The favorite sites of subdiffusive biased walks on a Galton-Watson tree

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Let $(X_n)_{n\geq 0}$ be a Markov chain on $\mathbb{T}$, and the local times process

$$L_n(x) := \sum_{i=1}^{n} 1_{\{X_i=x\}}, \qquad x\in\mathbb{T},\ n\geq 1.$$

The set of the favorite sites is defined as follows:

$$\mathbb{F}(n) := \Big\{x\in\mathbb{T} : L_n(x) = \max_{y\in\mathbb{T}} L_n(y)\Big\}.$$

Erdős and Révész considered $\mathbb{F}(n)$ of SRW on $\mathbb{Z}$, and conjectured that

(a) the set of favorite sites is tight;

(b) the cardinality of $\mathbb{F}(n)$ is eventually bounded by 2.

Erdős, P. and Révész, P. (1984). On the favourite points of a random walk. *Mathematical Structures – Computational Mathematics – Mathematical Modelling* **2**, pp. 152–157. Sofia.

A list of ten open problems on the favorite points in Chapter 13 of

Random Walk in Random and Non-Random Environments. by Révész, P. (2013). (Third edition). World Scientific, Singapore.

Conjecture (b) still remains open and the best result so far was obtained by Tóth.

Conjecture (a) was disproved by Bass and Griffin who showed the almost sure transience of $\mathbb{F}(n)$ for the simple random walk on $\mathbb{Z}$ as well as for the one-dimensional Brownian motion.

Later, the transience of $\mathbb{F}(n)$ was established by Bass, Eisenbaum and Shi (2000), Marcus (2001), Eisenbaum and Khoshnevisan (2002) for Lévy processes and even for fairly general Markov processes,

and by Hu and Shi (2000) for Sinai's one-dimensional RWRE.

background

Are $\mathbb{F}(n)$ always transient for general "non-trivial" null-recurrent Markov processes?

NO by Hu and Shi (2015) that $\mathbb{F}(n)$ is tight for a class of randomly biased walks on trees in the slow-movement regime.

The answer for the sub-diffusive regime is more complicated and is depending on some parameter $\kappa \in (1, \infty]$.
For a class of biased walk on tree, conditioned upon the set of non-extinction of the tree, $\mathbb{F}(n)$ is almost surely
bounded if $\kappa > 2$,
tight if $\kappa = 2$,
and may move to infinity almost surely if $1 < \kappa < 2$.

Model

The model of the randomly biased walk on trees, introduced by Lyons and Pemantle (1992).

$\mathbb{T}$ = a supercritical Galton–Watson tree rooted at $\varnothing$.
$\overleftarrow{x}$ be its parent for any vertex $x \in \mathbb{T}\backslash\{\varnothing\}$
$|x|$ the generation of the vertex x in $\mathbb{T}$.
For any $x, y \in \mathbb{T}$, we write $y < x$ if and only if y is an ancestor of x (and $y \le x$ iff $y < x$ or $y = x$).

For the sake of presentation, add a specific vertex $\overleftarrow{\varnothing}$, as the parent of $\varnothing$. $\overleftarrow{\varnothing}$ is not a vertex of $\mathbb{T}$, e.g., $\sum_{x\in\mathbb{T}} f(x)$ does not contain the term $f(\overleftarrow{\varnothing})$.

Model

Let $\omega := (\omega(x, \cdot),\, x \in \mathbb{T})$ be a sequence of vectors s.t. $\omega(x,\, y) \geq 0$ for all $x, y \in \mathbb{T}$ and $\sum_{y \in \mathbb{T}} \omega(x,\, y) = 1$ for $x \in \mathbb{T}$. Assume that

$$\omega(x,\, y) > 0 \iff \text{ either } \overleftarrow{x} = y \text{ or } \overleftarrow{y} = x.$$

Define $\omega(\overleftarrow{\varnothing}, \varnothing) := 1$ and modify the vector $\omega(\varnothing, \cdot)$ such that $\omega(\varnothing, \overleftarrow{\varnothing}) > 0$ and $\omega(\varnothing, \overleftarrow{\varnothing}) + \sum_{x:\, \overleftarrow{x} = \varnothing} \omega(\varnothing, x) = 1$.

For any vertex $x \in \mathbb{T}$, let $(x^{(1)}, \cdots, x^{(\nu_x)})$ be its children, where $\nu_x \geq 0$ is the number of children of x.

Model

Define $\mathbf{A}(x) := (A(x^{(i)}),\, 1 \le i \le \nu_x)$ by

$$A(x^{(i)}) := \frac{\omega(x,\, x^{(i)})}{\omega(x,\, \overleftarrow{x})}, \qquad 1 \le i \le \nu_x\,.$$

and suppose that $(\mathbf{A}(x))_{x\in\mathbb{T}}$ are i.i.d.

In general, we may construct a marked tree as in Neveu(1987) s.t. for any $k \ge 0$, conditionally on $\{\mathbf{A}(x), |x| \le k\}$, the random variables $(\mathbf{A}(y))_{|y|=k+1}$ are i.i.d. and distributed as $\mathbf{A}(\varnothing)$.

There is an obvious bijection between $(\mathbf{A}(x))_{x\in\mathbb{T}}$ and $(\mathbb{T},\omega)$ and we shall both notation interchangably.

Model

For given ω, the randomly biased walk $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{T} \cup \{\overleftarrow{\varnothing}\}$ with transition probabilities ω, starting from $\varnothing$; i.e. $X_0 = \varnothing$ and

$$P_\omega\big(X_{n+1} = y \,|\, X_n = x\big) = \omega(x,\, y).$$

Denote by **P** the law of $(\mathbb{T}, \omega)$ and define $\mathbb{P}(\cdot) := \int P_\omega(\cdot)\mathbf{P}(d\omega)$.

In the language of RWRE, P_ω is the quenched probability whereas $\mathbb{P}$ is the annealed probability.

Model

Assume that $\mathbf{P}(\nu = \infty) = 0$, $\mathbf{E}(\nu) \in (1, \infty]$, $\mathbf{E}\big(\sum_{i=1}^{\nu} A_i |\log A_i|\big) < \infty$ and

$$\mathbf{E}\Big(\sum_{i=1}^{\nu} A_i\Big) = 1, \qquad \mathbf{E}\Big(\sum_{i=1}^{\nu} A_i \log A_i\Big) < 0. \tag{1}$$

Then $(X_n)_{n\geq 0}$ is null-recurrent under (1), by Lyons and Pemantle (1992), Menshikov and Petritis (2002) and Faraud (2011).

Model

Suppose that:

$$\begin{cases} \text{either there exists a } \kappa \in (1, \infty) \text{ such that } \mathbf{E}\big(\sum_{i=1}^{\nu} A_i^{\kappa}\big) = 1; \\ \text{or } \mathbf{E}\big(\sum_{i=1}^{\nu} A_i^{t}\big) < 1 \text{ for any } t > 1, \text{ and define } \kappa := \infty \text{ in this case.} \end{cases} \tag{2}$$

When (1) and (2) are fulfilled, $(X_n)_{n\geq 0}$ may be diffusive or subdiffusive, e.g., if furthermore ν equals some integer, then a.s.,

$$\lim_{n\to\infty} \frac{1}{\log n} \log \max_{0\leq i\leq n} |X_i| = 1 - \max\Big(\frac{1}{2}, \frac{1}{\kappa}\Big). \tag{3}$$

Hu, Y. and Shi, Z. (2007). A subdiffusive behaviour of recurrent random walk in random environment on a regular tree. *Probability Theory and Related Fields* **138** pp. 521-549)

Model

When $\kappa \in (5, \infty]$, Faraud (2011) proved an invariance principle for $(|X_n|)_{n\geq 0}$.
Aïdékon and de Raphélis(2015+) proved that for any $\kappa \in (2, \infty]$, the tree visited by the walk, after renormalization, converges to the Brownian forest.
When $1 < \kappa \leq 2$, a similar convergence also holds, but towards the stable forest, and the height function of the walk also satisfies a central limit theorem, see de Raphélis(2016).

See Andreoletti and Debs (2014), Andreoletti and Chen (2015+) for the recent studies of the spread and local times of the biased walk in both subdiffusive and slow-movement regimes.
For further detailed references and open problems, see the survey paper by Ben Arous and Fribergh (2014+).

Denote by $\mathbf{P}^*$ the probability $\mathbf{P}$ conditioned on the non-extinction of the Galton-Watson tree $\mathbb{T}$:

$$\mathbf{P}^*(\bullet) := \mathbf{P}\Big(\bullet \mid \mathbb{T} \text{ is infinite}\Big),$$

and denote by $\mathbb{P}^*$ the (annealed) probability conditioned on the set of non-extinction of $\mathbb{T}$: $\mathbb{P}^*(\cdot) := \int P_\omega(\cdot)\mathbf{P}^*(d\omega)$.
Furthermore assume an integrability condition: when $1 < \kappa < \infty$, there exists some $\alpha > \kappa$ such that

$$\mathbf{E}\Big(\sum_{i=1}^{\nu} A_i\Big)^{\alpha} < \infty, \tag{4}$$

and when $\kappa = \infty$, we assume that (4) holds for some $\alpha > 2$.

Theorem

Assume (1), (2) and (4).
(i) If $\kappa \in (2, \infty]$, then $\mathbb{P}^*$-almost surely,

$$\mathbb{F}(n) \subset M, \qquad \textit{for all large } n, \tag{5}$$

where $M = \left\{ x \in \mathbb{T} : U(x) = \min_{y \in \mathbb{T}} U(y) \right\}$ is a.s. finite.
(ii) If $\kappa = 2$, then

$$\left(\sup_{x \in \mathbb{F}(n)} |x| \right)_{n \geq 1} \qquad \textit{is tight under } \mathbb{P}^*. \tag{6}$$

(iii) If $1 < \kappa < 2$, then for any $\varepsilon > 0$,

$$\limsup_{n \to \infty} n^{-\frac{\kappa-1}{\kappa}+\varepsilon} \inf_{x \in \mathbb{F}(n)} |x| \;=\; \infty, \qquad \mathbb{P}^*\text{-a.s.}, \tag{7}$$

$$\liminf_{n \to \infty} \sup_{x \in \mathbb{F}(n)} |x| \;<\; \infty, \qquad \mathbb{P}^*\text{-a.s.} \tag{8}$$

Discussions

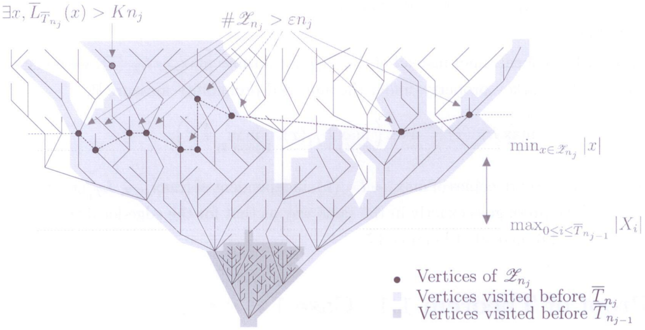

Figure 2: The event B_j (in red).

As mentioned before, $\mathbb{F}(n)$ is tight in the slow-movement regime (informally $\kappa = 1$).
In the case $1 < \kappa < 2$, $\mathbb{P}^*$-a.s., $\max_{0 \le i \le n} |X_i| = n^{\frac{\kappa-1}{\kappa} + o(1)}$ (see (3)), so (7) says that up to $n^{o(1)}$, $\mathbb{F}(n)$ could reach as far as the upper limits of the walk itself.
It is interesting to observe the phase transition at $\kappa = 2$.

The a.s. oscillation in the case $1 < \kappa < 2$ is surprising, because a priori, we cannot expect a localization of a null-recurrent walk on the tree, in contrast with the one-dimensional random walk in random environment on $\mathbb{Z}$. Moreover, when $1 < \kappa < 2$, $\mathbb{F}(n)$ is not tight under $\mathbb{P}^*$.

Discussions

To explain this phenomenon, we describe now the strategy of the proof of Theorem 1.

The main step will be the exploration of the Markov property on the space variable x of the local times process. It will be more convenient to consider the edge local times $(\overline{L}_n(x))_{n\geq 1, x\in\mathbb{T}}$ defined as follows:

$$\overline{L}_n(x) := \sum_{i=1}^{n} 1_{\{X_{i-1}=\overleftarrow{x}, X_i=x\}}, \qquad x \in \mathbb{T},\ n \geq 1. \tag{9}$$

Define a sequence of stopping times $(\overline{T}_n)_{n\geq 1}$ by induction: for any $n \geq 1$,

$$\overline{T}_n := \inf\{k > \overline{T}_{n-1} : X_{k-1} = \overleftarrow{\varnothing},\, X_k = \varnothing\}, \tag{10}$$

with $\overline{T}_0 := 0$. By definition, $\overline{T}_n - 1$ is exactly the n^{th} return time to $\overleftarrow{\varnothing}$ of the walk $(X_n)_{n\geq 0}$.

The key ingredient in the proof of Theorem 1 (cases $1 < \kappa \leq 2$) is the tail distribution of the maximum of (edge) local times considered at $\overline{T}_1$.

Denote by $f(x) \sim g(x)$ as $x \to x_0$ when $\lim_{x\to x_0} f(x)/g(x) = 1$ and $f(x) \asymp g(x)$ if
$0 < \liminf_{x\to x_0} f(x)/g(x) \leq \limsup_{x\to x_0} f(x)/g(x) < \infty$.

Theorem

Let $\kappa \in (1, \infty)$. Assume (1), (2) and (4). As $r \to \infty$, we have

$$\mathbb{P}\Big(\max_{x \in \mathbb{T}} \overline{L}_{\overline{T}_1}(x) \geq r \Big) \asymp \begin{cases} r^{-1}, & \text{if } 1 < \kappa < 2, \\ r^{-1}(\log r)^{-1/2}, & \text{if } \kappa = 2, \\ r^{-\kappa/2}, & \text{if } 2 < \kappa < \infty. \end{cases}$$

The same results hold when we replace $\max_{x \in \mathbb{T}} \overline{L}_{\overline{T}_1}(x)$ by $\max_{x \in \mathbb{T}} L_{\overline{T}_1}(x)$.

In the case $1 < \kappa < 2$, $\max_{x \in \mathbb{T}} \overline{L}_{\overline{T}_1}(x)$ has a Cauchy-type tail (independent of the value of κ).

Discussions

To see how the asymptotic behaviors of the favorite points ensue from Theorem 2, we introduce a set Z_k of vertices in $\mathbb{T}$:

$$Z_k := \left\{ x \in \mathbb{T} : \overline{L}_{\overline{T}_k}(x) = 1, \min_{\varnothing < y < x} \overline{L}_{\overline{T}_k}(y) \geq 2 \right\}, \qquad k \geq 1, \quad (11)$$

which is the set of the vertices being the first of their ancestry line to be of edge local time 1. This set is represented on Figure.

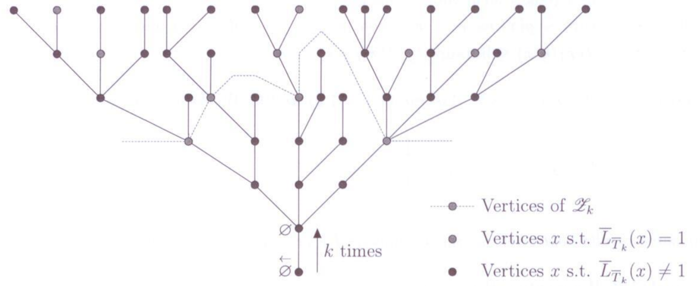

Figure 1: The set $\mathscr{Z}_k$.

For any fixed $k \geq 1$, by the strong Markov property, we get the following identity in law under the annealed probability measure $\mathbb{P}$:

$$\max_{x \in \mathbb{T}} \overline{L}_{\overline{T}_k}(x)$$

$$= \max \left(\max_{x \leq Z_k} \overline{L}_{\overline{T}_k}(x), \max_{1 \leq i \leq \# Z_k} \overline{L}^{*,i} \right), (12)$$

where $x \leq Z_k$ means that either $x \in Z_k$ or for any $\varnothing < y \leq x, \overline{L}_{\overline{T}_k}(y) \geq 2$, and $(\overline{L}^{*,i})_{i \geq 1}$ are i.i.d. copies of $\max_{x \in \mathbb{T}} \overline{L}_{\overline{T}_1}(x)$, independent of $(\max_{x \leq Z_k} \overline{L}_{\overline{T}_k}(x), \# Z_k)$.

A similar identity in law holds for the site local times $L_{\overline{T}_k}(x)$ instead of the edge local time $\overline{L}_{\overline{T}_k}(x)$.

Discussions

Consider for instance the case $1 < \kappa < 2$. $\# Z_k$ is of order k when $k \to \infty$, and it is not very hard to see that $\max_{x \leq Z_k} \overline{L}_{\overline{T}_k}(x)$ is also of order k.
By Theorem 2, $\overline{L}^{*,i}$ has the Cauchy-type tail, then an application of the extreme value theory based on (12) yields that along some subsequence $k \to \infty$, $\frac{1}{k} \max_{x \in \mathbb{T}} \overline{L}_{\overline{T}_k}(x) \to \infty$.

This implies the almost sure unboundedness of the favorite points.
On the other hand, the favorite points are either bounded or escape to infinity at a certain polynomial rate.
Combining the two facts we get the upper limits in (7).
The lower limits in (8) can be obtained in a similar way.

The potential

Introduce $V = (V(x))_{x \in \mathbb{T}}$ the *random potential* of the biased random walk $(X_n)_{n \geq 0}$, which will completely determine the behavior of (X_n). Define $V(\varnothing) := 0$ and

$$V(x) := -\sum_{y \in \rrbracket \varnothing, x \rrbracket} \log A(y), \qquad \text{for } x \in \mathbb{T} \backslash \{\varnothing\}, \tag{13}$$

where $\rrbracket \varnothing, x \rrbracket := \llbracket \varnothing, x \rrbracket \backslash \{\varnothing\}$, with $\llbracket \varnothing, x \rrbracket$ denoting the set of vertices (including x and $\varnothing$) on the unique shortest path connecting $\varnothing$ to x. The process $(V(x), x \in \mathbb{T})$ is a branching random walk, in the usual sense of Biggins (1977).

The potential

Define a symmetrized version of the potential which will naturally appear in the study of local times:

$$U(x) := V(x) - \log\left(\frac{1}{\omega(x, \overleftarrow{x})}\right), \qquad x \in \mathbb{T}. \tag{14}$$

Note that

$$e^{-U(x)} = \frac{1}{\omega(x, \overleftarrow{x})} e^{-V(x)} = e^{-V(x)} + \sum_{y \in \mathbb{T}:\, \overleftarrow{y} = x} e^{-V(y)}. \tag{15}$$

The potential

The Biggins-Hammersley-Kingman law of the large numbers implies that under the assumptions (1) and (2), $\mathbf{P}^*$-a.s., $\frac{1}{n}\min_{|x|=n} V(x)$ converges towards some positive constant.

Lemma

Assume (1), (2) *and* (4)*. As* $n \to \infty$,

$$\min_{|x|=n} U(x) \to \infty, \qquad \mathbf{P}^*\text{-}a.s.$$

Consequently, the set of minimums of U is finite a.s.:

$$M := \Big\{ x \in \mathbb{T} : U(x) = \min_{y \in \mathbb{T}} U(y) \Big\}. \tag{16}$$

Proof. According to (2), we may choose some constant $t > 1$ s.t.

$$\mathbf{E}\Big(\sum_{x\in\mathbb{T}:\,|x|=1} e^{-tV(x)}\Big) < 1, \qquad \text{and} \qquad c := \mathbf{E}\big(1+ \sum_{x\in\mathbb{T}:\,|x|=1} e^{-V(x)}\big)^t < \infty.$$

By the branching property, we have that for any $n \geq 1$,

$$\begin{aligned}
\mathbf{E}\Big(\sum_{x\in\mathbb{T}:\,|x|=n} e^{-tU(x)}\Big) &= \mathbf{E}\Big(\sum_{x\in\mathbb{T}:\,|x|=n} e^{-tV(x)}\big(1+\sum_{y:\,\overleftarrow{y}=x} e^{-(V(y)-V(x))}\big)^t\Big) \\
&= c\,\mathbf{E}\Big(\sum_{x\in\mathbb{T}:\,|x|=n} e^{-tV(x)}\Big) \\
&= c\,\Big(\mathbf{E}\big(\sum_{x\in\mathbb{T}:\,|x|=1} e^{-tV(x)}\big)\Big)^n,
\end{aligned}$$

whose sum on n converges. Therefore Borel-Cantelli's lemma yields that $\sum_{x:\,|x|=n} e^{-tU(x)} \to 0$, $\mathbf{P}$-a.s., and Lemma 3 follows. $\square$

Randomly Biased Walks on Trees
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Thank You !

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