On Feller and Strong Feller Properties of Regime-Switching Jump-Diffusion Processes with Countable Regimes

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Outline

Construction of the Process

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space
- Let (X, Λ) be a right continuous, strong Markov process with left-hand limits on $\mathbb{R}^d \times \mathbb{S}$, where $\mathbb{S} := \{0, 1, 2, \cdots, \}$.
- The first component X satisfies the following stochastic differential-integral equation

$$\begin{aligned} \mathrm{d}X(t) &= \sigma(X(t), \Lambda(t))\mathrm{d}B(t) + b(X(t), \Lambda(t))\mathrm{d}t \\ &\quad + \int_{U_0} c(X(t-), \Lambda(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \\ &\quad + \int_{U \setminus U_0} c(X(t-), \Lambda(t-), u)N(\mathrm{d}t, \mathrm{d}u), \end{aligned} \tag{1}$$

Construction of the Process (cont'd)

- The second component Λ is a discrete random process with an infinite state space $\mathbb{S} = \{0, 1, 2, \cdots, \}$ such that

$$\mathbb{P}\{\Lambda(t+\Delta) = l | \Lambda(t) = k, X(t) = x\} = \begin{cases} q_{kl}(x)\Delta + o(\Delta), & \text{if } k \neq l, \\ 1 + q_{kk}(x)\Delta + o(\Delta), & \text{if } k = l \end{cases} \tag{2}$$

  uniformly in $\mathbb{R}^d$, provided $\Delta \downarrow 0$.
- $q_{kl}(x) \geq 0$ and $\sum_{l \in \mathbb{S}} q_{kl}(x) = 0$ for all $x \in \mathbb{R}^d$ and $k \in \mathbb{S}$.

Construction of the Process (cont)

- $\sigma(x, k) \in \mathbb{R}^{d\times d}$, $b(x, k), c(x, k, u) \in \mathbb{R}^d$ for $x \in \mathbb{R}^d$, $k \in \mathbb{S}$ and $u \in U$.
- $\big(U, \mathcal{B}(U), \Pi\big)$ is a measurable space.
- $\Pi(\cdot)$ is a deterministic σ-finite characteristic measure on the measurable space $\big(U, \mathcal{B}(U)\big)$
- $U_0 \in \mathcal{B}(U)$ with $\Pi(U \setminus U_0) < \infty$.
- B is a d-dimensional Brownian motion.
- $N(\mathrm{d}t, \mathrm{d}u)$ is a Poisson random measure independent of $B(t)$.
- $\widetilde{N}(\mathrm{d}t, \mathrm{d}u) = N(\mathrm{d}t, \mathrm{d}u) - \Pi(\mathrm{d}u)\mathrm{d}t$ is the compensated Poisson random measure on $[0, \infty) \times U$.

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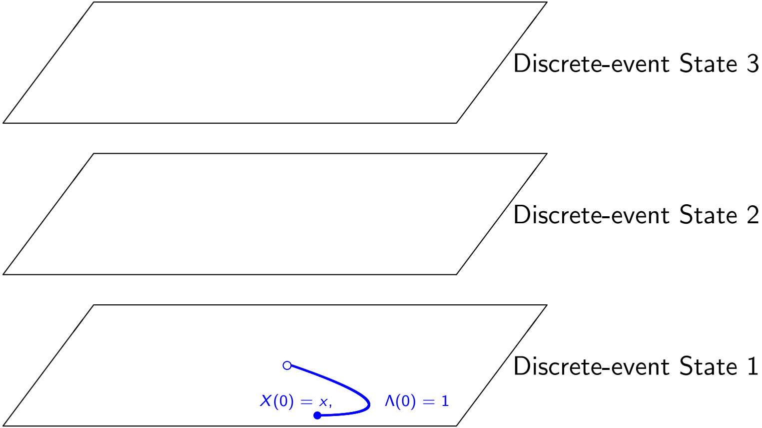
- $(X, \Lambda) \in \mathbb{R}^d \times \mathbb{S}$ is called a regime-switching jump diffusion process.
- It can be regarded as the usual jump diffusions living in random environments, with Λ providing the switching mechanism.

A “Sample Path” of a Regime-Switching Jump Diffusion



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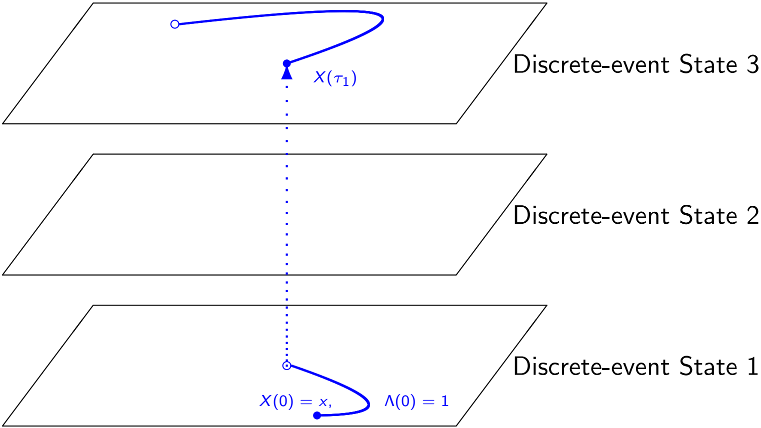

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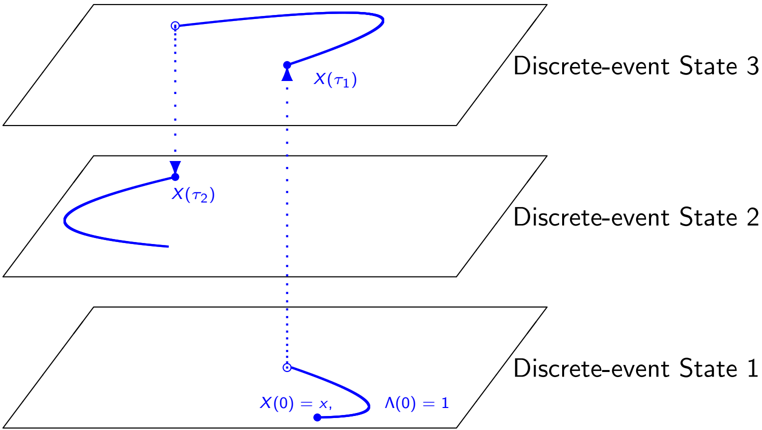

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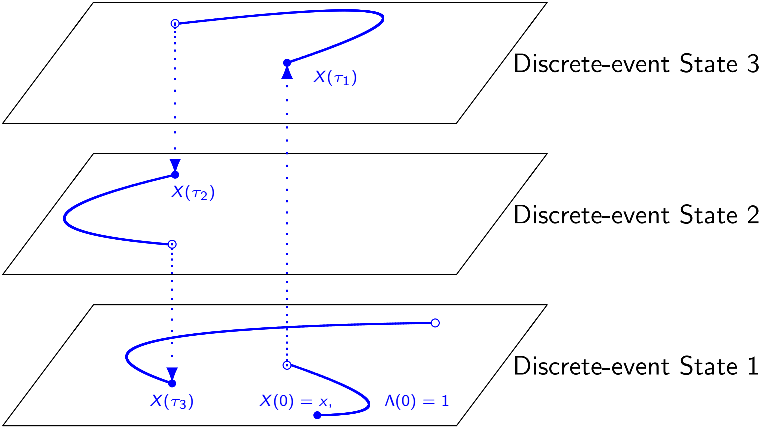

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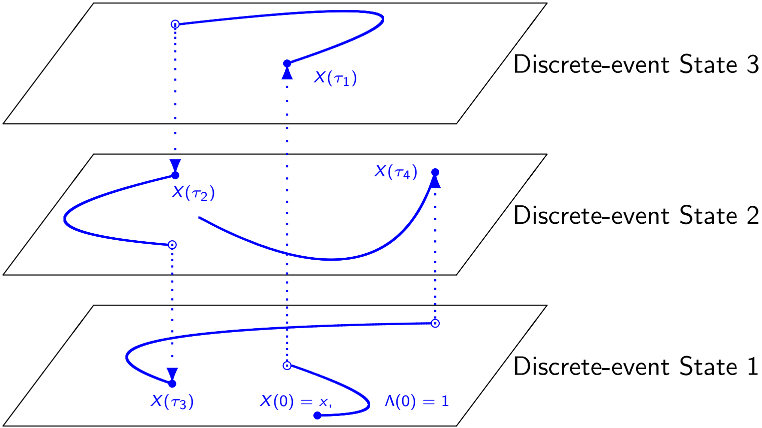

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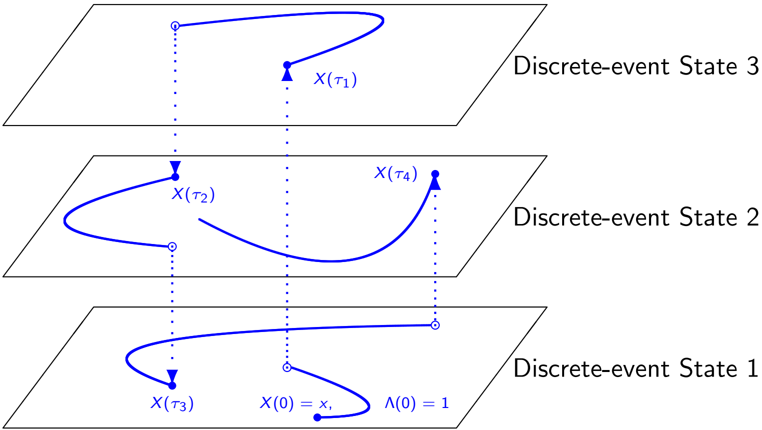

Figure: A "Sample Path" of a Regime-Switching Jump Diffusion

Flow Property: Smooth Dependence on the Initial Data

- Consider

$$X^x(t) = x + \int_0^t b(X^x(s))\mathrm{d}s + \int_0^t \sigma(X^x(s))\mathrm{d}W(s) \in \mathbb{R},$$

  in which b, σ satisfy the Lipschitz and linear growth conditions. Assume b, σ are sufficiently smooth.

- Then there exists a process ξ s.t. for any $t \geq 0$

$$\lim_{\delta \to 0} \mathbb{E}\left[\left| \frac{X^{x+\delta}(t) - X^x(t)}{\delta} - \xi(t) \right|^2 \right] = 0.$$

- $\xi(t)$ is the derivative of $X^x(t)$ with respect to its initial condition in the mean square sense and satisfies

$$\xi(t) = 1 + \int_0^t b'(X(s))\xi(s)\mathrm{d}s + \int_0^t \sigma'(X(s))\xi(s)\mathrm{d}W(s).$$

- Such a property readily carries over to regime-switching diffusion with Markovian switching:

$$X^x(t) = x + \int_0^t b(X^x(s), \Lambda^k(s))\mathrm{d}s + \int_0^t \sigma(X^x(s), \Lambda^k(s))\mathrm{d}W(s),$$

  in which $\Lambda^k \in \mathbb{S}$ is a continuous-time Markov chain independent of W.
- Here the generator of Λ is a constant q-matrix $Q = (q_{ij})$.
- What if $Q = Q(x)$ and so Λ and W are not necessarily independent?

A Simple Yet Interesting Example

- Consider the equation

$$\mathrm{d}X(t) = b(\Lambda(t))\mathrm{d}t, \qquad X(0) = x, \ \Lambda(0) = i \in \{1, 2\},$$

where $b(1) = 0$, $b(2) = 1$ and the switching process $\Lambda(t)$ has generator $Q = \begin{pmatrix} -f(x) & f(x) \\ 0 & 0 \end{pmatrix}$, where $f(x) > 0$ is a smooth function with compact support and $f(x) = x$ for $x \in [1, 2]$.

- Let $y > 0$ and $(X^{y,1}(t), \Lambda^{y,1}(t))$ be the solution with initial data $(y, 1)$. Let $\tau^y = \inf\{t \geq 0 : \Lambda^{y,1}(t) = 2\}$. Then

$$X^{y,1}(t) = y + t - t \wedge \tau^y.$$

- Then we can show that $n[X^{1+\frac{1}{n},1}(T) - X^{1,1}(T)]$ is not a Cauchy sequence in L^1 or L^2.

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- Then we can show that $n[X^{1+\frac{1}{n},1}(T) - X^{1,1}(T)]$ is not a Cauchy sequence in L^1 or L^2.
- Thus $X^{x,1}(t)$ cannot have a mean square derivative with respect to its initial data!

In this work

- Solution: Existence and Uniqueness
- Feller property
- Strong Feller property
- Exponential ergodicity

Related Work

1. Cloez, B. and Hairer, M. (2015). Exponential ergodicity for Markov processes with random switching. *Bernoulli*, 21(1):505–536.
2. Mao, X. and Yuan, C. (2006). *Stochastic differential equations with Markovian switching*. Imperial College Press, London.
3. Priola, E. and Wang, F.-Y. (2006). Gradient estimates for diffusion semigroups with singular coefficients. *J. Funct. Anal.*, 236(1):244–264.
4. Shao, J. (2015). Strong solutions and strong Feller properties for regime-switching diffusion processes in an infinite state space *SIAM J. Control Optim.*, 53(4): 2462–2479.
5. Wang, J. (2010). Regularity of semigroups generated by Lévy type operators via coupling. *Stochastic Process. Appl.*, 120(9):1680–1700.
6. Xi, F. (2009). Asymptotic properties of jump-diffusion processes with state-dependent switching. *Stochastic Process. Appl.*, 119(7):2198–2221.
7. Yin, G. G. and Zhu, C. (2010). *Hybrid Switching Diffusions: Properties and Applications*, volume 63 of *Stochastic Modelling and Applied Probability*. Springer, New York.

Existence and Uniqueness: Basic Assumptions

Assumption 1

Assume that $c(x,k,u)$ is $\mathcal{B}(\mathbb{R}^d \times \mathbb{S}) \times \mathcal{B}(U)$ measurable, and that for some constant $H > 0$,

$$|b(x,k)|^2 + |\sigma(x,k)|^2 + \int_U |c(x,k,u)|^2 \Pi(\mathrm{d}u) \le H(1+|x|^2),$$

$$|b(x,k) - b(y,k)|^2 + |\sigma(x,k) - \sigma(y,k)|^2 + \int_U |c(x,k,u) - c(y,k,u)|^2 \Pi(\mathrm{d}u) \le H|x-y|^2,$$

$$q_k(x) := \sum_{l \in \mathbb{S}} q_{kl}(x) \le Hk,$$

$$|q_{kl}(x) - q_{kl}(y)| \le H|x-y|,$$

$$\sum_{l \in \mathbb{S}} q_{kl}(x)(l-k)^2 \le H(x^2 + k^2 + 1),$$

for all $x, y \in \mathbb{R}^d$ and $k, l \in \mathbb{S}$.

Regime-Switching Jump Diffusion: Existence and Uniqueness

Proposition 2

Under Assumption 1, for each $(x, k) \in \mathbb{R}^d \times \mathbb{S}$*, the system* (1) *and* (2) *has a unique strong solution* $(X(t), \Lambda(t))$ *with initial condition* $(X(0), \Lambda(0)) = (x, k)$*. In addition, we have*

$$\mathbb{E}_{x,k}\left[\sup_{0 \le t \le T} (|X(t)|^2 + \Lambda(t)^2)\right] \le K,$$

where $K = K(x, T, H)$ *is a positive constant.*

The Infinitesimal Generator

The infinitesimal generator of (X, Λ) is given by

$$\mathcal{A}f(x,k) := \mathcal{L}_k f(x,k) + Q(x)f(x,k), \tag{3}$$

where

$$\begin{aligned}
\mathcal{L}_k f(x,k) := {} & \frac{1}{2}\mathrm{tr}\big(a(x,k)\nabla^2 f(x,k)\big) + \langle b(x,k), \nabla f(x,k)\rangle \\
& + \int_U \big(f(x+c(x,k,u),k) - f(x,k) \\
& \qquad - \langle \nabla f(x,k), c(x,k,u)\rangle \mathbf{1}_{\{u\in U_0\}}\big)\nu(\mathrm{d}u), \\
Q(x)f(x,k) := {} & \sum_{j\in\mathbb{S}} q_{kj}(x)[f(x,j) - f(x,k)].
\end{aligned}$$

Feller Property: Assumptions

Assumption 3

Assume that

$$\sum_{l \in \mathbb{S} \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \le H\,|x - y|\,,$$

$$\int_{U_0} |c(x,k,u) - c(z,k,u)|\,\nu(\mathrm{d}u) \le H|x - z|,$$

for all $x, z \in \mathbb{R}^d$ and $k \neq l \in \mathbb{S}$.

The Main Result: Feller Property

Theorem 4

Under Assumptions 1 and 3, the process (X, Λ) *has Feller property.*

The Wasserstein Metric

For two probability measures P_1 and P_2 on a complete separable metric space (E, d), define

$$W\left(P_1, P_2\right) = \inf_{\widetilde{P}} \int d(x, z) \widetilde{P}(\mathrm{d}x, \mathrm{d}z),$$

where $\widetilde{P}$ varies over all coupling probability measures with marginals P_1 and P_2, that is, for any $A \in \mathcal{B}(E)$, we have

$$\widetilde{P}(A \times E) = P_1(A), \text{ and } \widetilde{P}(E \times A) = P_2(A).$$

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- Denote by $\{P(t,x,k,A) : t \geq 0, (x,k) \in \mathbb{R}^d \times \mathbb{S}, A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{S})\}$ the transition probability family of the process (X, Λ).
- To obtain the Feller property, we need to show that $\forall t \geq 0$ and $k \in \mathbb{S}$, $P(t,x,k,\cdot)$ converges weakly to $P(t,z,k,\cdot)$ as $x \to z$.
- It suffices to prove that

$$W(P(t,x,k,\cdot), P(t,z,k,\cdot)) \to 0 \text{ as } x \to z,$$

The Coupling Operator

- For $x, z \in \mathbb{R}^d$ and $i, j \in \mathbb{S}$, we set

$$a(x,i,z,j) = \begin{pmatrix} a(x,i) & \sigma(x,i)\sigma(z,j)' \\ \sigma(z,j)\sigma(x,i)' & a(z,j) \end{pmatrix},$$
$$b(x,i,z,j) = \begin{pmatrix} b(x,i) \\ b(z,j) \end{pmatrix},$$

- Next, for $f(x,i,z,j) \in C_c^2(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}^d \times \mathbb{S})$, we define

$$\widetilde{\Omega}_{\mathrm{d}} f(x,i,z,j) = \frac{1}{2}\mathrm{tr}\big(a(x,i,z,j)D^2 f(x,i,z,j)\big) + \langle b(x,i,z,j), Df(x,i,z,j)\rangle,$$

- The is the coupling operator for the diffusion part of $\mathcal{A}$.

The Coupling Operator (cont'd)

$$
\begin{aligned}
&\widetilde{\Omega}_{\mathrm{j}} f(x,i,z,j) \\
&= \int_U \big[f(x + c(x,i,u), i, z + c(z,j,u), j) - f(x,i,z,j) \\
&\qquad - \langle D_x f(x,i,z,j), c(x,i,u) \rangle \mathbf{1}_{\{u \in U_0\}} \\
&\qquad - \langle D_z f(x,i,z,j), c(z,j,u) \rangle \mathbf{1}_{\{u \in U_0\}} \big] \nu(\mathrm{d}u),
\end{aligned}
$$

This is the coupling operator for the jump part of $\mathcal{A}$.

The Coupling Operator (cont'd)

$$
\begin{aligned}
&\widetilde{\Omega}_{\mathsf{s}} f(x,i,z,j)\\
&\quad = \sum_{l\in\mathbb{S}} [q_{il}(x) - q_{jl}(z)]^{+}(f(x,l,z,j) - f(x,i,z,j))\\
&\qquad + \sum_{l\in\mathbb{S}} [q_{jl}(z) - q_{il}(x)]^{+}(f(x,i,z,l) - f(x,i,z,j))\\
&\qquad + \sum_{l\in\mathbb{S}} [q_{il}(x) \wedge q_{jl}(z)](f(x,l,z,l) - f(x,i,z,j)).
\end{aligned}
$$

This is the coupling operator for the switching part of $\mathcal{A}$.

The Coupling Operator (cont'd)

$$
\begin{aligned}
&\widetilde{\Omega}_{\mathrm{s}} f(x,i,z,j) \\
&\quad = \sum_{l\in\mathbb{S}} [q_{il}(x) - q_{jl}(z)]^{+} (f(x,l,z,j) - f(x,i,z,j)) \\
&\qquad + \sum_{l\in\mathbb{S}} [q_{jl}(z) - q_{il}(x)]^{+} (f(x,i,z,l) - f(x,i,z,j)) \\
&\qquad + \sum_{l\in\mathbb{S}} [q_{il}(x) \wedge q_{jl}(z)] (f(x,l,z,l) - f(x,i,z,j)).
\end{aligned}
$$

This is the coupling operator for the switching part of $\mathcal{A}$.

Finally, the coupling operator to $\mathcal{A}$ can be written as

$$
\widetilde{\mathcal{A}} f(x,i,z,j) := \left[\widetilde{\Omega}_{\mathrm{d}} + \widetilde{\Omega}_{\mathrm{j}} + \widetilde{\Omega}_{\mathrm{s}}\right] f(x,i,z,j).
$$

A Useful Estimate

- Construct a sequence $\{\psi_n(r)\}_{n=1}^{\infty}$ of twice continuously differentiable functions satisfying
  1. $|\psi_n'(r)| \le 1$,
  2. $\lim_{n\to\infty} \psi_n(r) = |r|$ for $r \in \mathbb{R}$,
  3. $0 \le \psi_n''(r) \le 2n^{-1}H^{-1}r^{-2}$ for $r \ne 0$,
  4. for every $r \in \mathbb{R}$, the sequence $\{\psi_n(r)\}_{n=1}^{\infty}$ is nondecreasing.
- Consider the function

$$f_n(x,k,z,l) := \psi_n(|x-z|) + \mathbf{1}_{\{k\neq l\}}, \quad (x,k,z,l) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}^d \times \mathbb{S}.$$

- Then

$$\widetilde{\mathcal{A}} f_n(x,k,z,k) \le \frac{1}{n} + C|x-z|,$$

in which C is a positive constant.

The Coupling Argument

- Let $(\widetilde{X}(t), \widetilde{\Lambda}(t), \widetilde{Z}(t), \widetilde{\Xi}(t))$ denote the coupling process corresponding to the coupling operator $\widetilde{\mathcal{A}}$ with initial condition (x, k, z, k), in which $x \neq z$.
- Define $\zeta := \inf\{t \geq 0 : \widetilde{\Lambda}(t) \neq \widetilde{\Xi}(t)\}$. Note $\mathbb{P}\{\zeta > 0\} = 1$.
- Apply Itô's formula to the process $f_n(\widetilde{X}(\cdot), \widetilde{\Lambda}(\cdot), \widetilde{Z}(\cdot), \widetilde{\Xi}(\cdot))$ and pass to the limits as $n \to \infty$ to obtain:

$$\mathbb{E}\left[\left|\widetilde{X}(t \wedge \zeta) - \widetilde{Z}(t \wedge \zeta)\right|\right] \leq |x - z| \exp(Ct).$$

- Next use the function $g(x, k, z, l) := \mathbf{1}_{\{k \neq l\}}$ to obtain:

$$\begin{aligned}\mathbb{P}\{\zeta \leq t\} &= \mathbb{E}[g(\widetilde{X}(t \wedge \zeta), \widetilde{\Lambda}(t \wedge \zeta), \widetilde{Z}(t \wedge \zeta), \widetilde{\Xi}(t \wedge \zeta))] \\ &\leq K|x - z|e^{Ct}.\end{aligned}$$

The Coupling Argument (cont'd)

- Then

$$\mathbb{E}[|\widetilde{X}(t) - \widetilde{Z}(t) - \widetilde{X}(t \wedge \zeta) + \widetilde{Z}(t \wedge \zeta)|\mathbf{1}_{\{\zeta \le t\}}] \le K(1 + |x|^2 + |z|^2)^{\frac{1}{2}} |x - z|^{\frac{1}{2}},$$

and hence

$$\begin{aligned}\mathbb{E}[|\widetilde{X}(t) - \widetilde{Z}(t)|] &\le \mathbb{E}[|\widetilde{X}(t \wedge \zeta) - \widetilde{Z}(t \wedge \zeta)|] \\ &\quad + \mathbb{E}[|\widetilde{X}(t) - \widetilde{Z}(t) - \widetilde{X}(t \wedge \zeta) + \widetilde{Z}(t \wedge \zeta)|\mathbf{1}_{\{\zeta \le t\}}] \\ &\le K|x - z| + K(1 + |x|^2 + |z|^2)^{\frac{1}{2}} |x - z|^{\frac{1}{2}}.\end{aligned}$$

- Observe that $\mathbb{E}[\mathbf{1}_{\{\widetilde{\Lambda}(t) \neq \widetilde{\Xi}(t)\}}] \le \mathbb{P}\{\zeta \le t\} \le K|x - z|e^{Ct}$.
- Now let $f \in C_b(\mathbb{R}^d \times \mathbb{S})$, then we have

$$\begin{aligned}&\mathbb{E}\left[|f(\widetilde{X}(t), \widetilde{\Lambda}(t)) - f(\widetilde{Z}(t), \widetilde{\Xi}(t))|\right] \\ &\le \mathbb{E}\left[|f(\widetilde{X}(t), \widetilde{\Lambda}(t)) - f(\widetilde{Z}(t), \widetilde{\Lambda}(t))|\right] \\ &\quad + \mathbb{E}\left[|f(\widetilde{Z}(t), \widetilde{\Lambda}(t)) - f(\widetilde{Z}(t), \widetilde{\Xi}(t))|\right] \to 0 \text{ as } x \to z.\end{aligned}$$

Strong Feller Property: Assumptions

Assumption 5

Assume that

- there exists a positive integer κ such that $q_{kl}(x) = 0$ for all $k, l \in \mathbb{S}$ with $|k - l| \geq \kappa + 1$.
- the characteristic measure $\Pi(\cdot)$ is finite (i.e., $U_0 \equiv \emptyset$) and that for each $k \in \mathbb{S}$, the diffusion $X^{(k),0}$ satisfying

$$\mathrm{d}X^{(k),0}(t) = b(X^{(k),0}(t), k)\mathrm{d}t + \sigma(X^{(k),0}(t), k)\mathrm{d}B(t), \quad (4)$$

  has the strong Feller property and has a transition probability density with respect to the Lebesgue measure.

The Main Result: Strong Feller Property

Theorem 6

Assume Assumptions 1, 3, and 5. Then the process (X, Λ) has strong Feller property.

An Auxiliary Process

Consider the regime-switching jump diffusion (V, ψ):

$$\begin{aligned}\mathrm{d}V(t) &= b(V(t),\psi(t))\mathrm{d}t + \sigma(V(t),\psi(t))\mathrm{d}B(t) \\ &\quad + \int_U c(V(t-),\psi(t-),u)N(\mathrm{d}t,\mathrm{d}u),\end{aligned} \tag{5}$$

$$\mathbb{P}\{\psi(t+\Delta) = l | \psi(t) = k\} = \begin{cases} \widehat{q}_{kl}\Delta + o(\Delta), & \text{if } k \neq l, \\ 1 + \widehat{q}_{kk}\Delta + o(\Delta), & \text{if } k = l \end{cases} \tag{6}$$

provided $\Delta \downarrow 0$, where $\widehat{Q} = \left(\widehat{q}_{kl}\right)$ is a conservative Q-matrix:

$$\widehat{q}_{kl} = \begin{cases} -2\kappa & \text{if } k = l = 1, 2, \dots, \\ 1 & \text{if } k = 1, 2, \dots, \kappa, \text{ and } l = 1, 2, \dots, 2\kappa + 1 \text{ with } l \neq k, \\ 1 & \text{if } k = \kappa + 1, \kappa + 2, \dots, \text{ and } |l - k| \leq \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

For example, when $\kappa = 1$,

$$\widehat{Q} = (\widehat{q}_{kl}) = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Let $\mu(\cdot)$ be the product measure of the Gaussian probability measure on $\mathbb{R}^d$ and the Poisson probability measure on $\mathbb{S}$.

Lemma 7

Suppose that Assumptions 1 and 5 hold. Then (V, ψ) *has the strong Feller property and the transition probability* $\Gamma(t, (x, k), \cdot)$ *of* (V, ψ) *has density* $\gamma(t, (x, k), \cdot)$ *with respect to* $\mu(\cdot)$.

- Set $D := D([0,\infty), \mathbb{R}^d \times \mathbb{S})$ and denote by $\mathcal{D}$ the usual σ-field of D.
- For any $T > 0$, set $D_T := D([0,T], \mathbb{R}^d \times \mathbb{S})$ and denote by $\mathcal{D}_T$ the usual σ-field of D_T.
- Denote by $\mu_1(\cdot)$ the probability distribution induced by (X, Λ) and $\mu_2(\cdot)$ the probability distribution induced by (V, ψ) in the path space $\big(D, \mathcal{D}\big)$, respectively.
- Denote by $\mu_1^T(\cdot)$ the restriction of $\mu_1(\cdot)$ and $\mu_2^T(\cdot)$ the restriction of $\mu_2(\cdot)$ to $\big(D_T, \mathcal{D}_T\big)$, respectively.

- Thanks to [Xi, 2009, Lemma 4.2], $\mu_1^T(\cdot)$ is absolutely continuous with respect to $\mu_2^T(\cdot)$ with the Radon-Nikodym derivative:

$$
\begin{aligned}
& M_T\big(V(\cdot), \psi(\cdot)\big) \\
& \quad := \frac{\mathrm{d}\mu_1^T}{\mathrm{d}\mu_2^T}\big(V(\cdot), \psi(\cdot)\big) \\
& \quad = \prod_{i=0}^{n(T)-1} q_{\psi(\upsilon_i)\psi(\upsilon_{i+1})}\big(V(\upsilon_{i+1})\big) \\
& \qquad\qquad \times \exp\left(-\sum_{i=0}^{n(T)} \int_{\upsilon_i}^{\upsilon_{i+1}\wedge T} \left[q_{\psi(\upsilon_i)}(V(s)) - 2\kappa\right]\mathrm{d}s\right),
\end{aligned}
$$

where $q_k(x) = \sum_{l \neq k} q_{kl}(x)$.

For all $T > 0$, $x, y \in \mathbb{R}^d$ and $k \in \mathbb{S}$, we have

- $M_T\big(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot)\big)$ is integrable,
- $$\mathbb{E}\left[\left|M_T\big(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot)\big) - M_T\big(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot)\big)\right|\right] \to 0$$

  as $|x - y| \to 0$.

Proof of Theorem 6

For any $f \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{S})$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&\left|\mathbb{E}[f(X^{(x,k)}(t), \Lambda^{(x,k)}(t))] - \mathbb{E}[f(X^{(y,k)}(t), \Lambda^{(y,k)}(t))]\right| \\
&\quad \leq \mathbb{E}\Big[\Big| f(V^{(x,k)}(t), \psi^{(k)}(t)) \cdot M_t\big(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot)\big) \\
&\qquad\qquad - f(V^{(y,k)}(t), \psi^{(k)}(t)) \cdot M_t\big(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot)\big)\Big|\Big] \\
&\quad \leq \|f\| \cdot \mathbb{E}\left[\Big| M_t\big(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot)\big) - M_t\big(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot)\big)\Big|\right] \\
&\qquad + 2\|f\| \cdot \mathbb{E}\Big[M_t\big(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot)\big) \\
&\qquad\qquad\qquad \times I_{\{|f(V^{(x,k)}(t), \psi^{(k)}(t)) - f(V^{(y,k)}(t), \psi^{(k)}(t))| \geq \varepsilon\}}\Big] \\
&\qquad + \varepsilon \cdot \mathbb{E}\big[M_t\big(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot)\big)\big]
\end{aligned}$$

where $\|f\| := \sup\{|f(x,k)| : (x,k) \in \mathbb{R}^d \times \mathbb{S}\}$.

Lemma 8

Suppose that Assumptions 1 and 5 hold. For any given $t > 0$ *and bounded measurable function* f *on* $\mathbb{R}^d \times \mathbb{S}$*, we have that*

$$f(V^{(x,k)}(t), \psi^{(k)}(t)) \to f(V^{(y,k)}(t), \psi^{(k)}(t)) \quad \textit{in probability}$$

as $|x - y| \to 0$.

Exponential Ergodicity

For any positive function $f : \mathbb{R}^d \times \mathbb{S} \mapsto [1, \infty)$ and any signed measure ν defined on $\mathcal{B}(\mathbb{R}^d \times \mathbb{S})$, we write

$$\|\nu\|_f := \sup\{|\nu(g)| : g \in \mathcal{B}(\mathbb{R}^d \times \mathbb{S}) \text{ satisfying } |g| \leq f\},$$

where $\nu(g) := \sum_{l \in \mathbb{S}} \int_{\mathbb{R}^d} g(x, l)\nu(\mathrm{d}x, l)$.

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For a function $\infty > f \geq 1$ on $\mathbb{R}^d \times \mathbb{S}$, the process (X, Λ) is said to *f-exponentially ergodic* is there exists a probability measure $\pi(\cdot)$, a constant $\theta \in (0, 1)$ and a finite-valued function $\Theta(x, k)$ such that

$$\|P(t, (x, k), \cdot) - \pi(\cdot)\|_f \leq \Theta(x, k)\theta^t \tag{7}$$

for all $t \geq 0$ and all $(x, k) \in \mathbb{R}^d \times \mathbb{S}$.

Theorem 9

Suppose Assumptions 1, 3, and 5 hold. Assume that for any $k \neq l \in \mathbb{S}$, *there exist* $k_0, k_1, \cdots, k_r$ *in* $\mathbb{S}$ *with* $k_i \neq k_{i+1}$, $k_0 = k$ *and* $k_r = l$ *such that the set* $\{x \in \mathbb{R}^d : q_{k_i k_{i+1}}(x) > 0\}$ *has positive Lebesgue measure for* $i = 0, 1, \dots, r-1$. *In addition, assume that there exist positive numbers* α, β *and a nonnegative function* $V \in C^2(\mathbb{R}^d \times \mathbb{S})$ *satisfying*

(i) $V(x,k) \to \infty$ *as* $|x| \to \infty$ *for each* $k \in \mathbb{S}$,

(ii) $\mathcal{A}V(x,k) \leq -\alpha V(x,k) + \beta$, $(x,k) \in \mathbb{R}^d \times \mathbb{S}$.

Then the process (X, Λ) *is* f*-exponentially ergodic with* $f(x,k) = V(x,k) + 1$.

An Example

Consider a 1-D OU process

$$\mathrm{d}X(t) = \alpha(\Lambda(t))\mathrm{d}t + \sigma(\Lambda(t))\mathrm{d}B(t) + \int_{\mathbb{R}_0} \beta(\Lambda(t-))zN(\mathrm{d}t, \mathrm{d}z),$$

where $\alpha_k := \alpha(k), \beta_k := \beta(k), \sigma_k = \sigma(k) \in \mathbb{R}$ for each $k \in \mathbb{S}$, and $N(\mathrm{d}t, \mathrm{d}z)$ is Poisson random measure with characteristic measure $\nu(\mathrm{d}z) = \frac{1}{2}e^{-|z|}\mathrm{d}z$.
Suppose the switching component Λ is generated by the q-matrix

$$Q(x) = \begin{pmatrix} -q_{01}(x) & q_{01}(x) & 0 & 0 & 0 & \dots \\ q_{10}(x) & -(q_{10}(x) + q_{12}(x)) & q_{12}(x) & 0 & 0 & \dots \\ 0 & q_{21}(x) & -(q_{21}(x) + q_{23}(x)) & q_{23}(x) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $q_{k,k-1}(x)$ and $q_{k,k+1}(x)$ are positive and Lipschitz continuous functions.

Now suppose $\exists K_1, K_2 > 0$ so that the following conditions are satisfied:

(i) $2\alpha_0 + q_{01}(x) \leq -K_1 < 0$,

(ii) for each $k \in \mathbb{S}$, we have $\sigma_k > 0$, and $(k+1)\sigma_k^2 + 4(k+1)\beta_k^2 \leq K_2 < \infty$,

(iii) for all $k \in \mathbb{S} \setminus \{0\}$, we have $2(k+1)\alpha_k - q_{k,k-1}(x) + q_{k,k+1}(x) \leq -K_1(k+1) < 0$.

Then by Theorem 9, the process X is f-exponentially ergodic.

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Then by Theorem 9, the process X is f-exponentially ergodic. In particular, we can choose $\alpha_k, \beta_k, \sigma_k$ and $Q(x)$ so that

- $X^{(0)}$ is exponentially ergodic,
- $X^{(k)}$ is transient for each $k = 1, 2, \ldots,$
- X is f-exponentially ergodic.

Finally

Thank you!

Xi, F. (2009).
Asymptotic properties of jump-diffusion processes with state-dependent switching.
Stochastic Process. Appl., 119(7):2198–2221.