Moments of Continuous-State Branching Processes with or without Immigration

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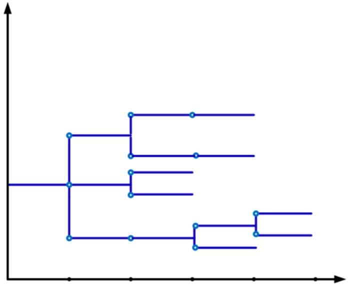
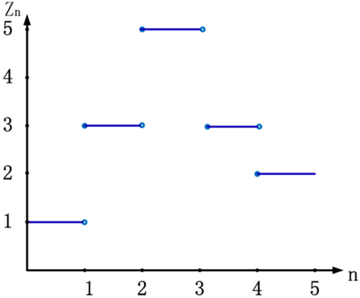
1. Galton-Waston processes

Let $\{\xi_{k,i}\}$ be a family of positive integer-valued i.i.d. random variables. Given Z_0, we can define a Galton-Watson branching process by:

$$Z_k = \sum_{i=1}^{Z_{k-1}} \xi_{k,i}, \qquad k \geq 1, \tag{1}$$

where $\xi_{k,i}$ = the number of children of i-th particle at generation $k-1$.





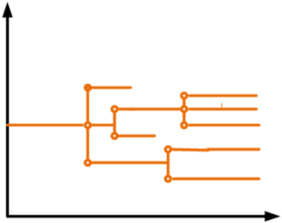
Rewrite the formulation (1),

$$
\begin{aligned}
\boldsymbol{Z_k} &= \boldsymbol{Z_{k-1} + \sum_{i=1}^{Z_{k-1}} (\xi_{k,i} - 1)}, \\
\boldsymbol{Z_n} &= \boldsymbol{Z_0 + \sum_{k=1}^{n} \sum_{j=1}^{Z_{k-1}} (\xi_{k,i} - 1)}. \qquad (2)
\end{aligned}
$$

In GW-process, the lifetime of each particle was one unit of time. A natural generalization is to allow the lifetimes to be random variables.

Continuous-time Markov branching process:

- the progenies of various particles are i.i.d.;
- each particle lives for an exponential time.



2. Continuous-time Markov branching processes

Let m be a finite measure on $\mathbb{N}$ satisfying $m(1) = 0$ and $m(\mathbb{N}) = a$ (reproduction measure). Given Z_0, we can define a Continuous-time Markov branching process by:

$$Z_t = Z_0 + \int_0^t \int_{\mathbb{N}} \int_0^{Z_{s-}} (z-1) N(ds, dz, du), \tag{3}$$

where $N(ds, dz, du)$ is a Poisson random measure with intensity $ds m(dz) du$ on $(0, \infty) \times \mathbb{N} \times (0, \infty)$.
Suppose that $\mu := a^{-1} \int_{\mathbb{N}} z m(dz) < \infty$. Then ($b = a(1 - \mu)$)

$$Z_t = Z_0 - b \int_0^t Z_s ds + \int_0^t \int_{\mathbb{N}} \int_0^{Z_{s-}} (z-1) \tilde{N}(ds, dz, du),$$

where $\tilde{N}(ds, dz, du) = N(ds, dz, du) - ds m(dz) du$.

GW-process $\Longleftrightarrow$ Continuous-time branching process $\Longleftrightarrow$ CB-process

3. CB-processes and CBI-processes

Suppose that $\sigma \geq 0$ and b are constants, and $(z \wedge z^2)m(dz)$ is a finite measure on $(0, \infty)$. Let

- $W(ds, du)$ = Gaussian white noise with intensity $dsdu$ on $\mathbb{R}_+^2$;
- $\tilde{M}(ds, dz, du)$ = compensated Poisson random measure with intensity $dsm(dz)du$ on $\mathbb{R}_+^3$.

Theorem (Dawson/Li '12) There is a pathwise unique positive (strong) solution to

$$\begin{aligned} X_t &= X_0 - b\int_0^t X_s ds + \sigma \int_0^t \int_0^{X_{s-}} W(ds, dz) \\ &\quad + \int_0^t \int_0^\infty \int_0^{X_{s-}} z\tilde{M}(ds, dz, du). \end{aligned}$$

And $(X_t)_{t\geq 0}$ is a continuous-state branching process (CB-process).

Suppose that $(1 \wedge z^2)m(dz)$ is a finite measure on $(0, \infty)$. Give a branching mechanism with the representation

$$\phi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-z\lambda} - 1 + z\lambda 1_{\{z\leq 1\}})m(dz), \quad \lambda \geq 0.$$

We assume

$$\int_{0+} \frac{1}{\phi(\lambda)} d\lambda = \infty.$$

Then the CB-process with branching mechanism ϕ is conservative. Given X_0, we consider the stochastic equation

$$\begin{aligned} X_t &= X_0 + \sigma \int_0^t \int_0^{X_{s-}} W(ds, dz) + \int_0^t \int_0^1 \int_0^{X_{s-}} z\tilde{M}(ds, dz, du) \\ &\quad - b\int_0^t X_s ds + \int_0^t \int_1^\infty \int_0^{X_{s-}} zM(ds, dz, du). \end{aligned} \tag{4}$$

Theorem 1

There is a unique positive strong solution to (4) and the solution $(X_t)_{t\geq 0}$ is a CB-process with branching mechanism ϕ.

Consider an immigration mechanism ψ given by

$$\psi(\lambda) = h\lambda + \int_0^\infty (1 - e^{-\lambda z}) n(dz), \qquad \lambda \geq 0,$$

where $h \geq 0$ is a constant and $(1 \wedge z)n(dz)$ is a finite measure on $\mathbb{R}_+$. Let $N(ds, dz)$ be a Poisson random measure on $(0, \infty)^2$ with intensity $ds n(dz)$. Given Y_0, we consider the stochastic equation

$$\begin{aligned} Y_t &= Y_0 + \sigma \int_0^t \int_0^{Y_{s-}} W(ds, du) + \int_0^t \int_0^1 \int_0^{Y_{s-}} z\tilde{M}(ds, dz, du) \\ &\quad + \int_0^t (h - bY_s) ds + \int_0^t \int_1^\infty \int_0^{Y_{s-}} zM(ds, dz, du) \\ &\quad + \int_0^t \int_0^\infty zN(ds, dz). \end{aligned} \tag{5}$$

Theorem 2

There is a unique positive strong solution to (5) and the solution $(Y_t)_{t \geq 0}$ is a CBI-process with branching mechanism ϕ and immigration mechanism ψ.

4. $f-$moments for classical branching processes

Suppose that f is a positive continuous function on $[0, \infty)$ satisfying the following:

Condition A. There exist constants $c \geq 0$ and $K > 0$ such that

(A1) f is convex on $[c, \infty)$;
(A2) $f(xy) \leq Kf(x)f(y)$ for all $x, y \in [c, \infty)$;
(A3) f is bounded in $[0, c)$.

- Important examples: $f(x) = x|logx|$ and $f(x) = x^n$.

Theorem (Athreya '69) Suppose that f satisfies Condition A. Let $\{Z_t : t \geq 0\}$ be a continuous-time branching process with $Z_0 = 1$ and reproduction measure m. Then for any $t > 0$ we have

$$\mathbf{P} f(Z_t) < \infty$$

if and only if

$$\sum m(k) f(k) < \infty.$$

5. $f-$moments for continuous-state branching processes

For CB-processes and CBI-processes,

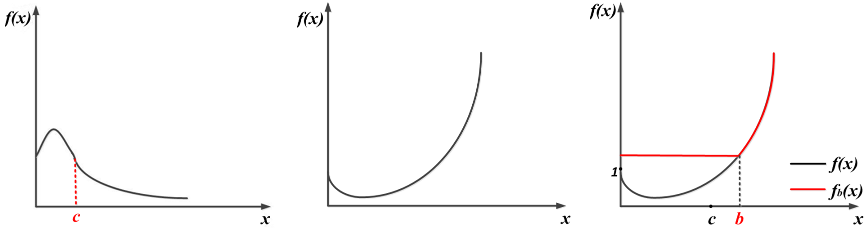
- Grey (1974) studied the existence of $xlogx$-moment of CB-processes;
- Bingham (1976) studied the existence of x^n-moment of CB-processes;
- A recursive formula for x^n-moments of multi-type CBI-processes was given by Barczy et al. (2015).

In our paper, we study the general f-moments of CB-processes with or without immigration.

Instead of Condition A, we introduce the following more convenient condition:

Condition B. There exists a constant $K > 0$ such that

(B1) $f(x)$ is convex and nondecreasing on $[0, \infty)$;
(B2) $f(xy) \le K f(x) f(y)$ for all $x, y \in [0, \infty)$;
(B3) $f(x) > 1$ for all $x \in [0, \infty)$.



A probability measure on $[0, \infty)$ has finite $f-$moment if and only if it has finite f_b-moment.

- Let $\{X_t(x) : t \geq 0\}$ be the solution of (4) with $X_0(x) = x$.
- Let $X_t^{(i)}(1) = X_t(i) - X_t(i-1)$. Then $\{X_t^{(i)}(1) : t \geq 0\}$, $i = 1, 2, \ldots$ are $i.i.d.$ CB-processes with $X_0^{(i)}(1) = 1$.
- Let $\lfloor x \rfloor$ denote the largest integer smaller than or equal to $x \geq 0$.

By the Markov property we have

$$\begin{aligned}
\mathbf{P} f(X_t) &= \mathbf{P}\Big[\mathbf{P}[f(X_t)|\mathscr{G}_0]\Big] \leq \mathbf{P}\Big[\mathbf{P}\Big[f\Big(\sum_{i=1}^{\lfloor X_0 \rfloor + 1} X_t^{(i)}(1)\Big)\Big|\mathscr{G}_0\Big]\Big] \\
&\leq K \mathbf{P} f(1 + X_0) \mathbf{P} f(X_t(1)) \\
&\leq \frac{1}{2} K^2 f(2)\big[f(1) + \mathbf{P} f(X_0)\big] \mathbf{P} f(X_t(1)).
\end{aligned}$$

Proposition 3

Suppose that f satisfies Condition B and $\mathbf{P} f(X_t(x)) < \infty$ for some $x > 0$ and $t > 0$. Let $\{X_t : t \geq 0\}$ be a CB-process with branching mechanism ϕ and arbitrary initial distribution. Then $\mathbf{P} f(X_t) < \infty$ if and only if $\mathbf{P} f(X_0) < \infty$.

Lemma 4

Suppose that f satisfies Condition B and $\int_1^\infty z^n m(dz) < \infty$ for every $n \geq 1$. Then for any $x > 0$ the function $t \to \mathbf{P} f(X_t(x))$ is locally bounded on $[0, \infty)$.

Consider the stochastic equation

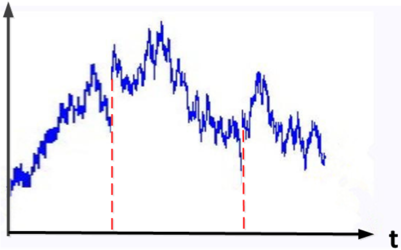
$$Z_t(x) = x - b\int_0^t Z_{s-}(x)ds + \sigma \int_0^t \int_0^{Z_{s-}(x)} W(ds, du) + \int_0^t \int_0^1 \int_0^{Z_{s-}(x)} z\tilde{M}(ds, dz, du). \tag{6}$$

Then $\{Z_t(x) : t \geq 0\}$ is a CB-process with branching mechanism

$$\phi_1(\lambda) = \beta\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^1 (e^{-\lambda z} - 1 + \lambda z)m(dz), \qquad \lambda \geq 0.$$

By Lemma 4, we can get $t \to \mathbf{P} f(Z_t(x))$ is locally bounded.

- Let $\tau_0(x) = 0$ and $\tau_n(x)$ denote the nth jump time with jump size in $(1, \infty)$ of $X_t(x)$ for $n \geq 1$. $\mathbf{P}(\tau_1(x) \in dt)$ was given by He and Li (2016).
- Let $\Delta_n = X_{\tau_n}(x) - X_{\tau_n -}(x)$. Then $\mathbf{P}(\Delta_n \in dz) = m(dz)/m(1, \infty)$.



Theorem 5

Suppose that f satisfies Condition A. Let $\{X_t : t \geq 0\}$ be a CB-process with $\mathbf{P}(X_0 > 0) > 0$. Then for any $t > 0$ we have

$$\mathbf{P}f(X_t) < \infty$$

if and only if

$$\mathbf{P}f(X_0) < \infty \text{ and } \int_1^\infty f(z)m(dz) < \infty.$$

Recall that a CBI-process is defined by the stochastic equation,

$$\begin{aligned} Y_t &= Y_0 + \sigma \int_0^t \int_0^{Y_{s-}} W(ds, du) + \int_0^t \int_0^1 \int_0^{Y_{s-}} z\tilde{M}(ds, dz, du) \\ &\quad + \int_0^t (h - bY_s)ds + \int_0^t \int_1^\infty \int_0^{Y_{s-}} zM(ds, dz, du) \\ &\quad + \int_0^t \int_0^1 zN(ds, dz) + \int_0^t \int_1^\infty zN(ds, dz). \end{aligned} \tag{7}$$

Theorem 6

Suppose that f *satisfies Condition A. Let* $\{Y_t : t \geq 0\}$ *be a CBI-process with* $\mathbf{P}(Y_0 > 0) > 0$. *Then for every* $t > 0$ *we have*

$$\mathbf{P}f(Y_t) < \infty$$

if and only if

$$\int_1^\infty f(z)(m+n)(dz) < \infty \text{ and } \mathbf{P}f(Y_0) < \infty.$$

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Thank you!