Branching structure within the (L, R)-random walk and its applications

Wenming Hong

(Beijing Normal University)

"3rd Workshop on Branching Processes and Related Topics"

BNU, May 8-12, 2017.

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Contents

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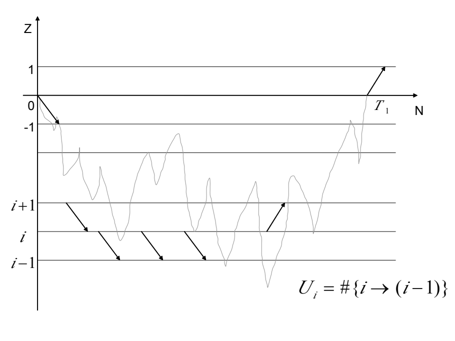

Figure: Branching structure of (1, 1)-RWRE

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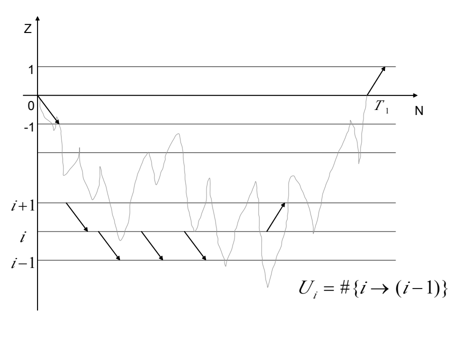

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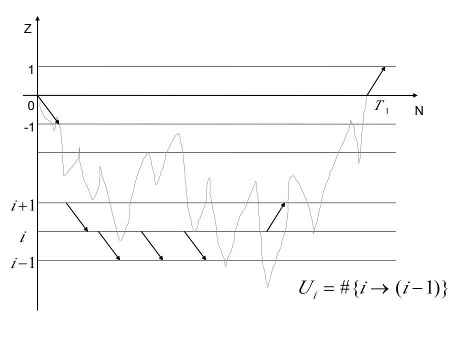

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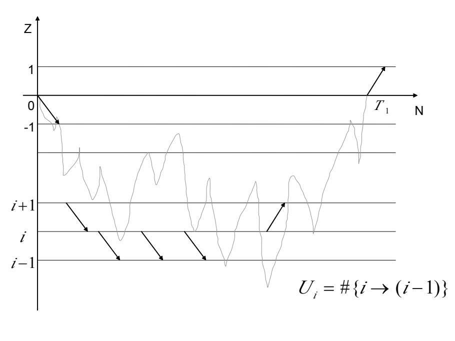

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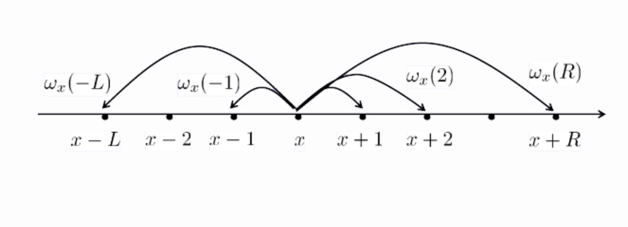
- How about the non-homogeneous and non-nearest Random walk ?

Model: (L, R)-RWRE



$$\omega_x(-L) + \cdots + \omega_x(-1) + \omega_x(1) + \cdots + \omega_x(R) = 1$$

Brémont, J. (2002). *Ann. of Probab.*
Key, E.S. (1984). *Ann. Prob.*
Zeitouni, O. (2004). *LNM 1837, 189-312,*

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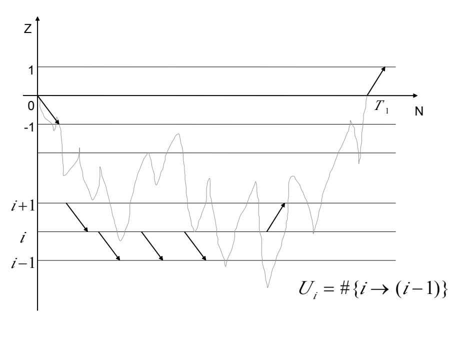

Figure: Branching structure of (1, 1)-RWRE

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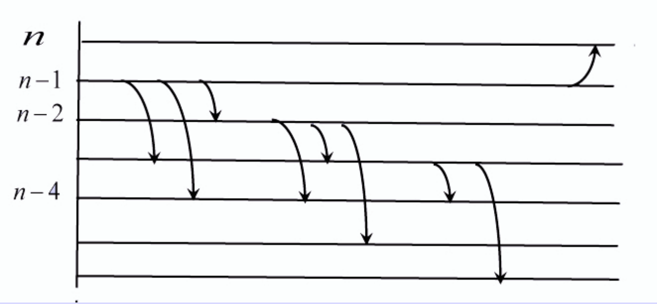
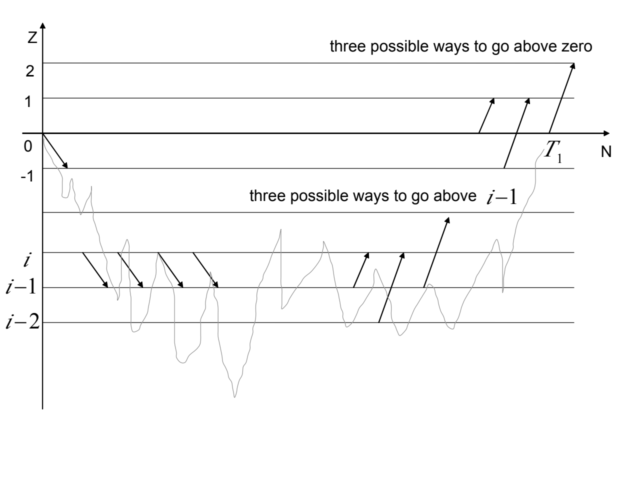


T_1 for (1, 2)-RW



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  - Lamperti Problem: stationary distribution $\pi_i = \frac{1}{E_i T_i}$.

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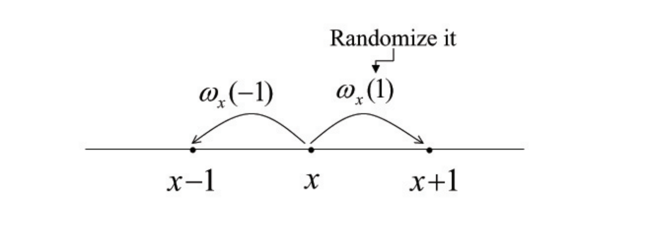
Branching Structure: (1,1)-RWRE (Kesten et al, 1975)

(1,1)-RWRE:



(1) $\rho_i = \frac{\omega_i(-1)}{\omega_i(1)}$.

(2) $\mathbb{E}(\log \rho_0) \le 0$, then P^o-a.s., $\limsup_{n\to\infty} X_n = \infty$.

(3) $T_1 := \inf[k \ge 0 : X_k = 1] < \infty$ P^o-a.s..

(4) $U_i := \#\{k < T_1 : X_{k-1} = i, X_k = i-1\}$.

Branching Structure: (1,1)-RWRE (Kesten et al, 1975)

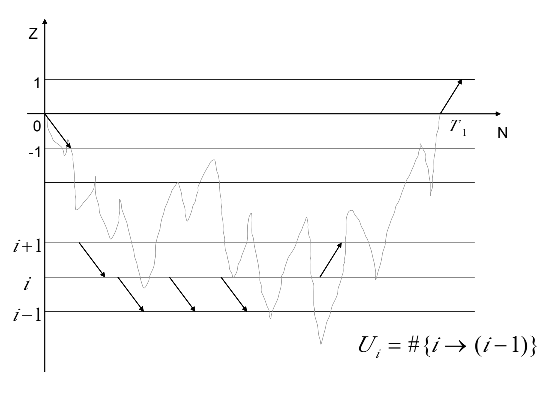


Figure: Branching structure of (1, 1)-RWRE

$$T_1 = 1 + 2\sum_{i \leq 0} U_i.$$

Branching Structure: (1,1)-RWRE

Theorem (Kesten et al, 1975)

Assume $X_n \to \infty$,
(a) $U_1 = 1$, $U_1, U_0, U_{-1}, \ldots$ *is under* P_ω^o *an inhomogeneous branching process with offspring distribution, for* $i \leq 0$,

$$P_\omega^o(U_{i-1} = k | U_i = 1) = \omega_i(-1)^k \omega_i(1), k = 0, 1, 2, \ldots. \qquad (1)$$

(b)

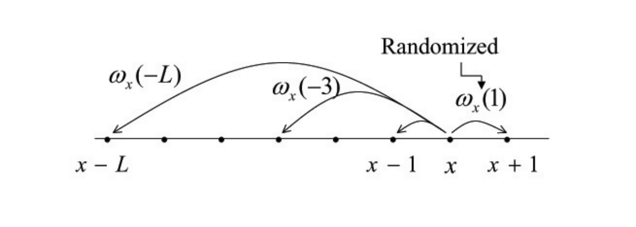
$$T_1 = 1 + 2 \sum_{i \leq 0} U_i.$$

Kesten, H., Kozlov, M. V., Spitzer, F. (1975). *Composotio Math.*;
Dwass (1975), *Proc.AMS.*
Harris (1952), *Tans.AMS*

Branching Structure: (L,1)-RWRE

(L-1) RWRE X_n:

$$P_\omega^o(X_{n+1} = x + l \big| X_n = x) = \omega_x(l)$$



$$\omega_\cdot = (\omega_x(-L), \cdots, \omega_x(-1), \omega_x(1))_{x \in Z} \sim \mathbb{P}.$$

Remark Key (1987) have pointed out the relationship between the $(L,1)$-RWRE and the multi-type branching process, but the branching structure have not been figured out.

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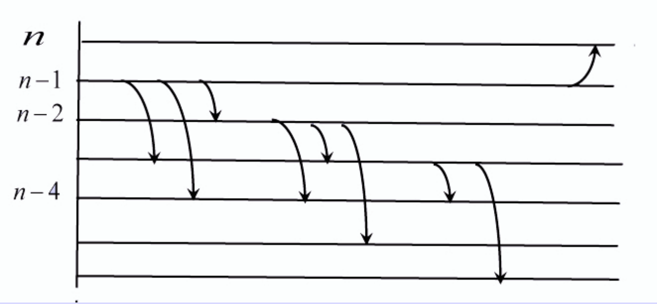

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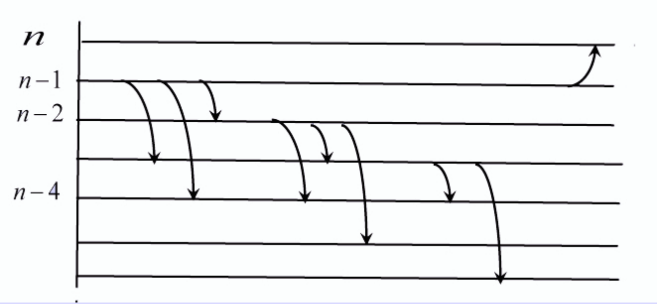

$$U_{i,l} = \#\{0 < k < T_1 : X_{k-1} > i, X_k = i - l + 1\},\ 1 \le l \le L$$

$$U_i := (U_{i,1}, U_{i,2}, \cdots, U_{i,L})$$

$$T_1 = 1 + \sum_{i=-\infty}^{0} |U_i| + \sum_{i=-\infty}^{0} U_{i,1} = 1 + \sum_{i=-\infty}^{0} U_i(2,1,...,1)^T.$$

Branching Structure: (L-1) RWRE

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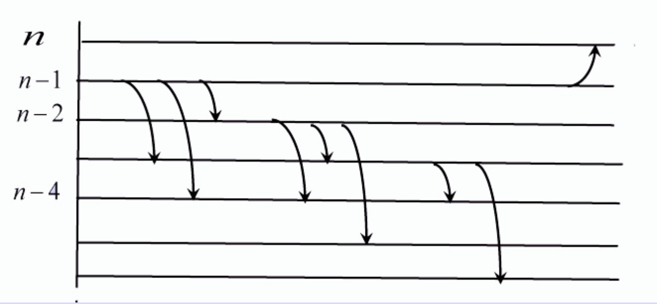

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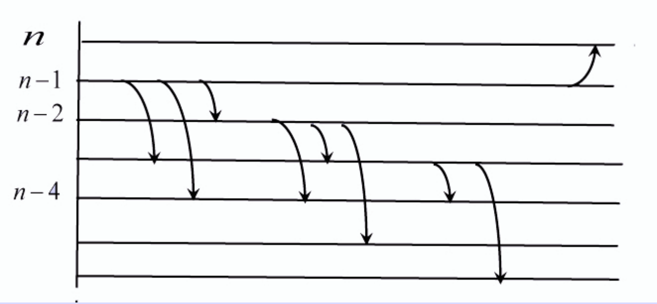

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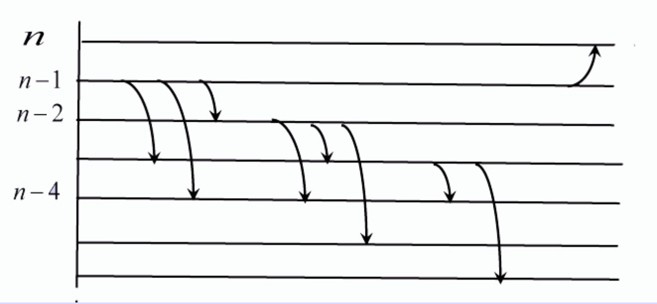

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$$T_1 = 1 + \sum_{i=-\infty}^{0} |U_i| + \sum_{i=-\infty}^{0} U_{i,1} = 1 + \sum_{i=-\infty}^{0} U_i(2,1,...,1)^T.$$

Theorem (H & Wang,2013)

Suppose that $\limsup_{n\to\infty} X_n = \infty.$, *one has that*

(a)

$$T_1 = 1 + \sum_{i=-\infty}^{0} |U_i| + \sum_{i=-\infty}^{0} U_{i,1} = 1 + \sum_{i=-\infty}^{0} U_i(2,1,...,1)^T; \tag{2}$$

(b) $U_1 = e_1$, $U_1, U_0, U_{-1}, \ldots$ *is an inhomogeneous multitype branching process with offspring distribution, for* $i \le 0$,

$$P_\omega(U_{i-1} = (u_1, ..., u_L) \big| U_i = e_1) = \frac{(u_1 + \cdots + u_L)!}{u_1! \cdots u_L!} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(1), \tag{3}$$

and for $2 \le l \le L$,

$$P_\omega(U_{i-1} = (u_1, ..., 1+u_{l-1}, ..., u_L) \big| U_i = e_l) = \frac{(u_1 + \cdots + u_L)!}{\cdots} \omega_i(-1)^{u_1} \cdots \omega_i(-L)^{u_L} \omega_i(1) \tag{4}$$

Branching Structure: (L,1)-RWRE (H & Wang,2013)

Theorem (H & Wang,2013, cont.)

(c) For fixed environment ω, the mean offspring matrix of the multitype branching process $\{U_i\}_{i\le 0}$ is

$$M_i = \begin{pmatrix} b_i(1) & \cdots & b_i(L-1) & b_i(L) \\ 1+b_i(1) & \cdots & b_i(L-1) & b_i(L) \\ \vdots & \ddots & \vdots & \vdots \\ b_i(1) & \cdots & 1+b_i(L-1) & b_i(L) \end{pmatrix}, \tag{5}$$

where $b_i(l) = \frac{\omega_i(-l)}{\omega_i(1)},\ 1 \le l \le L.$

Branching Structure: (1,2)-RWRE

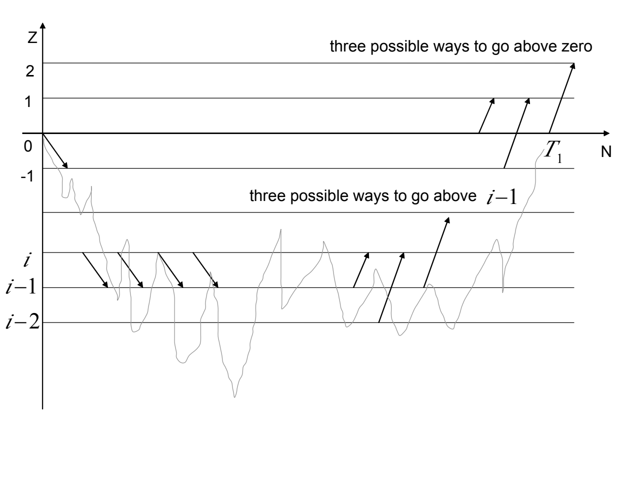

Figure: Branching structure of the (1, 2)-RWRE

Branching Structure: (1,2)-RWRE (Types)

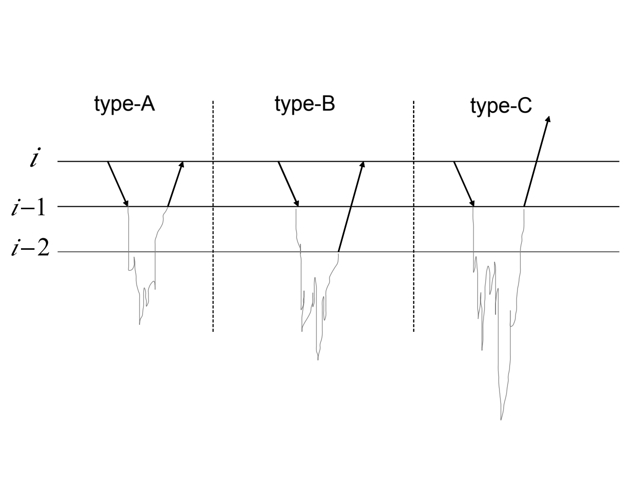

Figure: types

Branching Structure: (1,2)-RWRE (Types)

- $A(i)$:=#{downward steps of type A at i}= the numbers of steps from i to $i-1$ before time T_1 with crossing-back from $i-1$ to i.

- $B(i)$: =#{downward steps of type B at i}=the numbers of steps from i to $i-1$ before time T_1 with crossing-back from $i-2$ to i .

- $C(i)$: =#{downward steps of type C at i}= the numbers of steps from i to $i-1$ before time T_1 with crossing-back from $i-1$ to $i+1$.

- Set for $i \leq 0$,
$$U(i) = [A(i), B(i), C(i)],$$

- $T_1 = 1 + \sum_{i \leq 0} \Big(2A(i) + 2B(i) + C(i)\Big) = 1 + \sum_{i \leq 0} \Big\langle (2,2,1), U(i) \Big\rangle.$

Theorem (H & Zhang,2010)

Assume $X_n \to \infty$, $\mathbb{P}$-*a.s.*. *Then for* P-*a.s.* ω, $\Big(U(i) = [A(i), B(i), C(i)]\Big)_{i\le 0}$ *is an inhomogeneous multitype branching process with immigration*

$$U(1) = [1,0,0], \quad \textit{with probability} \quad \frac{p_1(0)}{1-\alpha(0)-\beta(0)},$$

$$U(1) = [0,1,0], \quad \textit{with probability} \quad \frac{\gamma(0)}{1-\alpha(0)-\beta(0)},$$

$$U(1) = [0,0,1], \quad \textit{with probability} \quad \frac{p_2(0)}{1-\alpha(0)-\beta(0)}.$$

Theorem (Cont.)

The offspring distribution is given by

$$P_\omega\Big(U(i) = [a,b,0] \,\Big|\, U(i+1) = [1,0,0]\Big) = [1-\alpha(i)-\beta(i)]C_{a+b}^{a}\alpha(i)^a\beta(i)^b,$$
$$P_\omega\Big(U(i) = [a,b,1] \,\Big|\, U(i+1) = [0,1,0]\Big) = [1-\alpha(i)-\beta(i)]C_{a+b}^{a}\alpha(i)^a\beta(i)^b,$$
$$P_\omega\Big(U(i) = [a,b,0] \,\Big|\, U(i+1) = [0,0,1]\Big) = [1-\alpha(i)-\beta(i)]C_{a+b}^{a}\alpha(i)^a\beta(i)^b,$$

(1,2)-RWRE Branching Structure: Probabilities

- for $j = 1, 2$, exit probabilities

$$\begin{aligned} &P_\omega^{i-1}[(-\infty, i-1), i-1+j] \\ := &P_\omega^{i-1}\{\text{reach } [i, +\infty) \text{ for the first time at the point } i-1+j\}. \end{aligned}$$

- $$\gamma(i) = q(i) \cdot P_\omega^{i-1}[(-\infty, i-1), i+1].$$

- $$\alpha(i) = q(i) \cdot P_\omega^{i-1}[(-\infty, i-1), i] \cdot \frac{p_1(i-1)}{p_1(i-1) + \gamma(i-1)},$$
$$\beta(i) = q(i) \cdot P_\omega^{i-1}[(-\infty, i-1), i] \cdot \frac{\gamma(i-1)}{p_1(i-1) + \gamma(i-1)}.$$

Example: calculate ET_1

- Example: We consider a (2-1) random walk. Fix an nonrandom $\omega_0 = (q_1, q_2, p)$ with $p + q_1 + q_2 = 1$, and $\omega := (..., \omega_0, \omega_0, \omega_0, ...) \in \Omega$. Let $\{X_n\}_{n\geq 0}$ be a (2-1) random walk with initial value $X_0 = 0$ and transition probabilities

$$P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} p & if \;\; j = i+1, \\ q_1 & if \;\; j = i-1, \\ q_2 & if \;\; j = i-2. \end{cases}$$

- Now, all M_i equal to

$$\begin{pmatrix} a & b \\ 1+a & b \end{pmatrix}$$

with $a = \frac{q_1}{p}$ and $b = \frac{q_2}{p}$.

Example: calculate ET_1

- We get two different eigenvalues of M,

$$\lambda_1 = \frac{a+b+\sqrt{(a+b)^2+4b}}{2} \text{ and } \lambda_2 = \frac{a+b-\sqrt{(a+b)^2+4b}}{2}.$$

-

$$\begin{aligned}
E_\omega(T_1) &= 1 + \sum_{i<0} E_\omega(U_i(2,1)^T) = 1 + \sum_{i=1}^{\infty} e_1 M_0 \cdots M_{-i+1}(2,1)^T \\
&= 1 + \sum_{n=1}^{\infty} e_1 M^n (2,1)^T \\
&= 1 + \frac{1}{\lambda_1 - \lambda_2} \sum_{n=1}^{\infty} (2\lambda_1^{n+1} - 2\lambda_2^{n+1} + \lambda_2\lambda_1^{n+1} - \lambda_1\lambda_2^{n+1}) \\
&= \cdots \\
&= \frac{1}{p - q_1 - 2q_2},
\end{aligned}$$

- Validate the Wald's equality: $EX_{T_1} = EX_1 \times ET_1$

Application (I)–invariant density and LLN

(1) For $n \geq 0$ define $\overline{\omega}(n) = \theta^{X_n}\omega$.

(2) Then $\{\overline{\omega}(n)\}$ is a Markov chain with transition kernel

$$\overline{P}(\omega, d\omega') = \omega_0(1)\delta_{\theta\omega=\omega'} + \sum_{l=1}^{L} \omega_0(-l)\delta_{\theta^{-l}\omega=\omega'}.$$

(3) Define $\pi(\omega) := \frac{1}{\omega_0(1)} \left(1 + \sum_{i=1}^{\infty} e_1 \overline{M}_i \cdots \overline{M}_1 e_1^T\right)$.

(4) Let $\tilde{\pi}(\omega) = \frac{\pi(\omega)}{\mathbb{E}(\pi(\omega))}$.

Theorem (H & Wang,2013)

Suppose that $\mathbb{E}(\pi(\omega)) < \infty$. *Then we have that*

(i) $\gamma_L < 0$;

(ii) $\tilde{\pi}(\omega)\mathbb{P}(d\omega)$ *is invariant under the kernel* $\overline{P}(\omega, d\omega')$, *that is*

$$\int 1_B \tilde{\pi}(\omega)\mathbb{P}(d\omega) = \iint 1_{\omega' \in B} \overline{P}(\omega, d\omega')\tilde{\pi}(\omega)\mathbb{P}(d\omega); \tag{6}$$

(iii) and $\mathbb{P}$*-a.s.*, $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}(\pi(\omega))}$.

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$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ q_1(1)+q_2(1) & 0 & p(1) & 0 & \dots \\ q_2(2) & q_1(2) & 0 & p(2) & \dots \\ 0 & q_2(3) & q_1(3) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $q_1(i) + q_2(i) + p_1(i) = 1$, and $q_1(i), q_2(i), p_1(i) > 0$.

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- $\{X_n\}_{n\geq 0}$ is positive recurrent

  $\Longleftrightarrow$ $E_0T_0<\infty$, and stationary distribution $\pi(i)=\frac{1}{E_iT_i}$.

  (hence for any i, $E_iT_i<\infty$, where $T_i:=\inf\{n>0, X_n=i\}$.)

Review: Facts for $(1,1)$ RW (birth-death chain)

- For $(1,1)$ RW: $q(i) + p(i) = 1$, and $q(i), p(i) > 0$; $i > 0$.

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Tools: Intrinsic branching structure within the $(L,1)-$ RW.

For $(2, 1)$ RW: recurrence

Notations(1):

- $e_1 = (1, 0)'$, $e_2 = (0, 1)'$, $u = e_1 + e_2$, $v = e_1 - e_2$;

For $(2,1)$ RW: recurrence

Notations(1):

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$$M(1) = \begin{pmatrix} \frac{q_1(1)+q_2(1)}{p(1)} & 0 \\ \frac{1}{p(1)} & 0 \end{pmatrix}, \ M(i) = \begin{pmatrix} \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} \\ 1 + \frac{q_1(i)}{p(i)} & \frac{q_2(i)}{p(i)} \end{pmatrix}, \ i > 1,$$

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(2, 1) RW: recurrence

Theorem (H, Zhou, Zhao, 2014)

$\{X_n\}_{n\geq 0}$ *is recurrence* $\Longleftrightarrow$ $\rho = \infty$.

proof By the "branching structure", we have

$$E_0 N(0) = \rho.$$

$\square$

$(2, 1)$ RW: positive recurrence

Notations(2):

- $\tilde{M}(i) := \begin{pmatrix} \frac{q_1(i)+q_2(i)}{p(i)} & \frac{q_2(i)}{p(i)} \\ 1 & 0 \end{pmatrix}, \ i \geq 1.$

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- Then [H and Zhang, 2010],

$$P^i[(n,i), i-1] = \frac{\langle e_1, [\tilde{M}(i) + \cdots + \tilde{M}(n) \cdots \tilde{M}(i)]v\rangle}{1 + \langle e_1, [\tilde{M}(i) + \cdots + \tilde{M}(n) \cdots \tilde{M}(i)]e_1\rangle},$$
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proof Because $E_0T_0 = 1 + E^1\tau_1$. □

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Theorem (H,Zhou, Zhao 2014)

If $E^1\tau_1 < \infty$, *Then, for* $i \geq 0$

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$$\begin{aligned} E^iT_i &= p(i)E^{i+1}\tau_{i+1} + (q_1(i) + q_2(i) \\ &\quad + p(i)P^{i+1}[(+\infty, i+1), i-1])E^{i-1}T_i + q_2(i)E^{i-2}T_{i-1} + 1; \end{aligned}$$

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$$E^i T_i$$

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and $E^0T_0 = E^1T_0 + 1$,

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&= p(i)E^{i+1}T_i + (q_1(i) + q_2(i))E^{i-1}T_i + q_2(i)E^{i-2}T_{i-1} + 1, \qquad (23)
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and $E^0 T_0 = E^1 T_0 + 1$,

$$
\begin{aligned}
& E^{i+1}T_i \\
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&= E^{i+1}\tau_{i+1} + P^{i+1}[(+\infty, i+1), i-1]E^{i-1}T_i.
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Then

$$
\begin{aligned}
E^i T_i &= p(i)E^{i+1}\tau_{i+1} + (q_1(i) + q_2(i) \\
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$\square$

Tail asymptotic of π_i

- Let $D := \{P = (p, q_1, q_2) : \sum_{i=1}^{2} q_i + p = 1; \sum_{i=1}^{2} i q_i > p; \forall i, q_i > 0, p > 0\}$

Tail asymptotic of π_i

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Reference

1. Bolthausen, E., Goldsheid, I. (2000), *Recurrence and transience of random walks in random environments on a strip. Commun. Math. Phys.* **214**, 429-447.
2. Bolthausen, E., Goldsheid, I. (2008), *Lingering random walks in random environment on a strip. Commun. Math. Phys.* **278**, 253-288.
3. Brémont, J. (2002). On some random walks on $\mathbb{Z}$ in random medium. *Ann. of Probab. 3, 1266-1312.*
4. Goldsheid, I. (2008):*Linear and sub-linear growth and the CLT for hitting times of a random walk in random environments on a strip. Probab.Theory Relat.Fields.* **141**, 471-511.
5. Hong,W.M. and Wang, H.M. (2009). Branching structure for an (L-1) random walk in random environment and its applications. *submitted* (arXiv:1003.3731)

Reference

[6] Hong,W.M. and Zhang,L. (2010). Branching structure for the transient (1;R)-random walk in random environment and its applications. *Infinite Dimensional Analysis,Quantum Probability and Related Topics*, Vol. 13, No. 4 (2010) 589-618.

[7] Wenming Hong and Huaming Wang(2011), Intrinsic branching structure within random walk on $\mathbb{Z}$, *submitted* (arXiv: 1012.0636).

[8] Wenming Hong and Meijuan Zhang(2012), Branching structure for the transient random walk on a strip in a random environment, *submitted* (arXiv: 1204.1104).

[9] Wenming Hong, Ke Zhou and Yiqiang Q. Zhao(2012), Explicit stationary distribution of the $(L, 1)$-reflecting random walk on the half line, *Acta Mathematica Sinica, English Series*, to appear (2013).

[10] Kesten, H., Kozlov, M.V., Spitzer, F. (1975). A limit law for random walk in a random environment. *Compositio Math. 30, pp. 145-168.*

Reference

[11] Key, E.S. (1984). Recurrence and transience criteria for random walk in a random environment. *Ann. Prob. 12, pp. 529-560.*

[12] oitershtein, A. (2008). *Transient random walks on a strip in a random environment. The Annals of Probability.* **36**, 2354-2387.

[13] Solomon, F. Random walks in random environments. *Ann. prob. 3, 1-31, (1975).*

[14] Sznitman, A.S. (2004). Topics in random walks in random environment. *ICTP Lect. Notes, XVII, Abdus Salam Int. Cent. Theoret. Phys., Trieste.*

[15] Zeitouni, O. Random walks in random environment. *LNM 1837, J. Picard (Ed.), 189-312, Springer-Verlag Berlin Heidelberg, (2004).*

[16] Seignourel, P. Discrete schemes for processes in random media. *Probab. Theory Relat. Fields.* **118** (2000), 293-322.

Thank you !