The extinction probability of a continuous state branching process with population dependent branching rate

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Outline of the Talk

1. Introduction
   - Continuous state branching process
   - Continuous state branching process with nonlinear branching
   - Survive or die out?
2. The extinction probability of nonlinear branching process
   - The results
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Continuous state branching process

- A continuous state branching process arises as a scaling limit of Galton-Watson processes.
- It is a nonnegative Markov process X satisfying the branching property, i.e. for any $\lambda, x, y > 0$

$$\mathbb{E}_{x+y} e^{-\theta X_t} = \mathbb{E}_x e^{-\theta X_t} \mathbb{E}_y e^{-\theta X_t}.$$

- Its Laplace transform is determined by

$$\mathbb{E}_x e^{-\theta X_t} = e^{-x u_t(\theta)}$$

where function $u_t(\theta)$ satisfies the differential equation

$$\frac{\partial u_t(\theta)}{\partial t} + \psi(u_t(\theta)) = 0$$

with $u_0(\theta) = \theta$ and

$$\psi(\lambda) = -q - a\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\pi(\mathrm{d}x)$$

for $q, \sigma \geq 0, a \in \mathbb{R}$ and for σ-finite measure π on $(0, \infty)$ satisfying $\int_0^\infty (z \wedge z^2)\pi(\mathrm{d}z) < \infty$.
X is critical if $q = a = 0$.

Continuous state branching process as solution of SDE

- By Dawson and Li (2006, 2012), a critical continuous state branching process solves the following SDE.
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$$X_t = \sigma \int_0^t \sqrt{X_{s-}} \mathrm{d}B_s + \int_0^t \int_0^\infty \int_0^{X_{s-}} x \tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u),$$

where $\tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u)$ is a compensated Poisson random measure on $[0, \infty) \times (0, \infty) \times [0, \infty)$ with compensator $\mathrm{d}s\pi(\mathrm{d}x)\mathrm{d}u$ with $\int_0^\infty x \wedge x^2 \pi(\mathrm{d}x) < \infty$ and B is an independent Brownian motion.

Branching process with population dependent branching rate

- We want to consider a more sophisticated branching process whose branching rate depends on the population size.
- For simplicity we only consider the critical branching process.
- For Feller branching processes, an example of such a process is a solution to the following sde

$$\mathrm{d}X_t = \sqrt{\gamma(X_t)X_t}\,\mathrm{d}B_t,$$

where γ is a nonnegative function on $[0, \infty)$.

- To introduce the population dependent branching rate, we can use a representation of Dawson and Li.
- An critical continuous state branching process with population dependent branching rate is the solution to the following sde.
- $$X_t = \sigma \int_0^t \sqrt{\gamma_1(X_{s-})} \mathrm{d}B_s + \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} x\tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u), \tag{1}$$

  where γ_1, γ_2 are nonnegative functions.
- Such a process is also called a nonlinear branching process.

If X is a weak solution to (1) with α-stable random measure $\tilde{N}$, then on an enlarged probability space, X is a weak solution to

$$X_t = \sigma \int_0^t \sqrt{\gamma_1(X_{s-})} \mathrm{d}B_s + \int_0^t \gamma_2(X_{s-})^{1/\alpha} \int_0^\infty x \tilde{N}(\mathrm{d}s, \mathrm{d}x). \quad (2)$$

By Li and Mytnik (2012), give that $\gamma_1(x) = x^{r_1}$ and $\gamma_2(x) = x^{r_2}$ for small $x \geq 0$ and $\gamma_i(x), i = 1, 2$ satisfy regularity conditions for other $x > 0$, equation (2) has a pathwise unique nonnegative strong solution for $r_1 \geq 1$ and $r_2 \geq \alpha - 1$.

- It is well known that a critical continuous state branching process dies out within a finite time; i.e. with probability one, $X_t = 0$ for t large enough.
- The branching causes the population to die out.
- For a nonlinear branching process whose branching rate goes to 0 as the population size goes to 0, a natural question is whether the process still dies out within a finite time.

A nonlinear branching process that survives forever

- We consider a nonlinear Feller branching process with branching rate $\gamma(x) = x^r$ for $r \geq 0$.
- The corresponding sde is

$$\mathrm{d}X_t = \sqrt{X_t^r X_t}\mathrm{d}B_t = X_t^{(r+1)/2}\mathrm{d}B_t. \tag{3}$$

- It is well known that for $r = 0$, $X_t = 0$ for t large enough.
- On the other hand, sde (3) is solvable if $r = 1$.

$$X_t = X_0 e^{B_t - \frac{1}{2}t}.$$

Clearly, $X_t \to 0$ as $t \to \infty$, but $X_t > 0$ for all t.
- What happens for $0 < r < 1$?

The main results

Recall that

$$X_t = \sigma \int_0^t \sqrt{\gamma_1(X_{s-})}\mathrm{d}B_s + \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} x\tilde{N}(\mathrm{d}s, \mathrm{d}x, \mathrm{d}u). \tag{4}$$

We assume that the above sde allows a unique weak solution. Let

$$\tau_a^- = \inf\{t : X_t = 0\}.$$

We first find a mild condition under which the process goes to 0.

Theorem

Given any solution X of SDE (4), if there exists $\delta > 0$ such that

$$\inf_{y \geq x} \gamma_1(y) > 0 \ \text{for all } x \in (0, \delta) \text{ when } \sigma > 0 \tag{5}$$

or

$$\inf_{y \geq x} \gamma_2(y) > 0 \ \text{for all } x \in (0, \delta) \text{ when } \pi \neq 0, \tag{6}$$

then $\tau_a^- < \infty$ $\mathbb{P}^x$-a.s. for all x, a with $x > a > 0$. Further, $\mathbb{P}^x$-a.s. $X_t \to 0$ as $t \to \infty$.

Given $X_t \to 0$, we need further conditions on $\gamma_1(x)$ and $\gamma_1(x)$ for small x to distinguish between survival and extinction.

The next theorem says if at least one of $\gamma_1(x)$ and $\gamma_2(x)$ do not decrease too fast near 0, then the process dies out with probability one.

Theorem

Let X be the unique weak solution to sde (4). Suppose there exist $\delta > 0$, $0 < \alpha < 2, r_1 < 2, r_2 < \alpha$ and $b_i > 0, i = 1, 2, 3$ such that either

$$\inf_{x \in (0,\delta)} \gamma_1(x) x^{-r_1} \geq b_1 \ \text{ for } \sigma > 0,$$

or

$$\inf_{x \in (0,\delta)} x^{\alpha-2} \int_0^x y^2 \pi(\mathrm{d}y) \geq b_3 \ \text{ and } \inf_{x \in (0,\delta)} \gamma_2(x) x^{-r_2} \geq b_2 \ \text{ for } \pi \neq 0,$$

then $\mathbb{P}_x\{\tau_0^- < \infty\} = 1$ for all $x > 0$.

The next result says that if both $\gamma_1(x)$ and $\gamma_1(x)$ decreases fast enough, then the process survives with probability one.

Theorem

Suppose that there exist constants $\delta > 0, 0 < \alpha < 2, r_2 > \alpha$, *and* $b_i > 0, i = 1, 2, 3$ *such that*

$$\sup_{x\in(0,\delta)} \gamma_1(x)x^{-2} \le b_1 \ \ \textit{if } \sigma > 0$$

and

$$\sup_{x\in(0,\delta)} x^{\alpha-2}\int_0^x y^2\pi(\mathrm{d}y) \le b_3, \quad \sup_{x\in(0,\delta)} \gamma_2(x)x^{-r_2} \le b_2 \ \ \textit{if } \pi \neq 0,$$

then $\mathbb{P}_x\{\tau_0^- = \infty\} = 1$ *for all* $x > 0$.

To handle the critical case of $r_2 = \alpha$ for $\alpha \in (1,2)$, we adopt a different approach under additional assumption on π.

Theorem

Suppose there are constants $C, \delta > 0$ and $\alpha \in (1,2)$ so that $\pi(\mathrm{d}z)$ is absolutely continuous with respect to Lebesgue measure when restricted to interval $(0,\delta)$ and

$$\pi(\mathrm{d}z) \le Cz^{-1-\alpha}\mathrm{d}z, \ \ z \in (0,\delta).$$

Further,

$$\sup_{x\in(0,\delta)} \gamma_1(x)x^{-2} < \infty, \quad \sup_{x\in(0,\delta)} \gamma_2(x)x^{-\alpha} < \infty.$$

Then $\mathbb{P}\{\tau_0^- = \infty\} = 1$.

Remark

For $\pi = 0$ *and* $\gamma_1(x) = x^{r_1}$, *then combining the above Theorems, for any* $x > 0$ *we have* $\mathbb{P}_x\{\tau_0^- < \infty\} = 1$ *for* $0 \leq r_1 < 2$ *and* $\mathbb{P}_x\{\tau_0^- < \infty\} = 0$ *but* $X_t \to 0$ $\mathbb{P}_x$*-a.s. for* $r_1 \geq 2$.

Remark

If $\sigma = 0$ *and* $\gamma_2(x) = x^{r_2}, \pi(\mathrm{d}x) = x^{1+\alpha}\mathrm{d}x$ *with* $1 < \alpha < 2$ *in sde (4), combining the Theorems we have for any* $x > 0$, $\mathbb{P}_x\{\tau_0^- < \infty\} = 1$ *for* $0 \leq r_2 < \alpha$ *and* $\mathbb{P}_x\{\tau_0^- < \infty\} = 0$ *but* $X_t \to 0$ $\mathbb{P}_x$*-a.s. for* $r_2 \geq \alpha$.

Let

$$\tau_a^- := \inf\{t : X_t = a\}$$

and

$$\tau_b^+ := \inf\{t : X_t > b\}$$

with the convention $\inf \emptyset = \infty$.

Proof of survival:

Method 1

For $0 < x < \delta$, we can apply Ito's formula to have

$$
\begin{aligned}
& e^{-\lambda X_{t\wedge\tau_\delta^+\wedge\tau_0^-}} \\
&= e^{-\lambda x} + \frac{\sigma^2}{2}\int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} \lambda^2 e^{-\lambda X_s}\gamma_1(X_s)\mathrm{d}s \\
&\quad + \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} e^{-\lambda X_{s-}}\gamma_2(X_{s-})\mathrm{d}s \int_0^{\delta-X_{s-}} \left[e^{-\lambda z} - 1 + \lambda z\right]\pi(\mathrm{d}z) + \text{mart.}
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_0^\delta \left[e^{-\lambda z} - 1 + \lambda z\right]\pi(\mathrm{d}z) \le C\int_0^\delta \left[(\lambda z)\wedge(\lambda z)^2\right] z^{-1-\alpha}\mathrm{d}z \\
&= C\lambda^\alpha \int_0^{\lambda\delta}\left[z\wedge z^2\right] z^{-1-\alpha}\mathrm{d}z = C(\alpha)\lambda^\alpha,
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathbb{E}_x e^{-\lambda X_{t\wedge\tau_\delta^+\wedge\tau_0^-}} \\
&= e^{-\lambda x} + \mathbb{E}_x \frac{\sigma^2}{2}\int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} \lambda^2 e^{-\lambda X_s}\gamma_1(X_s)\mathrm{d}s \\
&\quad + \mathbb{E}_x \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} e^{-\lambda X_{s-}}\gamma_2(X_{s-})\mathrm{d}s \int_0^{\delta - X_{s-}} \left[e^{-\lambda z} - 1 + \lambda z\right]\pi(\mathrm{d}z) \\
&\le e^{-\lambda x} + \frac{\sigma^2}{2} C\mathbb{E}_x \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} (\lambda X_s)^2 e^{-\lambda X_s}\mathrm{d}s \\
&\quad + C\mathbb{E}_x \int_0^{t\wedge\tau_\delta^+\wedge\tau_0^-} (\lambda X_s)^\alpha e^{-\lambda X_s}\mathrm{d}s.
\end{aligned}
$$

Letting $\lambda \to \infty$, we have

$$\mathbb{P}_x\{X_{t\wedge\tau_\delta^+\wedge\tau_0^-} = 0\} = 0.$$

Letting $t \to \infty$,

$$\mathbb{P}_x\{\tau_0^- < \tau_\delta^+\} = 0.$$

We can show that

$$\lim_{x\to 0+} \mathbb{P}_x\{\tau_\delta^+ = \infty\} = 1.$$

Then

$$\lim_{x\to 0+} \mathbb{P}_x\{\tau_0^- < \infty\} = 0.$$

Given any $x_0 > 0$, $\mathbb{P}_{x_0}\{\tau_0^- < \infty\} = 0$.

Method 2:
Alteratively, we can prove the survival using a martingale argument with martingale

$$\frac{1}{X_t}\exp\left\{-\int_0^t \frac{\gamma(X_{s-})}{X_{s-}^2}\mathrm{d}s \right.$$
$$\left. -\int_0^t \gamma_2(X_{s-})X_{s-}\mathrm{d}s\int_0^\infty \left(\frac{1}{X_{s-}+x}-\frac{1}{X_{s-}}+\frac{x}{X_{s-}^2}\right)\pi(\mathrm{d}x)\right\}$$

together with Borel-Cantelli arguments.

Proof of extinction:

We consider martingale

$$\exp\left\{-\lambda X_t - \frac{\lambda^2\sigma^2}{2}\int_0^t \gamma(X_{s-})\mathrm{d}s \right.$$
$$\left. - \int_0^t \gamma_2(X_{s-})\mathrm{d}s \int_0^\infty \left[e^{-\lambda(X_{s-}+y)} - e^{-\lambda X_{s-}} + \lambda y e^{-\lambda X_{s-}}\right]\pi(\mathrm{d}y)\right\}$$

and use Borel-Cantelli arguments.

Thank you for your attention!