a\,(b - \lambda_s)\,ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) + \sigma_Z \int_0^t \int_0^{r_0} \int_{\mathbb{R}^+} \zeta \widetilde{N}(ds, du, d\zeta) \tag{5}$$

where the processes W and $\widetilde{N}$ are the same as in (1).

- the above LOU process can be written as

$$\lambda_t = r_0 + \int_0^t a\,(b - \lambda_s)\,ds + \sigma \int_0^t \sqrt{\lambda_s} dB_s + \sigma_Z \sqrt[\alpha]{r_0} Z_t$$

- The implicit negative drift leads to a linear decay for λ_t while an exponential decay for r_t: when σ_Z increases, the decreasing drift plays a more important role in α-CIR than in LOU.

Comparison between α-CIR and LOU (continued)

- Separating small and large jumps in LOU, we get

$$\lambda_t = r_0 + \int_0^t a\Big(b - \frac{\sigma_Z r_0 \Theta(\alpha, y)}{a} - \lambda_s\Big) ds + \sigma \int_0^t \int_0^{r_0} W(ds, du)$$
$$+ \sigma_Z \int_0^t \int_0^{r_0} \int_0^y \zeta \widetilde{N}(ds, du, d\zeta) + \sigma_Z \int_0^t \int_0^{r_0} \int_y^\infty \zeta N(ds, du, d\zeta)$$

  where

$$\Theta(\alpha, y) = \frac{2}{\pi} \alpha \Gamma(\alpha - 1) \frac{\sin(\pi\alpha/2)}{y^{\alpha-1}}.$$

- In a similar way, the α-CIR process can be written as

$$r_t = r_0 + \int_0^t \widetilde{a}(\alpha, y)\Big(\widetilde{b}(\alpha, y) - r_s\Big) ds + \sigma \int_0^t \int_0^{r_s} W(ds, du)$$
$$+ \sigma_Z \int_0^t \int_0^{r_{s-}} \int_0^y \zeta \widetilde{N}(ds, du, d\zeta) + \sigma_Z \int_0^t \int_0^{r_{s-}} \int_y^\infty \zeta N(ds, du, d\zeta)$$

  where

$$\widetilde{a}(\alpha, y) = a + \sigma_Z \Theta(\alpha, y), \quad \widetilde{b}(\alpha, y) = \frac{ab}{a + \sigma_Z \Theta(\alpha, y)}$$

Continuous state branching process with immigration (CBI)

CBI (Kawazu & Watanabe 1971) of branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$: Markov process X with state space $\mathbb{R}_+$ verifying

$$\mathbb{E}_x\left[e^{-pX_t}\right] = \exp\left[-xv(t,p) - \int_0^t \Phi\big(v(s,p)\big)ds\right],$$

where $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p$$

and Ψ and Φ are functions on $\mathbb{R}_+$ given by

$$\Psi(q) = \beta q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(du),$$
$$\Phi(q) = \gamma q + \int_0^\infty (1 - e^{-qu})\nu(du),$$

with $\sigma, \gamma \geq 0$, $\beta \in \mathbb{R}$ and π, ν being two Lévy measures such that $\int_0^\infty (u \wedge u^2)\pi(du) < \infty$ and $\int_0^\infty (1 \wedge u)\nu(du) < \infty$.

Link with the CBI processes

Let r be an α-CIR $(a, b, \sigma, \sigma_Z, \alpha)$ process. Then r is a CBI with

$$\text{branching mechanism: } \Psi(q) = aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha \qquad (6)$$

$$\text{immigration rate: } \Phi(q) = abq. \qquad (7)$$

Consequences:

- Let $r^{(\alpha)}$ be α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process, $\alpha \in (1, 2]$. Then $r^{(\alpha)} \xrightarrow{\mathcal{L}} r^{(2)}$ in $D(\mathbb{R}_+)$ as $\alpha \to 2$.
- Laplace transform (cf. Filipović (2001)):

$$\mathbb{E}\Big[e^{-\xi r_t - p\int_0^t r_s ds}\Big] = \exp\Big(- r_0 v(t,\xi,p) - \int_0^t \Phi\big(v(s,\xi,p)\big)ds\Big),$$

$$\text{with } \partial_t v(t,\xi,p) = -\Psi(v(t,\xi,p)) + p, \quad v(0,\xi,p) = \xi.$$

- As $t \to +\infty$, r_t has a limite distribution r_∞ (cf. Keller-Ressel and Steiner (2008)), given by

$$\mathbb{E}[e^{-pr_\infty}] = \exp\left\{-\int_0^p \frac{\Phi(q)}{\Psi(q)}dq\right\}, \quad p \geq 0.$$

Equivalent martingale measure for bond pricing

- Let r be an α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ processes under the initial probability $\mathbb{P}$.
- Fix $\eta \in \mathbb{R}$ and $\theta \in \mathbb{R}_+$, and define

$$U_t := \eta \int_0^t \int_0^{r_s} W(ds, du) + \int_0^t \int_0^{r_{s-}} \int_0^\infty (e^{-\theta\zeta} - 1) \widetilde{N}(ds, du, d\zeta).$$

- Change of probability: $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(U)$, with $\mathcal{E}(U)$ the Doléans-Dade exponential of U (Kallsen & Muhle-Karbe, 2010).
- r is an α-CIR$(a', b', \sigma, \sigma_Z, \alpha)$ type process under $\mathbb{Q}$ with

$$a' = a - \sigma\eta - \frac{\alpha\sigma_Z}{\cos(\pi\alpha/2)}\theta^{\alpha-1}, \ \ b' = ab/a',$$

and a modified Lévy measure

$$\mu'(d\zeta) = -\frac{e^{-\theta\zeta} 1_{\{\zeta>0\}}}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}} d\zeta.$$

r remains to be a CBI process under $\mathbb{Q}$.

Application to bond pricing

For simplicity, we assume that the short rate r is given by an α-CIR$(a, b, \sigma, \sigma_Z, \mu, \alpha)$ model under $\mathbb{Q}$.

- Zero-coupon bond price:

$$B(t,T) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big)\,|\,\mathcal{F}_t\Big] = \exp\Big(-r_t v(T-t) - ab\int_0^{T-t} v(s)ds\Big)$$

  where $v(\cdot)$ is given by

$$\frac{\partial v(t)}{\partial t} = 1 - \Psi(v(t)), \quad v(0) = 0,$$

  with $\Psi(q) = aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha$.

- We have

$$v(t) = f^{-1}(t) \text{ where } f(t) = \int_0^t \frac{dx}{1-\Psi(x)} \tag{8}$$

Proposition

The function $v(\cdot)$ is increasing with respect to $\alpha \in (1, 2]$. In particular, the bond price $B(0, T)$ is decreasing with respect to α.

- Empirical studies underline that CIR model systematically overestimates short interest rates, e.g. Brown and Dybvig (1986) and Gibbons and Ramaswamy (1993)
- The above proposition shows that the α-CIR model is suitable to describe the phenomenon of low interest rate trend with jumps.
- The explanation is based on the self-exciting property: a smaller α is related to a deeper (negative) compensation and hence a stronger mean-reversion. Then as the interest rate becomes low, the self-exciting property will imply a decreasing frequency of jumps and enforce the tendency of low interest rate.
- In other CIR+jump models e.g. Duffie and Gârleanu (2001), Keller-Ressel and Steiner (2008), LOU etc., the bond prices are in general smaller than the CIR ones.

Bond prices

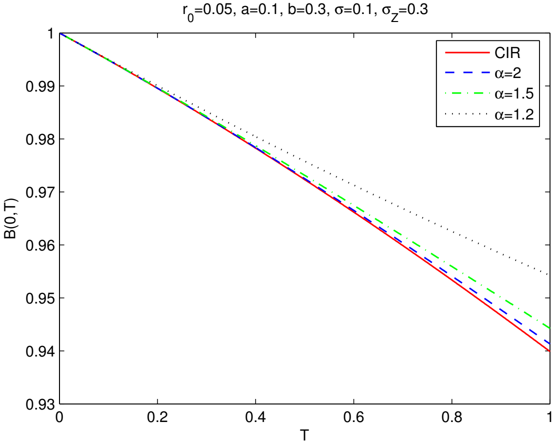


Figure: Bond price is decreasing w.r.t. α, curve CIR (in red) corresponds to $\sigma_Z = 0$

Bond yield behavior

- The zero-coupon yield $Y(t,\theta)$ is given by $Y(t,0) = r_t$ and

$$Y(t,\theta) = -\frac{1}{\theta}\log B(t,t+\theta) = r_t\frac{v(\theta)}{\theta} + \frac{ab\int_0^\theta v(s)ds}{\theta}, \quad \theta > 0$$

- The long-term yield, which is the asymptotic level of the yield curve when $\theta \to \infty$, is given by

$$b_{\text{asym}} = abx_0$$

where x_0 is the unique positive solution of $\Psi(x) = 1$.

Application to path-dependent option

Let r be an α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process with initial value $r_0 > 0$.

- Put option written on the running minimum of the bond yield (Lookback option):

$$P\Big(\inf_{u\in[0,T]} Y(u,u+\kappa), 0, T, K\Big) := \mathbb{E}\Big[e^{-\int_0^T r_s ds}\Big(K - \inf_{u\in[0,T]} Y(u,u+\kappa)\Big)_+\Big]$$

where

$$Y(t,t+\kappa) = -\frac{1}{\kappa}\ln B(t,t+\kappa) = \frac{1}{\kappa}\Big(r_t f^{-1}(\kappa) + ab\int_0^\kappa f^{-1}(s)ds\Big).$$

- Laplace transform with respect to the maturity:

$$L_\theta(0,\kappa,K;r_0) = \int_0^\infty e^{-\theta T} P\Big(\inf_{u\in[0,T]} Y(u,u+\kappa), 0, T, K\Big)dT.$$

Path-dependent option on bond yield (continued)

We have

$$L_\theta\left(0,\kappa,K;r_0\right)=\frac{f^{-1}(\kappa)}{\kappa}\int_0^{\overline{K}}\frac{H_\varepsilon(\theta,r_0)}{H_\varepsilon(\theta,y)}M(\theta,y)dy,$$

where

$$\overline{K}:=\frac{\kappa K-ab\int_0^\kappa f^{-1}(s)ds}{f^{-1}(\kappa)},$$

$$H_\varepsilon(\theta,y)=\int_{q_1}^\infty\frac{e^{-yz}dz}{\Psi(z)-1}\exp\Big(\int_{q_1+\varepsilon}^z\frac{abu+\theta}{\Psi(u)-1}du\Big),$$

with $\Psi(q_1)=1$ and ε is an arbitrary positive number, and

$$M(\theta,y)=\int_0^\infty e^{-\theta u}B_y(0,u)du$$

with $B_y(0,u)$ being the bond price with initial short rate y.

Jump behavior

- The jumps, especially the large jumps capture the significant changes in the interest rate and may imply the downgrade risk of credit quality.
- Fix $y > 0$. Consider the jumps of the process r which are larger than $\sigma_Z y$ and the associated truncated process $r^{(y)}$ as

$$r_t^{(y)} = r_0 + \int_0^t \widetilde{a}(\alpha, y)\big(\widetilde{b}(\alpha, y) - r_s\big)ds + \sigma \int_0^t \int_0^{r_s} W(ds, du)$$
$$+ \sigma_Z \int_0^t \int_0^{r_{s-}} \int_0^y \zeta \widetilde{N}(ds, du, d\zeta).$$

- It is also a CBI process which coincides with r up to the first large jump $\tau_y := \inf\{t > 0 : \Delta r_t > \sigma_Z y\}$ and has the branching mechanism given by

$$\Psi^{(y)} = \Psi + \sigma_Z^\alpha \int_y^\infty (1 - e^{-q\zeta})\mu(d\zeta).$$

Laplace transform of the jump counter process

Let J_t^y denote the number of jumps of r with jump size larger than $\sigma_Z y$ in $[0,t]$, i.e.

$$J_t^y := \sum_{0\le s\le t} 1_{\{\Delta r_s > \sigma_Z y\}}.$$

Then for any $p \ge 0$ and $t \ge 0$,

$$\mathbb{E}\Big[e^{-pJ_t^y}\Big] = \exp\Big(-l(p,y,t)r_0 - ab\int_0^t l(p,y,s)ds\Big)$$

where $l(p,y,t)$ is the unique solution of the following equation

$$\frac{\partial l(p,y,t)}{\partial t} = \sigma_Z^\alpha \int_y^\infty \big(1 - e^{-p-l(p,y,t)\zeta}\big)\mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}(l(p,y,t)),$$

with initial condition $l(p,y,0) = 0$.

Probability law of the first large jump

We have

$$\mathbb{P}(\tau_y > t) = \exp\Big(- I(y,t) r_0 - ab \int_0^t I(y,s) ds \Big)$$

where $I(y,t)$ is the unique solution of

$$\frac{dI}{dt}(y,t) = \sigma_Z^\alpha \int_y^\infty \mu(d\zeta) - \Psi^{(y)}(I(y,t)),$$

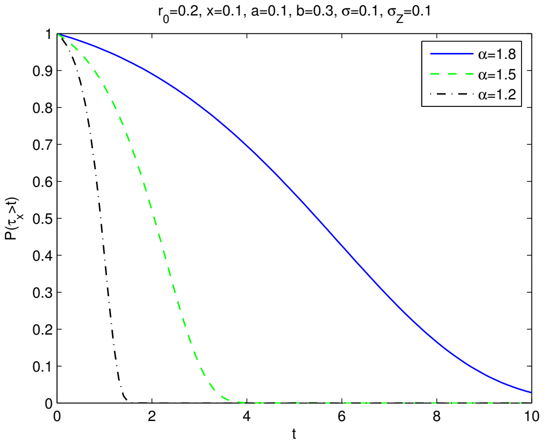
with initial condition $I(y,0) = 0$.

- Equivalent form:

$$\mathbb{P}(\tau_y > t) = \mathbb{E}\Big[\exp\Big\{ - \sigma_Z^\alpha \Big(\int_y^\infty \mu(d\zeta) \Big) \Big(\int_0^t r_s^{(y)} ds \Big) \Big\} \Big].$$

  which is a bond price written on the auxiliary rate $r^{(y)}$ weighted by the measure μ restricted on (y, ∞).

Probability function $\mathbb{P}(\tau_y > t)$ for the first big jump



Thanks for your attention !