Continuous-state branching processes, extremal processes and super-individuals

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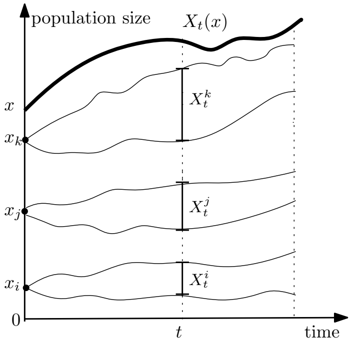
Introduction

Consider a random continuous branching population model (with no spatial motion and no interaction between the individuals).

Each individual reproduces independently with a same law:

For all $t > 0$, $X_t(x) = \sum_{x_i \leq x} X_t^i$

$(X_t(x), t \geq 0)$ is a CSBP started from x



How are organized the growth and the decay *locally* in the population? Are there some initial individuals whose progenies are growing or decaying faster than the others?

Definition (Branching Property and CSBP)

A positive Markov process $(X_t(x), t \geq 0)$ *with* $X_0(x) = x \geq 0$ *is a CSBP if for any* $y \in \mathbb{R}_+$

$$(X_t(x+y), t \geq 0) \stackrel{d}{=} (X_t(x), t \geq 0) + (\tilde{X}_t(y), t \geq 0)$$

where $(\tilde{X}_t(y), t \geq 0)$ *is an independent copy of* $(X_t(y), t \geq 0)$.

Theorem (Characterization: Jirina (58), Lamperti (67))

For any $\lambda > 0$*, there exists a map* $t \mapsto v_t(\lambda)$ *s.t.*

$$\mathbb{E}[e^{-\lambda X_t(x)}] = \exp(-x v_t(\lambda)) \text{ and } v_{s+t}(\lambda) = v_s \circ v_t(\lambda),$$

satisfying $\frac{\mathrm{d}v_t(\lambda)}{\mathrm{d}t} = -\Psi(v_t(\lambda)),\ v_0(\lambda) = \lambda$*, with* Ψ *of the form*

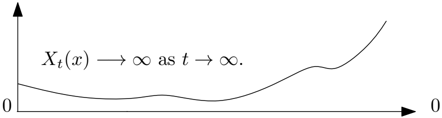
$$\Psi(q) = \frac{\sigma^2}{2}q^2 + \gamma q + \int_0^{+\infty} \left(e^{-qx} - 1 + qx1_{\{x \leq 1\}}\right) \pi(\mathrm{d}x)$$

Asymptotic behaviors

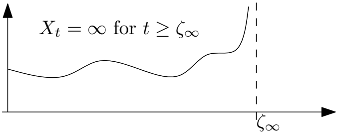
Let $(X_t(x), t \geq 0)$ a CSBP(Ψ).

- $\mathbb{E}[X_t(x)] = xe^{-\Psi'(0+)t}$ and $\mathbb{E}[X_t(x)] < \infty$ iff $\Psi'(0+) \neq -\infty$.
- If $\Psi'(0+) \in [-\infty, 0[$, $X_t(x) \underset{t\to+\infty}{\longrightarrow} +\infty$ with probability > 0.
- If $\Psi'(0+) \geq 0$, $X_t(x) \underset{t\to+\infty}{\longrightarrow} 0$ with probability 1.

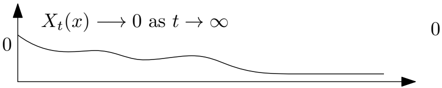
Infinite growth : $\Psi'(0+) \in [-\infty, 0[$ & $\int_0 \frac{du}{|\Psi(u)|} = \infty$.



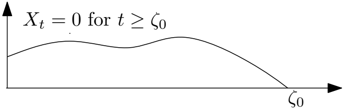
Explosion : $\Psi'(0+) = -\infty$ & $\int_0 \frac{du}{|\Psi(u)|} < \infty$.



Infinite decay (persistence): $\int^\infty \frac{du}{\Psi(u)} = \infty$



Absorption: $\int^\infty \frac{du}{\Psi(u)} < \infty$



Continuous population model: infinite variation case

Assume Ψ "of infinite variation" ($\sigma > 0$ or $\int_0^1 x\pi(dx) = \infty$). For $t > 0$, let ℓ_t the Lévy measure of the subordinator $(X_t(x), x \geq 0)$: $(\ell_t, t > 0)$ forms an *entrance law* for the semi-group of the CSBP(Ψ). Let N_Ψ the associated characteristic measure on $\mathcal{D}(\mathbb{R}_+^\star, \bar{\mathbb{R}}_+)$.

Definition (See e.g. Li 2010, Duquesne and Labbé 2014)

Consider $\mathcal{N} = \sum_{i \in I} \delta_{(x_i, X^i)}$ *a PPP over* $\mathbb{R}_+ \times \mathcal{D}$ *with intensity* $\mathrm{d}x \otimes N_\Psi(\mathrm{d}X)$. *For all* $x \geq 0$, *let* $X_0(x) = x$ *and for all* $t > 0$,

$$X_t(x) = \sum_{x_i \leq x} X_t^i$$

- *for all* $t \geq 0$ $(X_t(x), x \geq 0)$ *is a driftless càdlàg subordinator with Laplace exponent* $\lambda \mapsto v_t(\lambda)$
- *for any* $y \geq x$, $(X_t(y) - X_t(x), t \geq 0)$ *is a CSBP*(Ψ) *started from* $y - x$, *independent of* $(X_t(x), t \geq 0)$.

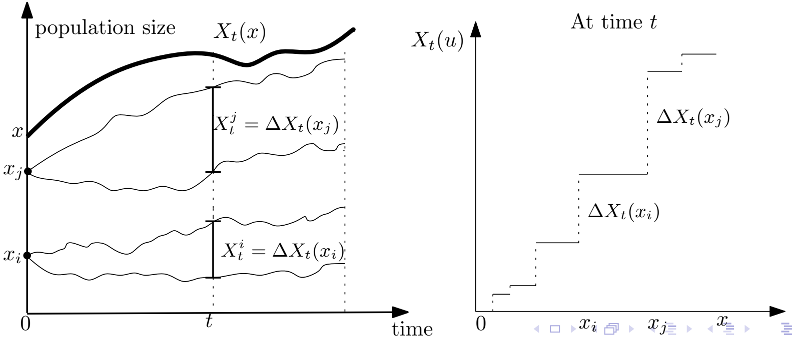
Flow of CSBPs

The flow $(X_t(x), t \geq 0, x \geq 0)$ provides a **continuous population** (Bertoin Le Gall 2000):

- The individual y is a *descendant* at time t of the individual x living at time 0 if

$$X_t(x-) < y < X_t(x)$$

- $\Delta X_t(x) := X_t(x) - X_t(x-)$ is the progeny of x at time t.



Continuous population model: finite variation case

Assume Ψ "of finite variation" ($\sigma = 0$ and $\int_0^1 x\pi(dx) < \infty$). Let $\mathbf{d} := \lim_{z\to\infty} \frac{\Psi(z)}{z} \in \mathbb{R}$.

Definition (See Duquesne Labbé 2014)

Consider $\mathcal{N} = \sum_{i\in I} \delta_{(x_i,t_i,X^i)}$ a PPP with intensity

$$dx \otimes e^{-\mathbf{d}t} dt \otimes \int_0^\infty \pi(dr)\mathbb{P}_r^\Psi(dX).$$

For all $x \geq 0$, and for all $t \geq 0$,

$$X_t(x) = e^{-\mathbf{d}t} x + \sum_{x_i \leq x} 1_{\{t_i \leq t\}} X^i_{t-t_i}.$$

- for all $t \geq 0$ $(X_t(x), x \geq 0)$ is a càdlàg subordinator with Laplace exponent $\lambda \mapsto v_t(\lambda)$
- for any $y \geq x$, $(X_t(y) - X_t(x), t \geq 0)$ is a CSBP(Ψ) started from $y - x$, independent of $(X_t(x), t \geq 0)$.

Super-individuals

Definition

The individual x is a **super-individual** *if its progeny overwhelms the total progeny of all individuals below it:*

$$\lim_{t\to+\infty}\frac{\Delta X_t(x)}{X_t(x-)}=+\infty \ a.s.$$

Denote by $\mathcal{S}$ the set of super-individuals

$$\mathcal{S}:=\left\{x>0;\ \lim_{t\to+\infty}\frac{\Delta X_t(x)}{X_t(x-)}=+\infty\right\}.$$

We will see that super-individuals exist in CSBPs with **infinite mean** and in *subcritical* CSBPs with **infinite variation**.

There is an order between super-individuals: if $x_1, x_2 \in \mathcal{S}$ and $x_1 \leq x_2$, then $\frac{\Delta X_t(x_1)}{\Delta X_t(x_2)} \leq \frac{X_t(x_2-)}{\Delta X_t(x_2)} \underset{t\to\infty}{\longrightarrow} 0$.

Supercritical case

Supercritical CSBP with finite mean

Definition (Bertoin, Fontbona, Martinez 2008)

An individual x is **prolific** *if $\Delta X_t(x) \underset{t\to\infty}{\longrightarrow} +\infty$ a.s.*

$$\mathcal{P} := \{x > 0; \Delta X_t(x) \underset{t\to\infty}{\longrightarrow} +\infty\}$$

Proposition (Grey 74+Bertoin et al. 2008+ Duquesne Labbé 2014)

Assume $\Psi'(0+) \in (-\infty, 0)$. Let $\lambda \mapsto v_{-t}(\lambda)$, the inverse of $\lambda \mapsto v_t(\lambda)$. Almost-surely, for all $x > 0$,

$$v_{-t}(\lambda)X_t(x) \underset{t\to\infty}{\longrightarrow} W_x^\lambda \text{ and } v_{-t}(\lambda)X_t(x-) \underset{t\to\infty}{\longrightarrow} W_{x-}^\lambda$$

with $(W_x^\lambda, x \geq 0) = \left(\sum_{x_i \leq x} W_i^\lambda, x \geq 0\right)$ a càdlàg subordinator.

- $\mathcal{P} = \{x > 0; \Delta W_x^\lambda > 0\}$
- $\lim_{t\to\infty} \frac{\Delta X_t(x)}{X_t(x-)} = \frac{\Delta W_x^\lambda}{W_{x-}^\lambda} = \infty \iff W_{x-}^\lambda = 0$

$\Longrightarrow$ $\mathcal{S} \cap \mathcal{P}$ *is a singleton (or empty).*

Supercritical CSBP with infinite mean

Theorem (preliminary version, Grey 77, F. Ma 16+)

Suppose $\Psi'(0+) = -\infty$ *and* $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = -\infty$. *Fix* $\lambda_0 \in (0, \rho)$, *and define* $G(y) := \exp\left(-\int_y^{\lambda_0} \frac{\mathrm{d}u}{\Psi(u)}\right)$ *for* $y \in (0, \rho)$. *Then, for all* $x \geq 0$, *almost-surely*

$$e^{-t} G\left(\frac{1}{X_t(x)} \wedge \rho\right) \underset{t \to +\infty}{\longrightarrow} Z_x.$$

- G *is decreasing and slowly varying at* 0
- $\{Z_x = 0\} = \{X_t(x) \underset{t\to\infty}{\longrightarrow} 0\}$
- $\mathbb{P}(Z_x \leq z) = \exp(-xG^{-1}(z))$ *with* $G^{-1}(z) = v_{\log(1/z)}(\lambda_0)$.

Example (Neveu's mechanism)

$\Psi(u) = u \log u$ for which $\rho = 1$. Fix $\lambda_0 = \frac{1}{e}$, $G(z) = \log(1/z)$

Question

What is the nature of the process $(Z_x, x \geq 0)$*?*

Definition (extremal process="subordinator for the max operator")

A process $(Z_x, x \geq 0)$ *is an extremal-F process if*

$$\begin{cases} Z_{x+y} = \max(Z_x, Z'_y) \text{ a.s. where } Z'_y \perp\!\!\!\perp (Z_u)_{0 \leq u \leq x} \text{ and } Z'_y \overset{d}{=} Z_y \\ \mathbb{P}(Z_x \leq z) = F(z)^x \end{cases}$$

Lemma

$(Z_x, x \geq 0)$ *is an extremal-F process with* $F(z) = e^{-v_{\log(1/z)}(\lambda_0)}$.

Proof.

$X_t(x+y) = X'_t(y) + X_t(x)$ with $X'_t(y) = X_t(x+y) - X_t(x)$.

$\Longrightarrow e^{-t}G(\frac{1}{X_t(x+y)}) \geq e^{-t}G(\frac{1}{X_t(x)}) \vee e^{-t}G(\frac{1}{X'_t(y)})$

$\Longrightarrow Z_{x+y} \geq \max(Z_x, Z'_y)$ a.s. but $Z_{x+y} \overset{d}{=} \max(Z_x, Z'_y)$ $\square$

Fact ("Lévy-Itô decomposition" of extremal processes)

Consider a PPP with intensity $dx \otimes \mu$ *over* $\mathbb{R}_+ \times \mathbb{R}$. *The process of its records is a càdlàg extremal-F process with* $F(z) = e^{-\bar{\mu}(z)}$.

Theorem (F. Ma 2016+, supercritical part)

Suppose $\Psi'(0+) = -\infty$ *and* $\int_0 \frac{\mathrm{d}u}{\Psi(u)} = -\infty$. *Almost-surely, for all* $x \geq 0$

$$e^{-t} G\left(\frac{1}{X_t(x)} \wedge \rho\right) \underset{t\to+\infty}{\longrightarrow} Z_x = \sup_{x_i \leq x} Z_i$$

where

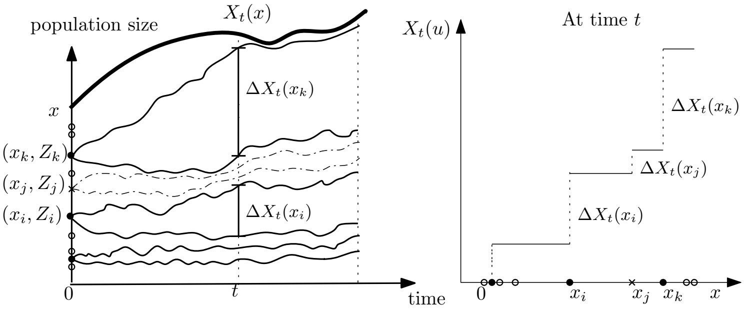
$$Z_i := \lim_{t\to\infty} e^{-t} G\left(\frac{1}{X_t^i} \wedge \rho\right) \text{ and } \mathcal{M} := \sum_{i\in I^*} \delta_{(x_i, Z_i)}$$

is a PPP$(\mathrm{d}x \otimes \mu)$ *with intensity* $\bar{\mu}(z) = v_{\log(1/z)}(\lambda_0)$, μ *has no atom and* $\mu(0,\infty) = \rho \in (0,\infty]$.

Interpretation of $(Z_x, x \geq 0)$, Poisson representation, super-prolific individuals

Proposition

$$\mathcal{P} = \{x_i; Z_i > 0, i \in I\} \text{ and } \mathcal{S} \cap \mathcal{P} = \{x > 0; \Delta Z_x > 0\} \text{ a.s.}$$



$\circ$ denotes a *non prolific* initial individual

$\times$ denotes (x_i, Z_i) not a partial record (*prolific non superprolific*)

$\bullet$ denotes (x_i, Z_i) partial record: (*superprolific*)

Subcritical case

Assume $\Psi'(0+) \geq 0$. Since for all $x \geq 0$, $X_t(x) \underset{t\to\infty}{\longrightarrow} 0$ a.s. there is no prolific individual in the population. Recall

$$\mathcal{S} := \left\{ x > 0; \lim_{t\to+\infty} \frac{\Delta X_t(x)}{X_t(x-)} = +\infty \right\}.$$

A super-individual is an individual whose *decay is much slower than the decay of all individuals below it.*

Subcritical CSBP with finite variation

Definition (variation)

For any Ψ,

$$\lim_{u\to+\infty}\frac{\Psi(u)}{u}=:\mathbf{d}=+\infty 1_{\{\sigma>0\}}+\gamma+\int_0^1 x\pi(\mathrm{d}x)\in\mathbb{R}\cup\{+\infty\}.$$

Proposition (Grey 74 + Duquesne Labbé 2014)

Assume $\mathbf{d}\in\mathbb{R}$. *For all* x, $(X_t(x), t\geq 0)$ *is persistent and almost-surely, for all* $x\geq 0$

$$v_{-t}(\lambda)X_t(x)\underset{t\to+\infty}{\longrightarrow}V_x^{\lambda} \text{ and } v_{-t}(\lambda)X_t(x-)\underset{t\to+\infty}{\longrightarrow}V_{x-}^{\lambda}.$$

where $(V_x^{\lambda}, x\geq 0)$ *is a càdlàg subordinator. Thus* $\mathcal{S}$ *is empty.*

Subcritical CSBP with infinite variation

Theorem (F. Ma 2016+, subcritical part)

Suppose $\mathbf{d} = +\infty$ *and* $\int^{+\infty} \frac{\mathrm{d}u}{\Psi(u)} = +\infty$*. Fix* $\lambda_0 \in (0, +\infty)$ *and define* $G(y) := \exp\left(-\int_{\lambda_0}^{y} \frac{\mathrm{d}u}{\Psi(u)}\right)$ *on* $(0, +\infty)$*. Almost-surely, for all* $x \geq 0$

$$e^t G\left(\frac{1}{X_t(x)}\right) \underset{t\to+\infty}{\longrightarrow} Z_x = \sup_{x_i \leq x} Z_i$$

where $\mathcal{M} := \sum_{i\in I} \delta_{(x_i, Z_i)}$ *a PPP*$(\mathrm{d}x \otimes \mu)$ *with*

$$\bar{\mu}(z) = v_{\log(z)}(\lambda_0).$$

μ *has no atom and* $\mu(0,\infty) = \infty$*.*

Proposition

$$\mathcal{S} = \{x > 0; \Delta Z_x > 0\} \text{ a.s.}$$

Neveu case

Consider $(X_t(x), t \geq 0)$ a CSBP of Neveu: $\Psi(u) = u \log u$. It is non-explosive with infinite mean and persistent with infinite variation. For any fixed x

$$e^{-t} \log X_t(x) \underset{t \to +\infty}{\longrightarrow} Z_x \text{ a.s.}$$

where Z_x has a Gumbel law over $\mathbb{R}$ (this was observed by Neveu in 1992). Combining our results, we get:

Proposition

Almost-surely for all $x \geq 0$,

$$e^{-t} \log X_t(x) \underset{t \to +\infty}{\longrightarrow} Z_x = \sup_{x_i \leq x} Z_i \in \mathbb{R}$$

where $\sum_{i \in I} \delta_{(x_i, Z_i)}$ is a PPP on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}x \otimes e^{-z}\mathrm{d}z$. $(Z_x, x \geq 0)$ is an extremal-Λ process with $\forall z \in \mathbb{R}$, $\Lambda(z) = e^{-e^{-z}}$.

Eve property

Duquesne and Labbé (2014) have considered the following question:

Question

*Does the population (encoded by a flow of CSBPs) concentrates on the progeny of a single individual? In other words, fix the initial size x, is there an individual $e \in [0, x]$ (**the Eve**), such that*

$$\frac{\Delta X_t(e)}{X_t(x)} \underset{t\to\infty}{\longrightarrow} 1 \ a.s.?$$

Corollary (Duquesne and Labbé 2014)

In the case of infinite variation and infinite mean, the population has an Eve. In our framework, the Eve corresponds to the last super-individual in $[0, x]$.

Remarks and Conclusion

- In the cases of absorption in ∞ or in 0, extremal processes still arise, but through the times of explosion and absorption. Super-individuals are those who explode the first or die the last: $\zeta(x) := \inf\{t \geq 0; X_t(x) = 0\} = \sup_{x_i \leq x} \zeta_i$

$$\frac{\Delta X_t(x)}{X_t(x-)} = \infty \text{ for some } t \text{ if } \zeta(x) > \zeta(x-).$$

- In the infinite mean or infinite variation case,"the infinite divisibility of the flow $(X_t(x), t \geq 0, x \geq 0)$ becomes the *max*-infinite divisibility of the process $(Z_x, x \geq 0)$", this was observed by Cohn and Pakes (1978) for Galton-Watson processes [1] with infinite mean.
- The "super-individuals" are *partial records* and do not represent the successive Eves.

[1] They study the convergence rates towards Z

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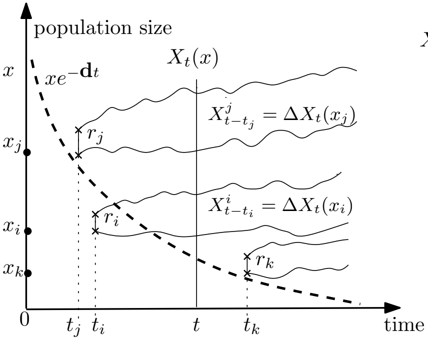
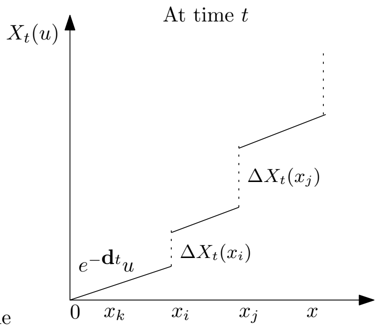
Thank you


population size
$X_t(x)$
x
$xe^{-\mathbf{d}t}$
x_j
r_j
$X^j_{t-t_j} = \Delta X_t(x_j)$
$X^i_{t-t_i} = \Delta X_t(x_i)$
x_i
r_i
x_k
r_k
0
t_j
t_i
t
t_k
time


At time t
$X_t(u)$
$\Delta X_t(x_j)$
$\Delta X_t(x_i)$
$e^{-\mathbf{d}t}u$
0
x_k
x_i
x_j
x

Convergence to Z_x. Sketch of proof

Fix x.

1. For all $\lambda \in (0,\rho)$, $v_{-t}(\lambda)X_t(x) \underset{t\to\infty}{\longrightarrow} W_x^\lambda \in \{0,\infty\}$ a.s. and $\mathbb{P}(W_x^\lambda = \infty) = e^{-x\lambda}$.
2. If $\lambda' \geq \lambda$ then $W_x^{\lambda'} \geq W_x^\lambda$
3. $\Lambda_x := \inf\{\lambda \in (0,\rho) \cap \mathbb{Q};\, W_x^\lambda = +\infty\}$ is a random variable!
4. Let $\lambda' \in (0,\rho)$. If $\lambda < \Lambda_x < \lambda'$, then for large t :

$$\begin{aligned} & v_{-t}(\lambda) \leq 1/X_t(x) \text{ and } v_{-t}(\lambda') \geq 1/X_t(x) \\ & \Longrightarrow G(v_{-t}(\lambda)) \geq G(1/X_t(x)) \geq G(v_{-t}(\lambda')) \\ & \Longrightarrow G(\lambda) \leq e^{-t} G(1/X_t(x)) \leq G(\lambda') \end{aligned}$$

5. If $\Lambda_x = \rho$ then $X_t(x) \underset{t\to\infty}{\longrightarrow} 0$.

This yields the a.s convergence: $e^{-t}G(1/X_t(x) \wedge \rho) \underset{t\to\infty}{\longrightarrow} Z_x$