Asymptotic properties of maximum likelihood estimator for the growth rate for a jump-type CIR process

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(joint work with M. Ben Alaya, A. Kebaier and G. Pap)

Outline of my talk

- (diffusion-type) Cox-Ingersoll-Ross (CIR) process
- a jump-type CIR process driven by a Wiener process and a subordinator, a special case: basic affine jump diffusion (BAJD)
- existence of a pathwise unique strong solution
- a classification: subcritical, critical and supercritical cases based on the asymptotic behavior of the expectation
- explicit joint Laplace transform of the process and its integrated process
- derivation of MLE of the growth rate (a drift parameter) of the model based on continuous time observations
- consistency and asymptotic behavior of MLE as sample size tends to infinity according to the cases subcritical, critical and supercritical

Diffusion-type CIR process, I

Cox-Ingersoll-Ross (CIR) process (Feller (1951) and Cox, Ingersoll and Ross (1985)):

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t, \qquad t \geqslant 0,}$$

where Y_0 is a non-negative initial value, $a \geqslant 0$, $b \in \mathbb{R}$, $\sigma > 0$, and $(W_t)_{t\geqslant 0}$ is a standard Wiener process, independent of Y_0.

Diffusion-type = sample paths are continuous almost surely.

Y is also called a square root process or a Feller process.

The existence of a pathwise unique non-negative strong solution can be found in Ikeda and Watanabe (1981).
The key points are that $|\sqrt{x} - \sqrt{y}| \leqslant \sqrt{|x-y|}$, $x, y \geqslant 0$, and $\int_0^\varepsilon \frac{1}{(\sqrt{x})^2}\,\mathrm{d}x = \infty$, $\forall\,\varepsilon > 0$.

Diffusion-type CIR process, II

We note that

- if $a \geqslant \frac{\sigma^2}{2}$, then $\mathrm{P}(Y_t > 0 \text{ for all } t > 0) = 1$.
- if $0 < a < \frac{\sigma^2}{2}$, then Y hits 0 with probability $p \in (0, 1)$ in case of $b < 0$, and with probability 1 in case of $b \geqslant 0$ (and zero is reflecting due to $a > 0$).

In financial mathematics, CIR process is used to describe the evolution of interest rates.

This process is well-studied, e.g., explicit characteristic function, density function and several results on estimation of (a, b), see later on.

A jump-type CIR process

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \qquad t \geqslant 0,}$$

where Y_0 is an a.s. non-negative initial value, $a \geqslant 0$, $b \in \mathbb{R}$, $\sigma > 0$, $(W_t)_{t\geqslant 0}$ is a standard Wiener process, and $(J_t)_{t\geqslant 0}$ is a subordinator (an increasing Lévy process) with zero drift and with Lévy measure m concentrating on $(0, \infty)$ such that

$$\int_0^\infty z\, m(\mathrm{d}z) \in [0, \infty), \qquad \text{(A1)}$$

that is,

$$\mathrm{E}(\mathrm{e}^{uJ_t}) = \exp\left\{ t \int_0^\infty (\mathrm{e}^{uz} - 1)\, m(\mathrm{d}z) \right\}$$

for $t \geqslant 0$ and $u \in \mathbb{C}$ with $\mathrm{Re}(u) \leqslant 0$.

We suppose that Y_0, $(W_t)_{t\geqslant 0}$ and $(J_t)_{t\geqslant 0}$ are independent.

It will turn out that b can be interpreted as a growth rate.

A special case: Basic Affine Jump-Diffusion (BAJD)

It was introduced by Duffie and Gârleanu (2001):

- only in the subcritical case with parametrization $a = \kappa\theta$ and $b = \kappa$, where $\kappa > 0$ (subcritical) and $\theta \geqslant 0$, i.e., the drift takes the form $\kappa(\theta - Y_t)$,
- the Lévy measure m takes form

$$m(\mathrm{d}z) = c\lambda \mathrm{e}^{-\lambda z} \mathbb{1}_{(0,\infty)}(z)\, \mathrm{d}z$$

with some constants $c \geqslant 0$ and $\lambda > 0$.

Then J is a compound Poisson process, its first jump time $\sim Exp(c)$ and its jump size $\sim Exp(\lambda)$.

Aim

To study the asymptotic properties of the MLE of $b \in \mathbb{R}$ under the conditions:

- $a \geqslant 0$, $\sigma > 0$ and the Lévy measure m are known,
- based on continuous time observations $(Y_t)_{t \in [0,T]}$ with $T \in (0, \infty)$,
- known non-random initial value $y_0 \geqslant 0$: $\mathrm{P}(Y_0 = y_0) = 1$,
- sample size tends to ∞, i.e., $T \to \infty$.

It will turn out that for the calculation of the MLE of b, one does not need to know σ and m.

At the moment, we can not handle the MLE of a supposing that b is known or the joint MLE of (a, b).
Reason: limit behavior of $\int_0^t \frac{1}{Y_s} \, \mathrm{d}s$ as $t \to \infty$, is not known to us.

Some history on parameter estimation of (a, b)

Overbeck and Rydén (1997) studied conditional least squares estimator (LSE) of (a, b) for diffusion-type CIR process based on discrete and continuous time observations as well.

Overbeck (1998) studied MLE of (a, b) for diffusion-type CIR process based on continuous time observations.

Mai (2012) studied MLE of b supposing that a is known for our jump-type CIR process, but only in the subcritical case $(b > 0)$ and under ergodicity assumption. Own contribution.

Ben Alaya and Kebaier (2012-13) completed the results of Overbeck (1998) giving explicit forms of the joint Laplace transforms of the building blocks of the MLE in question as well.

Li and Ma (2013) investigated (weighted) conditional LSE of (a, b) of a so-called stable CIR model based on discrete time observations in the subcritical case.

Huang, Ma and Zhu (2011): (weighted) conditional LSE for drift parameters of general CBI processes under 2nd order moment assumptions on the branching and immigration mechanisms.

Existence and uniqueness of a strong solution

Recall that

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \qquad t \geqslant 0.}$$

Proposition. Let η_0 be a random variable independent of $(W_t)_{t\geqslant 0}$ and $(J_t)_{t\geqslant 0}$ satisfying $\mathrm{P}(\eta_0 \geqslant 0) = 1$ and $\mathrm{E}(\eta_0) < \infty$. Then for all $a \geqslant 0$, $b \in \mathbb{R}$, $\sigma > 0$ and Lévy measure m on $(0, \infty)$ satisfying (A1),

- there is a pathwise unique strong solution $(Y_t)_{t\geqslant 0}$ such that $\mathrm{P}(Y_0 = \eta_0) = 1$ and $\mathrm{P}(Y_t \geqslant 0 \text{ for all } t \geqslant 0) = 1$. (It is a consequence of Dawson and Li (2006).)
- if $\mathrm{P}(\eta_0 > 0) = 1$ or $a > 0$, then $\mathrm{P}\big(\int_0^t Y_s\,\mathrm{d}s > 0\big) = 1$, $t > 0$.

Remark

(i) Y is a CBI process having a branching mechanism

$$R(u) = \frac{\sigma^2}{2} u^2 - bu, \qquad u \in \mathbb{C} \text{ with } \operatorname{Re}(u) \leqslant 0,$$

and an immigration mechanism

$$F(u) = au + \int_0^\infty (\mathrm{e}^{uz} - 1)\, m(\mathrm{d}z), \qquad u \in \mathbb{C} \text{ with } \operatorname{Re}(u) \leqslant 0.$$

The jump part has effects only on the immigration mechanism.

(ii) The infinitesimal generator of Y takes the form

$$(\mathcal{A}f)(y) = (a - by)f'(y) + \frac{\sigma^2}{2} y f''(y) + \int_0^\infty (f(y+z) - f(y))\, m(\mathrm{d}z),$$

where $y \geqslant 0$, $f \in C_c^2(\mathbb{R}_+, \mathbb{R})$, and f' and f'' denote the first and second order derivatives of f, where

$$(\mathcal{A}f)(y) := \lim_{t \downarrow 0} \frac{\mathrm{E}(f(Y_t) \mid Y_0 = y) - f(y)}{t}, \quad y \geqslant 0$$

for suitable functions $f : \mathbb{R}_+ \to \mathbb{R}$.

Asymptotic behavior of expectation

Proposition. Let $a \geqslant 0$, $b \in \mathbb{R}$, $\sigma > 0$, and let m be a Lévy measure on $(0, \infty)$ satisfying (A1). Let $(Y_t)_{t \geqslant 0}$ be the unique strong solution satisfying $\mathrm{P}(Y_0 \geqslant 0) = 1$ and $\mathrm{E}(Y_0) < \infty$. Then

$$\mathrm{E}(Y_t) = \begin{cases} \mathrm{e}^{-bt}\mathrm{E}(Y_0) + \left(a + \int_0^\infty z\, m(\mathrm{d}z)\right) \frac{1-\mathrm{e}^{-bt}}{b} & \text{if } b \neq 0, \\ \mathrm{E}(Y_0) + \left(a + \int_0^\infty z\, m(\mathrm{d}z)\right) t & \text{if } b = 0, \end{cases} \qquad t \geqslant 0.$$

Consequently, if $b > 0$, then

$$\lim_{t \to \infty} \mathrm{E}(Y_t) = \left(a + \int_0^\infty z\, m(\mathrm{d}z)\right) \frac{1}{b},$$

if $b = 0$, then

$$\lim_{t \to \infty} t^{-1} \mathrm{E}(Y_t) = a + \int_0^\infty z\, m(\mathrm{d}z),$$

if $b < 0$, then

$$\lim_{t \to \infty} \mathrm{e}^{bt} \mathrm{E}(Y_t) = \mathrm{E}(Y_0) - \left(a + \int_0^\infty z\, m(\mathrm{d}z)\right) \frac{1}{b}.$$

Classification of jump-type CIR processes

Based on the asymptotic behavior of $\mathsf{E}(Y_t)$ as $t \to \infty$, we introduce a classification of jump-type CIR processes.

Definition. Let $a \geqslant 0$, $b \in \mathbb{R}$, $\sigma > 0$, and let m be a Lévy measure on $(0, \infty)$ satisfying (A1). Let $(Y_t)_{t \geqslant 0}$ be the unique non-negative strong solution satisfying $\mathsf{P}(Y_0 \geqslant 0) = 1$ and $\mathsf{E}(Y_0) < \infty$. We call $(Y_t)_{t \geqslant 0}$

$$\begin{cases} \text{subcritical} & \text{if } b > 0, \\ \text{critical} & \text{if } b = 0, \\ \text{supercritical} & \text{if } b < 0. \end{cases}$$

Stationarity and ergodicity in the subcritical case, I

Theorem. Let $a \geqslant 0$, $b > 0$, $\sigma > 0$, and let m be a Lévy measure on $(0, \infty)$ satisfying (A1). Let $(Y_t)_{t \geqslant 0}$ be the unique strong solution satisfying $\mathrm{P}(Y_0 \geqslant 0) = 1$ and $\mathrm{E}(Y_0) < \infty$.

(i) Then $(Y_t)_{t \geqslant 0}$ converges in law to its unique stationary distribution π having Laplace transform

$$\int_0^\infty \mathrm{e}^{uy}\, \pi(\mathrm{d}y) = \exp\left\{ \int_u^0 \frac{av + \int_0^\infty (\mathrm{e}^{vz} - 1)\, m(\mathrm{d}z)}{\frac{\sigma^2}{2} v^2 - bv}\, \mathrm{d}v \right\}, \qquad u \leqslant 0.$$

Moreover, π has a finite expectation given by

$$\int_0^\infty y\, \pi(\mathrm{d}y) = \left(a + \int_0^\infty z\, m(\mathrm{d}z) \right) \frac{1}{b} \ \left(= \lim_{t \to \infty} \mathrm{E}(Y_t)\right).$$

In the special case $m = 0$ (diffusion-type CIR process),

$$\int_0^\infty \mathrm{e}^{uy}\, \pi(\mathrm{d}y) = \left(1 - \frac{\sigma^2}{2b} u \right)^{-\frac{2a}{\sigma^2}}, \qquad u \leqslant 0,$$

i.e., π has Gamma distribution with parameters $\frac{2a}{\sigma^2}$ and $\frac{2b}{\sigma^2}$.

Stationarity and ergodicity in the subcritical case, II

(ii) If, in addition, $a > 0$ and the extra moment condition

$$\int_0^1 z \log\left(\frac{1}{z}\right) m(\mathrm{d}z) < \infty$$

holds, then the process $(Y_t)_{t \geqslant 0}$ is exponentially ergodic, namely, there exist constants $\beta \in (0,1)$ and $C > 0$ such that

$$\|\mathrm{P}_{Y_t | Y_0 = y_0} - \pi\|_{\mathrm{TV}} \leqslant C(y_0 + 1)\beta^t, \qquad t, y_0 \geqslant 0,$$

where $\|\mu\|_{\mathrm{TV}}$ denotes the total-variation norm of a signed measure μ on $\mathbb{R}_+$ defined by $\|\mu\|_{\mathrm{TV}} := \sup_{A \in \mathcal{B}(\mathbb{R}_+)} |\mu(A)|$, and $\mathrm{P}_{Y_t | Y_0 = y_0}$ is the conditional distribution of Y_t given $Y_0 = y_0$.

Moreover, for all Borel measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ with $\int_0^\infty |f(y)| \, \pi(\mathrm{d}y) < \infty$, we have

$$\frac{1}{T} \int_0^T f(Y_s) \, \mathrm{d}s \xrightarrow{\text{a.s.}} \int_0^\infty f(y) \, \pi(\mathrm{d}y) \qquad \text{as } T \to \infty.$$

References for stationarity and ergodicity in the subcritical case

For the existence of a unique stationary distribution,
see Pinsky (1972), Keller-Ressel and Steiner (2008), Li (2011) and Keller-Ressel and Mijatović (2012).

For the exponential ergodicity,
see Jin, Rüdiger and Trabelsi (2016).

Interpretation of b as a growth rate

In the subcritical case $(b > 0)$:

$$\mathrm{E}(Y_t) = \int_0^\infty y\,\pi(\mathrm{d}y) + \mathrm{e}^{-bt}\Big(\mathrm{E}(Y_0) - \int_0^\infty y\,\pi(\mathrm{d}y)\Big), \quad t \geqslant 0,$$

and so $\lim_{t\to\infty} \mathrm{E}(Y_t) = \int_0^\infty y\,\pi(\mathrm{d}y)$, and b can be interpreted as the rate at which $\mathrm{E}(Y_t)$ tends to $\int_0^\infty y\,\pi(\mathrm{d}y)$ as $t \to \infty$.

In the critical and supercriticial cases b also determines the speed at which $\mathrm{E}(Y_t)$ converges to ∞ as $t \to \infty$.

So, in all cases b can be interpreted as the growth rate.

b is also called a speed of adjustment in the subcritical case.

On the parameter σ

We do not estimate the parameter σ, since it is a measurable function (statistic) of $(Y_t)_{t \in [0,T]}$ for any $T > 0$, following from

$$\frac{1}{\frac{1}{n}\sum_{i=1}^{\lfloor nT \rfloor} Y_{\frac{i-1}{n}}} \left[\sum_{i=1}^{\lfloor nT \rfloor} \left(Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}} \right)^2 - \sum_{u \in [0,T]} (\Delta Y_u)^2 \right] \xrightarrow{\mathrm{P}} \frac{\langle Y^{\mathrm{cont}} \rangle_T}{\int_0^T Y_u \, \mathrm{d}u} = \sigma^2$$

as $n \to \infty$, where

- $\Delta Y_u := Y_u - Y_{u-}$, $u > 0$, and $\Delta Y_0 := 0$,
- $Y_t^{\mathrm{cont}} = \sigma \int_0^t \sqrt{Y_u} \, \mathrm{d}W_u$, $t \geqslant 0$, denotes the continuous martingale part of Y,
- the convergence holds almost surely as well along a suitable subsequence.

From now on, we suppose $\mathrm{P}(Y_0 = y_0) = 1$, where $y_0 \geqslant 0$ is known (deterministic initial value).

Joint Laplace transform of Y_t and $\int_0^t Y_s \, ds$

Theorem. Let $a \geqslant 0$, $b \in \mathbb{R}$, $\sigma > 0$, and let m be a Lévy measure on $(0, \infty)$ satisfying (A1). Let $(Y_t)_{t \geqslant 0}$ be the unique strong solution satisfying $\mathrm{P}(Y_0 = y_0) = 1$ with some $y_0 \geqslant 0$. Then for all $u, v \leqslant 0$,

$$\mathrm{E}\left[\exp\left\{uY_t + v\int_0^t Y_s \, ds\right\}\right]$$
$$= \exp\left\{\psi_{u,v}(t)y_0 + \int_0^t \left(a\psi_{u,v}(s) + \int_0^\infty \left(\mathrm{e}^{z\psi_{u,v}(s)} - 1\right) m(dz)\right) ds\right\}$$

for $t \geqslant 0$, where $\psi_{u,v} : [0, \infty) \to (-\infty, 0]$ takes the form

$$\psi_{u,v}(t) = \begin{cases} \dfrac{u\gamma_v \cosh\left(\frac{\gamma_v t}{2}\right) + (-ub + 2v)\sinh\left(\frac{\gamma_v t}{2}\right)}{\gamma_v \cosh\left(\frac{\gamma_v t}{2}\right) + (-\sigma^2 u + b)\sinh\left(\frac{\gamma_v t}{2}\right)} & \text{if } v < 0 \text{ or } b \neq 0, \\ \dfrac{u}{1 - \frac{\sigma^2 u}{2} t} & \text{if } v = 0 \text{ and } b = 0, \end{cases}$$

where $\gamma_v := \sqrt{b^2 - 2\sigma^2 v}$.

Note that this Laplace transform is an exponentially affine function of the initial value $(y_0, 0)$.

Remark.

(i) We have $\int_0^t \psi_{u,v}(s)\,\mathrm{d}s$ takes the form

$$\begin{cases} \frac{b}{\sigma^2}\,t - \frac{2}{\sigma^2}\log\Big(\cosh\left(\frac{\gamma_v t}{2}\right) + \frac{-\sigma^2 u + b}{\gamma_v}\sinh\left(\frac{\gamma_v t}{2}\right)\Big) & \text{if } v < 0 \text{ or } b \neq 0, \\ -\frac{2}{\sigma^2}\log\left(1 - \frac{\sigma^2 u}{2}t\right) & \text{if } v = 0 \text{ and } b = 0. \end{cases}$$

(ii) For affine process, Keller-Ressel (2008), and for CBI processes, Jiao, Ma and Scotti (2016+), derived the joint Laplace transform of the process and its integrated process containing the (not necessarily explicit) solution of a Riccati type DE.

(iii) Our proof is based on the fact that $(Y_t, \int_0^t Y_s\,\mathrm{d}s)$, $t \geqslant 0$, is a 2-dimensional CBI process yielding that

$$\mathrm{E}\left[\exp\left\{uY_t + v\int_0^t Y_s\,\mathrm{d}s\right\}\right] = \exp\left\{\psi_{u,v}(t)y_0 + \int_0^t\left(a\psi_{u,v}(s) + \int_0^\infty (\mathrm{e}^{z\psi_{u,v}(s)} - 1)\,m(\mathrm{d}z)\right)\mathrm{d}s\right\}$$

for $t \geqslant 0$, $u, v \leqslant 0$, where $\psi_{u,v} : [0,\infty) \to (-\infty, 0]$ is the unique locally bounded solution to the Riccati DE

$$\psi_{u,v}'(t) = \frac{\sigma^2}{2}\psi_{u,v}(t)^2 - b\psi_{u,v}(t) + v, \qquad t \geqslant 0, \qquad \psi_{u,v}(0) = u.$$

Corollary.

(i) With $v = 0$, we have the Laplace transform $\mathrm{E}\big(\mathrm{e}^{uY_t}\big)$.
For all $u \leqslant 0$, the function $\psi_{u,0} : [0,\infty) \to (-\infty, 0]$ is

$$\psi_{u,0}(t) = \begin{cases} \frac{2ub\mathrm{e}^{-bt}}{\sigma^2 u(\mathrm{e}^{-bt}-1)+2b} & \text{if } b \neq 0, \\ \frac{u}{1-\frac{\sigma^2 u}{2}t} & \text{if } b = 0, \end{cases} \qquad t \geqslant 0.$$

(ii) With $u = 0$, we have the Laplace trans. $\mathrm{E}[\exp\{v\int_0^t Y_s\,\mathrm{d}s\}]$.
For all $v \leqslant 0$, the function $\psi_{0,v} : [0,\infty) \to (-\infty, 0]$ is

$$\psi_{0,v}(t) = \begin{cases} \frac{2v\sinh\left(\frac{\gamma_v t}{2}\right)}{\gamma_v\cosh\left(\frac{\gamma_v t}{2}\right)+b\sinh\left(\frac{\gamma_v t}{2}\right)} & \text{if } v < 0 \text{ or } b \neq 0, \\ 0 & \text{if } v = 0 \text{ and } b = 0, \end{cases} \qquad t \geqslant 0.$$

Existence and uniqueness of MLE, I

Recall that

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \qquad t \geqslant 0,}$$

with known $a \geqslant 0$, $\sigma > 0$, Lévy measure m satisfying (A1) and deterministic initial value $Y_0 = y_0 \geqslant 0$.

We consider $b \in \mathbb{R}$ as an unkown parameter.

Let P_b denote the probability measure induced by $(Y_t)_{t\geqslant 0}$ on the measurable space $(D(\mathbb{R}_+,\mathbb{R}), \mathcal{D}(\mathbb{R}_+,\mathbb{R}))$ endowed with the natural filtration $(\mathcal{D}_t(\mathbb{R}_+,\mathbb{R}))_{t\in\mathbb{R}_+}$, where

$D(\mathbb{R}_+,\mathbb{R})$ is the space of $\mathbb{R}$-valued càdlàg functions on $\mathbb{R}_+$,

$\mathcal{D}_t(\mathbb{R}_+,\mathbb{R})$ is roughly speaking the σ-algebra generated by the past up to time t.

For all $T > 0$, let $\mathrm{P}_{b,T} := \mathrm{P}_b|_{\mathcal{D}_T(\mathbb{R}_+,\mathbb{R})}$.

Existence and uniqueness of MLE, II

Proposition. Let $b, \widetilde{b} \in \mathbb{R}$. Then, for all $T > 0$, the probability measures $\mathsf{P}_{b,T}$ and $\mathsf{P}_{\widetilde{b},T}$ are absolutely continuous with respect to each other, and, under P,

$$\log\left(\frac{\mathrm{d}\mathsf{P}_{b,T}}{\mathrm{d}\mathsf{P}_{\widetilde{b},T}}(\widetilde{Y})\right) = -\frac{b-\widetilde{b}}{\sigma^2}(\widetilde{Y}_T - y_0 - aT - J_T) - \frac{b^2-\widetilde{b}^2}{2\sigma^2}\int_0^T \widetilde{Y}_s \,\mathrm{d}s,$$

where $\widetilde{Y}$ is the process corresponding to the parameter $\widetilde{b}$.

A proof is based on a careful application of results of Jacod and Mémin (1976) and Jacod and Shiryaev (2003), which provide a formula for the loglikelihood function in question in terms of the semimartingale characteristics and the continuous martingale part of Y. Further, in our case, one can use the explicit form of the continuous martingale part in question: $\sigma\int_0^t \sqrt{Y_s}\,\mathrm{d}W_s$, $t \geqslant 0$.

Existence and uniqueness of MLE, III

By an MLE $\widehat{b}_T$ of b based on the observations $(Y_t)_{t\in[0,T]}$, we mean

$$\widehat{b}_T := \operatorname*{arg\,max}_{b\in\mathbb{R}} \left(-\frac{b-\widetilde{b}}{\sigma^2}(Y_T - y_0 - aT - J_T) - \frac{b^2-\widetilde{b}^2}{2\sigma^2}\int_0^T Y_s \, \mathrm{d}s \right),$$

which will turn out to be not dependent on $\widetilde{b}$ (i.e., the fixed reference measure $\mathrm{P}_{\widetilde{b},T}$ does not play a role).

We can find the global maximum point above explicitly, since an explicit formula is available for the loglikelihood function and it is a quadratic expression of b.

Existence and uniqueness of MLE, IV

Proposition. Let $a \geqslant 0$, $b \in \mathbb{R}$, $\sigma > 0$, $y_0 \geqslant 0$, and let m be a Lévy measure on $(0, \infty)$ satisfying (A1). If $a > 0$ or $y_0 > 0$, then for each $T > 0$, there exists a unique MLE $\widehat{b}_T$ of b a.s. having the form

$$\widehat{b}_T = -\frac{Y_T - y_0 - aT - J_T}{\int_0^T Y_s \, ds},$$

provided that $\int_0^T Y_s \, ds > 0$ (which holds a.s.).

Remark. (i) For $t \in [0, T]$, J_t is a measurable function of $(Y_u)_{u \in [0,T]}$:

$$J_t = \sum_{s \in [0,t]} \Delta J_s = \sum_{s \in [0,t]} \Delta Y_s, \qquad t \geqslant 0.$$

(ii) For the calculation of the MLE above, one does not need to know σ and m.

Difference of the MLE and the true parameter

Using our SDE, we have

$$\widehat{b}_T - b = -\frac{Y_T - y_0 - aT - J_T + b\int_0^T Y_s\,\mathrm{d}s}{\int_0^T Y_s\,\mathrm{d}s}$$

$$= -\sigma\,\frac{\int_0^T \sqrt{Y_s}\,\mathrm{d}W_s}{\int_0^T Y_s\,\mathrm{d}s} = -\sigma^2\,\frac{Y_t^{\mathrm{cont}}}{\langle Y^{\mathrm{cont}}\rangle_t},$$

provided that $\int_0^T Y_s\,\mathrm{d}s > 0$ (which holds a.s.).

Mathematical task: using the explicit form of $\widehat{b}_T$ and $\widehat{b}_T - b$, let us describe the asymptotics of $\widehat{b}_T$ as $T \to \infty$.

Asymptotics of MLE: subcritical case $(b > 0)$

Theorem. Let $a > 0$, $b > 0$, $\sigma > 0$, m be a Lévy measure on $(0, \infty)$ satisfying (A1), and $\mathrm{P}(Y_0 = y_0) = 1$ with some $y_0 \geqslant 0$. Then the MLE $\widehat{b}_T$ of b is asymptotically normal, i.e.,

$$\sqrt{T}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\sigma^2 b}{a + \int_0^\infty z\, m(\mathrm{d}z)}\right) \qquad \text{as } T \to \infty.$$

Especially, $\widehat{b}_T$ is weakly consistent, i.e., $\widehat{b}_T \xrightarrow{\mathrm{P}} b$ as $T \to \infty$.

With a random scaling,

$$\frac{1}{\sigma}\left(\int_0^T Y_s \,\mathrm{d}s\right)^{1/2} (\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1) \qquad \text{as } T \to \infty.$$

Under the additional moment condition $\int_0^1 z \log\left(\frac{1}{z}\right) m(\mathrm{d}z) < \infty$, $\widehat{b}_T$ is strongly consistent, i.e., $\widehat{b}_T \xrightarrow{\text{a.s.}} b$ as $T \to \infty$.

A proof is based on

- the decomposition

$$\sqrt{T}(\widehat{b}_T - b) = -\sigma \frac{\frac{1}{\sqrt{T}} \int_0^T \sqrt{Y_s}\,\mathrm{d}W_s}{\frac{1}{T}\int_0^T Y_s\,\mathrm{d}s}, \qquad T > 0.$$

- by the explicit form of the Laplace transform of $\int_0^T Y_s\,\mathrm{d}s$,

$$\frac{1}{T}\int_0^T Y_s\,\mathrm{d}s \xrightarrow{\mathrm{P}} \frac{1}{b}\left(a + \int_0^\infty z\,m(\mathrm{d}z)\right) = \int_0^\infty y\,\pi(\mathrm{d}y) \qquad \text{as } \; T \to \infty,$$

- a limit theorem for continuous local martingales due to van Zanten (2000), presented below.

- under the moment assumption $\int_0^1 z \log\left(\frac{1}{z}\right) m(\mathrm{d}z) < \infty$, we have $\frac{1}{T}\int_0^t Y_s\,\mathrm{d}s \xrightarrow{\text{a.s.}} \int_0^\infty y\,\pi(\mathrm{d}y)$ as $T \to \infty$, yielding $\int_0^T Y_s\,\mathrm{d}s \xrightarrow{\text{a.s.}} \infty$ as $T \to \infty$, and then one can use a SLLN for continuous local martingales.

A limit theorem for continuous local martingales

Theorem (van Zanten (2000)). Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathrm{P}\big)$ be a filtered probability space satisfying the usual conditions. Let $(\boldsymbol{M}_t)_{t\geqslant 0}$ be a d-dimensional square-integrable continuous local martingale w.r.t the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ such that $\mathrm{P}(\boldsymbol{M}_0 = \boldsymbol{0}) = 1$. Suppose that there exists a function $\boldsymbol{Q} : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ such that

- $\boldsymbol{Q}(t)$ is an invertible (non-random) matrix for all $t \geqslant 0$,
- $\lim_{t\to\infty} \|\boldsymbol{Q}(t)\| = 0$,
- $\boldsymbol{Q}(t)\langle \boldsymbol{M}\rangle_t \, \boldsymbol{Q}(t)^\top \xrightarrow{\mathrm{P}} \boldsymbol{\eta}\boldsymbol{\eta}^\top$ as $t \to \infty$, where $\boldsymbol{\eta}$ is a $d \times d$ (possibly) random matrix.

Then

$$\boldsymbol{Q}(t)\boldsymbol{M}_t \xrightarrow{\mathcal{L}} \boldsymbol{\eta}\boldsymbol{Z} \qquad \text{as } t \to \infty,$$

where $\boldsymbol{Z}$ is a d-dimensional standard normally distributed random vector independent of $\boldsymbol{\eta}$.

That is, $\boldsymbol{Q}(t)\boldsymbol{M}_t$ has a mixed normal limit distribution as $t \to \infty$.

Asymptotics of MLE: critical case $(b = 0)$

Theorem. Let $a \geqslant 0$, $b = 0$, $\sigma > 0$, m be a Lévy measure on $(0, \infty)$ satisfying (A1), and $\mathrm{P}(Y_0 = y_0) = 1$ with some $y_0 \geqslant 0$. Suppose that $a > 0$ or $a = 0$, $y_0 > 0$, $\int_0^\infty z\, m(\mathrm{d}z) > 0$. Then

$$T(\widehat{b}_T - b) = T\widehat{b}_T \xrightarrow{\mathcal{L}} \frac{a + \int_0^\infty z\, m(\mathrm{d}z) - \mathcal{Y}_1}{\int_0^1 \mathcal{Y}_s \, \mathrm{d}s} \qquad \text{as } T \to \infty,$$

where $(\mathcal{Y}_t)_{t \geqslant 0}$ is the critical (diffusion type) CIR process

$$\mathrm{d}\mathcal{Y}_t = \left(a + \int_0^\infty z\, m(\mathrm{d}z) \right) \mathrm{d}t + \sigma \sqrt{\mathcal{Y}_t}\, \mathrm{d}\mathcal{W}_t, \quad t \geqslant 0, \quad \text{with } \mathcal{Y}_0 = 0,$$

where $(\mathcal{W}_t)_{t \geqslant 0}$ is a standard Wiener process.

As a consequence, $\widehat{b}_T$ is weakly consistent.

With a random scaling,

$$\frac{1}{\sigma} \left(\int_0^T Y_s \, \mathrm{d}s \right)^{1/2} (\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \frac{a + \int_0^\infty z\, m(\mathrm{d}z) - \mathcal{Y}_1}{\sigma \left(\int_0^1 \mathcal{Y}_s \, \mathrm{d}s \right)^{1/2}} \qquad \text{as } T \to \infty.$$

A proof is based on

- the decomposition

$$T\widehat{b}_T = -\frac{\frac{Y_T}{T} - \frac{y_0}{T} - a - \frac{J_T}{T}}{\frac{1}{T^2}\int_0^T Y_s \, \mathrm{d}s}, \qquad T > 0.$$

- by SLLN for Lévy processes, $\frac{J_T}{T} \xrightarrow{\text{a.s.}} \mathrm{E}(J_1) = \int_0^\infty z \, m(\mathrm{d}z)$ as $T \to \infty$.
- $$\left(\frac{1}{T} Y_T, \frac{1}{T^2}\int_0^T Y_s \, \mathrm{d}s\right) \xrightarrow{\mathcal{L}} \left(\mathcal{Y}_1, \int_0^1 \mathcal{Y}_s \, \mathrm{d}s\right) \qquad \text{as} \quad T \to \infty,$$

  where the Laplace transform of the limit law takes the form

$$\mathrm{E}\left(\mathrm{e}^{u\mathcal{Y}_1 + v\int_0^1 \mathcal{Y}_s \, \mathrm{d}s}\right) = \begin{cases} \left(\cosh\left(\frac{\gamma_v}{2}\right) - \frac{\sigma^2 u}{\gamma_v}\sinh\left(\frac{\gamma_v}{2}\right)\right)^{-\frac{2}{\sigma^2}\left(a + \int_0^\infty z \, m(\mathrm{d}z)\right)} & \text{if} \quad u \leqslant 0, \, v < 0, \\ \left(1 - \frac{\sigma^2 u}{2}\right)^{-\frac{2}{\sigma^2}\left(a + \int_0^\infty z \, m(\mathrm{d}z)\right)} & \text{if} \quad u \leqslant 0, \, v = 0, \end{cases}$$

  where $\gamma_v = \sqrt{-2\sigma^2 v}$, $v \leqslant 0$.

Asymptotics of MLE: supercritical case $(b < 0)$

Theorem. Let $a > 0$, $b < 0$, $\sigma > 0$, m be a Lévy measure on $(0, \infty)$ satisfying (A1), and $\mathrm{P}(Y_0 = y_0) = 1$ with some $y_0 \geqslant 0$. Then $\widehat{b}_T$ is strongly consistent, and asymptotically mixed normal, namely

$$\mathrm{e}^{-bT/2}(\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \sigma Z \left(-\frac{V}{b}\right)^{-1/2} \qquad \text{as } T \to \infty,$$

where V is a positive r. v. having Laplace transform

$$\mathrm{E}(\mathrm{e}^{uV}) = \exp\left\{\frac{u y_0}{1 + \frac{\sigma^2 u}{2b}}\right\} \left(1 + \frac{\sigma^2 u}{2b}\right)^{-\frac{2a}{\sigma^2}} \exp\left\{\int_0^\infty \left(\int_0^\infty \left(\exp\left\{\frac{zu\mathrm{e}^{by}}{1 + \frac{\sigma^2 u}{2b}\mathrm{e}^{by}}\right\} - 1\right) m(\mathrm{d}z)\right) \mathrm{d}y\right\}$$

for all $u \leqslant 0$, and Z is a standard normally distributed r. v., independent of V.

With a random scaling, we have

$$\frac{1}{\sigma}\left(\int_0^T Y_s \, \mathrm{d}s\right)^{1/2} (\widehat{b}_T - b) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1) \qquad \text{as } T \to \infty.$$

A stochastic representation of V

$$V \stackrel{\mathcal{L}}{=} \widetilde{\mathcal{V}} + \widetilde{\widetilde{\mathcal{V}}},$$

where

- $\widetilde{\mathcal{V}}$ and $\widetilde{\widetilde{\mathcal{V}}}$ are independent random variables,
- $e^{bt}\widetilde{\mathcal{Y}}_t \xrightarrow{\text{a.s.}} \widetilde{\mathcal{V}}$ as $t \to \infty$, where $(\widetilde{\mathcal{Y}}_t)_{t \geqslant 0}$ is the (diffusion-type) supercritical CIR process

  $$\mathrm{d}\widetilde{\mathcal{Y}}_t = (a - b\widetilde{\mathcal{Y}}_t)\,\mathrm{d}t + \sigma\sqrt{\widetilde{\mathcal{Y}}_t}\,\mathrm{d}\widetilde{\mathcal{W}}_t, \qquad t \geqslant 0, \qquad \text{with} \quad \widetilde{\mathcal{Y}}_0 = y_0,$$

  where $(\widetilde{\mathcal{W}}_t)_{t \geqslant 0}$ is a standard Wiener process,
- $e^{bt}\widetilde{\widetilde{\mathcal{Y}}}_t \xrightarrow{\text{a.s.}} \widetilde{\widetilde{\mathcal{V}}}$ as $t \to \infty$, where $(\widetilde{\widetilde{\mathcal{Y}}}_t)_{t \geqslant 0}$ is the jump-type supercritical CIR process

  $$\mathrm{d}\widetilde{\widetilde{\mathcal{Y}}}_t = -b\widetilde{\widetilde{\mathcal{Y}}}_t\,\mathrm{d}t + \sigma\sqrt{\widetilde{\widetilde{\mathcal{Y}}}_t}\,\mathrm{d}\widetilde{\widetilde{\mathcal{W}}}_t + \mathrm{d}J_t, \qquad t \geqslant 0, \qquad \text{with} \quad \widetilde{\widetilde{\mathcal{Y}}}_0 = 0,$$

  where $(\widetilde{\widetilde{\mathcal{W}}}_t)_{t \geqslant 0}$ is a standard Wiener process indep. of $\widetilde{\mathcal{W}}$.
- $\widetilde{\mathcal{V}} \stackrel{\mathcal{L}}{=} \mathcal{Z}_{-\frac{1}{b}}$, where $\mathrm{d}\mathcal{Z}_t = a\,\mathrm{d}t + \sigma\sqrt{\mathcal{Z}_t}\,\mathrm{d}\mathcal{W}_t$, $t \geqslant 0$ with $\mathcal{Z}_0 = y_0$.

A proof is based on

- the decomposition

$$e^{-bT/2}(\widehat{b}_T - b) = -\sigma \frac{e^{bT/2} \int_0^T \sqrt{Y_s}\,\mathrm{d}W_s}{e^{bT} \int_0^T Y_s\,\mathrm{d}s}, \qquad T > 0.$$

- there exists a non-negative random variable V such that

$$e^{bT} Y_T \xrightarrow{\text{a.s.}} V \qquad \text{and} \qquad e^{bT} \int_0^T Y_u\,\mathrm{d}u \xrightarrow{\text{a.s.}} -\frac{V}{b} \qquad \text{as } T \to \infty,$$

  following from submartingale convergence theorem applied to $(e^{bT} Y_T)_{t \geqslant 0}$, and from the integral Toeplitz lemma.
- positivity of V following from the absolute continuity of $\widetilde{\mathcal{V}}$ due to $\widetilde{\mathcal{V}} \overset{\mathcal{L}}{=} \mathcal{Z}_{-\frac{1}{b}}$.
- a limit theorem for continuous local martingales due to van Zanten (2000).
- SLLN for Lévy processes: $\frac{J_T}{T} \xrightarrow{\text{a.s.}} \mathrm{E}(J_1) = \int_0^\infty z\, m(\mathrm{d}z)$ as $T \to \infty$ (used for strong consistency).

Remarks on the limit theorems

(i) In the subcritical case, the limit distribution of $\sqrt{T}(\widehat{b}_T - b)$, and in the critical case, the limit distribution of $T(\widehat{b}_T - b)$, does not depend on the intial value y_0.

But, in the supercritical case, the limit law of $e^{-bT/2}(\widehat{b}_T - b)$ does depend on the initial value y_0.

(ii) Unified theory: there is a common (random) normalization for the MLE $\widehat{b}_T$ to have a non-trivial limit in all cases.

Namely, for all $b \in \mathbb{R}$,

$$\frac{1}{\sigma}\left(\int_0^T Y_s \, ds\right)^{1/2}(\widehat{b}_T - b) \quad \text{converges in distribution as} \quad T \to \infty,$$

and the limit distribution is standard normal for the non-critical cases, while it is non-normal for the critical case.

Possible future research questions for this model

For the jump-type CIR process

$$\boxed{\mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t + \mathrm{d}J_t, \qquad t \geqslant 0,}$$

one could investigate

- the MLE of a supposing that b is known based on continuous time observations. For this, e.g., we should find the limit behavior of $\int_0^t \frac{1}{Y_s}\,\mathrm{d}s$ as $t \to \infty$.
- the MLE of (a, b) based on continuous time observations,
- statistical tests for deciding on the null hypothesis $H_0 : b = b_0$ against $H_1 : b \neq b_0$, where $b_0 \in \mathbb{R}$ is given.

This talk is based on:

BARCZY, M., BEN ALAYA, M., KEBAIER, A., PAP, G.,
Asymptotic properties of maximum likelihood estimator for the growth rate for a jump-type CIR process based on continuous time observations. *Submitted.*
Arxiv: 1609.05865 (2016).

Thank you for your attention!

CIR process and the jump-type CIR processes considered are CBI processes.

Heston process is a two-factor affine process.

Affine processes are common generalizations of

- CBI processes

and

- Ornstein-Uhlenbeck-type (OU-type) processes.

Two-factor affine processes

Definition. A time-homogeneous Markov process $(Z_t)_{t\geqslant 0}$ with state space $[0,\infty)\times\mathbb{R}$ is called a two-factor affine process if its (conditional) characteristic function takes the form

$$\boxed{\mathrm{E}(\mathrm{e}^{i\langle u, Z_t\rangle} \mid Z_0 = z) = \exp\{\langle z, G(t,u)\rangle + H(t,u)\}}$$

for $z \in [0,\infty)\times\mathbb{R}$, $u \in \mathbb{R}^2$ and $t \geqslant 0$, where $G(t,u) \in \mathbb{C}^2$ and $H(t,u) \in \mathbb{C}$. (Here $\langle \alpha, \beta\rangle := \alpha_1\beta_1 + \alpha_2\beta_2$ for $\alpha, \beta \in \mathbb{C}^2$.)

For any $t \geqslant 0$, the (cond.) characteristic function of Z_t depends exponentially affine on the initial value z.

Duffie, Filipović and Schachermayer (2003): there exist (two-factor) affine processes for so-called admissible set of parameters.

Two-factor affine diffusion processes

Dawson and Li (2006) derived a jump-type SDE for (some) two-factor (not necessarily diffusion-type) affine processes.

We specialize this result to the diffusion case.

Theorem. (Dawson and Li (2006)) Let $a \geqslant 0$, $\sigma_1, \sigma_2 > 0$, $\sigma_3 \geqslant 0$, $b, \alpha, \beta, \gamma \in \mathbb{R}$, $\varrho \in [-1, 1]$, and let us consider the SDE:

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma_1\sqrt{Y_t}\,\mathrm{d}W_t, \\ \mathrm{d}X_t = (\alpha - \beta Y_t - \gamma X_t)\,\mathrm{d}t + \sigma_2\sqrt{Y_t}\,\mathrm{d}(\varrho W_t + \sqrt{1-\varrho^2}B_t) + \sigma_3\,\mathrm{d}Q_t, \end{cases}$$

where $t \geqslant 0$, and $(W_t)_{t\geqslant 0}$, $(B_t)_{t\geqslant 0}$ and $(Q_t)_{t\geqslant 0}$ are independent stand. Wiener processes. Then it has a pathwise unique strong solution being a two-factor affine diffusion process.

Conversely, every two-factor affine diffusion process is a strong solution of such an SDE.

Diffusion-type Heston model (1993)

Let $a \geqslant 0$, $b, \alpha, \beta \in \mathbb{R}$, $\sigma_1 > 0$, $\sigma_2 > 0$, $\varrho \in (-1, 1)$, and let us consider the SDE:

$$\boxed{\begin{cases} \mathrm{d}Y_t = (a - bY_t)\,\mathrm{d}t + \sigma_1\sqrt{Y_t}\,\mathrm{d}W_t, \\ \mathrm{d}X_t = (\alpha - \beta Y_t)\,\mathrm{d}t + \sigma_2\sqrt{Y_t}\big(\varrho\,\mathrm{d}W_t + \sqrt{1-\varrho^2}\,\mathrm{d}B_t\big), \end{cases} \qquad t \geqslant 0,}$$

where $(W_t)_{t\geqslant 0}$ and $(B_t)_{t\geqslant 0}$ are independent standard Wiener processes.

Y is a CBI process: Cox–Ingersoll–Ross (CIR) process, square root process, Feller process.