Stochastic Hamiltonian flows with rough coefficients

Xicheng Zhang

(Wuhan University)

The 12th Workshop on Markov Processes and Related Topics
Jiangsu Normal University
2016.07.13-17

Backgound

- Consider the following second order stochastic differential equation

$$\mathrm{d}\dot{X}_t = b_t(X_t, \dot{X}_t)\mathrm{d}t + \sigma_t(X_t, \dot{X}_t)\mathrm{d}W_t, \quad (X_0, \dot{X}_0) = (x, \mathrm{v}) \in \mathbb{R}^{2d}, \quad (1.1)$$

where $b_t(x, \mathrm{v}) : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d$ and $\sigma_t(x, \mathrm{v}) : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are Borel measurable functions, $\dot{X}_t$ denotes the first order derivative of X_t with respect to t, and W_t is a d-dimensional standard Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Backgound

- Consider the following second order stochastic differential equation

$$\mathrm{d}\dot{X}_t = b_t(X_t, \dot{X}_t)\mathrm{d}t + \sigma_t(X_t, \dot{X}_t)\mathrm{d}W_t, \quad (X_0, \dot{X}_0) = (x, \mathrm{v}) \in \mathbb{R}^{2d}, \quad (1.1)$$

  where $b_t(x, \mathrm{v}) : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d$ and $\sigma_t(x, \mathrm{v}) : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are Borel measurable functions, $\dot{X}_t$ denotes the first order derivative of X_t with respect to t, and W_t is a d-dimensional standard Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- When $\sigma = 0$, it is the classical Newtonian mechanic equation.

Backgound

- Consider the following second order stochastic differential equation

$$\mathrm{d}\dot{X}_t = b_t(X_t, \dot{X}_t)\mathrm{d}t + \sigma_t(X_t, \dot{X}_t)\mathrm{d}W_t, \quad (X_0, \dot{X}_0) = (x, \mathrm{v}) \in \mathbb{R}^{2d}, \quad (1.1)$$

  where $b_t(x, \mathrm{v}) : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d$ and $\sigma_t(x, \mathrm{v}) : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ are Borel measurable functions, $\dot{X}_t$ denotes the first order derivative of X_t with respect to t, and W_t is a d-dimensional standard Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- When $\sigma = 0$, it is the classical Newtonian mechanic equation.

- When $\sigma \neq 0$, it describes the motion of a particle perturbed by some random external force.

Backgound

- If we let $Z_t := (X_t, \dot{X}_t)$, then Z_t solves the following degenerate SDE:

$$\mathrm{d}Z_t = (\dot{X}_t, b_t(Z_t))\mathrm{d}t + (0, \sigma_t(Z_t)\mathrm{d}W_t), \quad Z_0 = z = (x, \mathrm{v}) \in \mathbb{R}^{2d},$$

Backgound

- If we let $Z_t := (X_t, \dot{X}_t)$, then Z_t solves the following degenerate SDE:

$$\mathrm{d}Z_t = (\dot{X}_t, b_t(Z_t))\mathrm{d}t + (0, \sigma_t(Z_t)\mathrm{d}W_t), \quad Z_0 = z = (x, \mathrm{v}) \in \mathbb{R}^{2d},$$

whose time-dependent infinitesimal generator is given by

$$\mathscr{L}_t^{a,b} f(x, \mathrm{v}) := \mathrm{tr}(a_t \cdot \nabla_{\mathrm{v}}^2 f)(x, \mathrm{v}) + \mathrm{v} \cdot \nabla_x f(x, \mathrm{v}) + b_t \cdot \nabla_{\mathrm{v}} f(x, \mathrm{v}),$$

where $\nabla_{\mathrm{v}}^2 f(x, \mathrm{v})$ stands for the Hessian matrix, and

$$a_t(x, \mathrm{v}) := \tfrac{1}{2}(\sigma_t \sigma_t^*)(x, \mathrm{v}).$$

Backgound

- If we let $Z_t := (X_t, \dot{X}_t)$, then Z_t solves the following degenerate SDE:

$$\mathrm{d}Z_t = (\dot{X}_t, b_t(Z_t))\mathrm{d}t + (0, \sigma_t(Z_t)\mathrm{d}W_t), \quad Z_0 = z = (x, \mathrm{v}) \in \mathbb{R}^{2d},$$

whose time-dependent infinitesimal generator is given by

$$\mathscr{L}_t^{a,b} f(x, \mathrm{v}) := \mathrm{tr}(a_t \cdot \nabla_{\mathrm{v}}^2 f)(x, \mathrm{v}) + \mathrm{v} \cdot \nabla_x f(x, \mathrm{v}) + b_t \cdot \nabla_{\mathrm{v}} f(x, \mathrm{v}),$$

where $\nabla_{\mathrm{v}}^2 f(x, \mathrm{v})$ stands for the Hessian matrix, and

$$a_t(x, \mathrm{v}) := \tfrac{1}{2}(\sigma_t \sigma_t^*)(x, \mathrm{v}).$$

- In the literature $\mathscr{L}_t^{a,b}$ is also called kinetic Fokker-Planck operator.

Backgound

- It is a classical fact that if b and σ are Lipschitz continuous, then SDE (1.1) has a unique strong solution.

Backgound

- It is a classical fact that if b and σ are Lipschitz continuous, then SDE (1.1) has a unique strong solution.
- Recently, Chaudru de Raynal [1] showed the strong well-posedness of SDE (1.1) under the following Hölder assumptions on b:

$$|b_t(x, \mathrm{v}) - b_t(x', \mathrm{v}')| \leqslant C(|x - x'|^\alpha + |\mathrm{v} - \mathrm{v}'|^\beta),$$

where $\alpha \in (\frac{2}{3}, 1)$ and $\beta \in (0, 1)$, and the following Lipschitz assumption on σ

$$|\sigma_t(x, \mathrm{v}) - \sigma_t(x', \mathrm{v}')| \leqslant C(|x - x'| + |\mathrm{v} - \mathrm{v}'|).$$

[1] Chaudru de Raynal: Strong existence and uniqueness for stochastic differential equation with Hölder drift and degenerate noise. AIHP, to appear.

Backgound

- F.Y. Wang and Z. [1] extended Chaudru de Raynal's result to the following assumptions on b:

$$|b_t(x,\mathrm{v}) - b_t(x',\mathrm{v}')| \leqslant C(|x-x'|^{2/3}\phi_1(|x-x'|) + \phi_2(|\mathrm{v}-\mathrm{v}'|)),$$

where $\phi_i, i = 1,2$ are Dini's functions like $\phi_i(s) = 1 \vee \log s^{-1}$. Meanwhile, we also showed the C^1-diffeomorphism flow property.

[1] F.Y. Wang and X. Zhang: Degenerate SDE with Hölder-Dini drift and non-Lipschitz noise coefficient. SIAM J. Math. Anal., to appear.

Backgound

- In the non-degenerate case, Krylov and Röcker [1] showed the strong uniqueness when σ is constant and

$$b \in L^q_{loc}(\mathbb{R}_+; L^p(\mathbb{R}^d)), \quad \frac{d}{p} + \frac{2}{q} < 1.$$

Their result was extended by Z. in [2] [3] to Sobolev diffusion σ. Related results can be found in Fedrizzi and Flandoli [4] $\cdots$

[1] Krylov N.V. and Röckner M.: Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Relat. Fields*, **131**, 154-196 (2005).

[2] Zhang X.: Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Elect. Jour. of Prob.*, Vol. **16**, no. 38,1096-1116 (2011).

[3] Zhang X.: Stochastic differential equations with Sobolev diffusion and singular drift. To appear in *Annals of Applied Probability*.

[4] Fedrizzi E., Flandoli F.: Noise prevents singularities in linear transport equations. *J. of Funct. Anal.*, **264**, 1329-1354 (2013).

Main results

Our main result is:

Main Theorem (Stochastic Hamiltonian flow with singular drift)

(H) Suppose that σ is uniformly elliptic, i.e., for some $K \geqslant 1$ and all $(t, x, \mathrm{v}) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$,

$$K^{-1}|\xi| \leqslant |\sigma_t^*(x, \mathrm{v})\xi| \leqslant K|\xi|, \quad \forall \xi \in \mathbb{R}^d, \qquad \textbf{(UE)}$$

where σ^* denotes the transpose of matrix σ, and for some $p > 4d + 2$,

$$\sup_{t \geqslant 0} \|\nabla \sigma_t\|_p^p + \int_0^\infty \|(\mathbb{I} - \Delta_x)^{\frac{1}{3}} b_t\|_p^p \mathrm{d}t < \infty.$$

Main results

Main Theorem (Continuing)

(i) For any $z = (x, \mathrm{v}) \in \mathbb{R}^{2d}$, SDE (1.1) admits a unique solution $Z_t(z) = (X_t, \dot{X}_t)$ so that $(t, z) \mapsto Z_t(z)$ has a bi-continuous version.

Main results

Main Theorem (Continuing)

(i) For any $z = (x, \mathrm{v}) \in \mathbb{R}^{2d}$, SDE (1.1) admits a unique solution $Z_t(z) = (X_t, \dot{X}_t)$ so that $(t, z) \mapsto Z_t(z)$ has a bi-continuous version.

(ii) $\{Z_t(z), z \in \mathbb{R}^{2d}, t \geqslant 0\}$ forms a stochastic homeomorphism flow, and $z \mapsto Z_t(z)$ is weakly differentiable a.s., and

$$\operatorname{ess.}\sup_{z} \mathbb{E}|\nabla Z_t(z)|^q < \infty.$$

Main Theorem (Continuing)

(i) For any $z = (x, \mathrm{v}) \in \mathbb{R}^{2d}$, SDE (1.1) admits a unique solution $Z_t(z) = (X_t, \dot{X}_t)$ so that $(t, z) \mapsto Z_t(z)$ has a bi-continuous version.

(ii) $\{Z_t(z), z \in \mathbb{R}^{2d}, t \geqslant 0\}$ forms a stochastic homeomorphism flow, and $z \mapsto Z_t(z)$ is weakly differentiable a.s., and

$$\mathrm{ess.}\sup_z \mathbb{E}|\nabla Z_t(z)|^q < \infty.$$

Remark: After this work was finished, I was informed by Priola that very recently, Fedrizzi, Flandoli, Priola and Vovelle also obtained the strong well-posedness together with their flow property of SDE (1.1) under the conditions $\sigma_t(z) = \mathbb{I}$ and $b_t(z) = b(z)$ possessing the following regularity

$$\|(\mathbb{I} - \Delta_x)^{s/2} b\|_p < \infty$$

for some $s > 2/3$ and $p > 6d$.

Main results

Corollary (Local existence and uniqueness)

Suppose that for any $R > 0$, there exists a constant $K_R \geqslant 1$ such that for all $(t, x, \mathrm{v}) \in [0, T] \times B_R \times B_R$ and $\xi \in \mathbb{R}^d$,

$$K_R^{-1}|\xi| \leqslant |\sigma_t^*(x, \mathrm{v})\xi| \leqslant K_R|\xi|,$$

and for some $p > 4d + 2$,

$$\sup_{t\in[0,T]} \|\nabla(\sigma_t\chi_R)\|_p^p + \int_0^T \|(\mathbb{I} - \Delta_x)^{\frac{1}{3}}(b_s\chi_R)\|_p^p \mathrm{d}s \leqslant K_R,$$

where $\chi_R : \mathbb{R}^{2d} \to [0, 1]$ is a smooth function with $\chi_R = 1$ for $|x| \leqslant R$ and $\chi_R = 0$ for $|x| > 2R$. Then for any fixed $(x, \mathrm{v}) \in \mathbb{R}^{2d}$, SDE (1.1) admits a unique local solution $(X_t, \dot{X}_t)$ up to the explosion time ζ.

Main results

Let μ_t be the distribution of X_t. By Itô's formula, μ_t solves the following Fokker-Planck equation in the distributional sense:

$$\partial_t \mu_t = (\mathscr{L}_t^{a,b})^* \mu_t, \quad \mu_0 = \delta_z, \tag{2.1}$$

where δ_z is the Dirac measure at z. More precisely, for any $f \in C_c^2(\mathbb{R}^{2d})$,

$$\partial_t \mu_t(f) = \mu_t(\mathscr{L}_t^{a,b} f), \ \mu_0(f) = f(z),$$

where $\mu_t(f) = \int f \mathrm{d}\mu_t = \mathbb{E} f(Z_t)$.

Main results

Theorem (Uniqueness of Fokker-Planck equations)

Suppose that σ satisfies **(UE)** and for any $T > 0$,

$$\lim_{|z-z'|\to 0} \sup_{t\in[0,T]} \|\sigma_t(z) - \sigma_t(z')\| = 0,$$

and $b \in L^q_{loc}(\mathbb{R}_+; L^q(\mathbb{R}^{2d}))$ for some $q \in (2(2d+1), \infty]$. Then for any $\nu \in \mathcal{P}(\mathbb{R}^{2d})$, there exists a unique probability measure-valued solution $\mu_t \in \mathcal{P}(\mathbb{R}^{2d})$ to (2.1) in the distributional sense in the class that $t \mapsto \mu_t$ is weakly continuous with $\mu_0 = \nu$ and

$$\int_0^t \int_{\mathbb{R}^{2d}} (|\mathrm{v}| + |b_s(x, \mathrm{v})|)\mu_s(\mathrm{d}x, \mathrm{d}\mathrm{v})\mathrm{d}s < \infty, \quad t > 0.$$

Basic idea

- Consider the following one dimensional stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t,$$

where W_t is a one-dimensional Brownian motion, and $b(x)$ is bounded measurable and L^1-integrable, i.e. $b \in L^1 \cap L^\infty$.

Basic idea

- Consider the following one dimensional stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t,$$

where W_t is a one-dimensional Brownian motion, and $b(x)$ is bounded measurable and L^1-integrable, i.e. $b \in L^1 \cap L^\infty$.

- Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. By Itô's formula, we have

$$\mathrm{d}f(X_t) = [f'(X_t)b(X_t) + \frac{1}{2}f''(X_t)]\mathrm{d}t + f'(X_t)\mathrm{d}W_t.$$

- Let us solve the following ODE:

$$f'(x)b(x) + \frac{1}{2}f''(x) = 0.$$

- Let us solve the following ODE:

$$f'(x)b(x) + \frac{1}{2}f''(x) = 0.$$

It is easily seen that

$$f'(x) = e^{-2\int_0^x b(y)\mathrm{d}y}, \qquad f(x) = \int_0^x e^{-2\int_0^{x'} b(y)\mathrm{d}y}\mathrm{d}x'.$$

- Let us solve the following ODE:

$$f'(x)b(x) + \frac{1}{2}f''(x) = 0.$$

It is easily seen that

$$f'(x) = e^{-2\int_0^x b(y)\mathrm{d}y}, \qquad f(x) = \int_0^x e^{-2\int_0^{x'} b(y)\mathrm{d}y}\mathrm{d}x'.$$

- Since $b \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, one can see that $x \mapsto f(x)$ is a C^1-diffeomorphism.

- Let f^{-1} be the inverse function of $x \mapsto f(x)$. Then $Y_t = f(X_t)$ solves the following SDE:

$$\mathrm{d}Y_t = f' \circ f^{-1}(Y_t)\mathrm{d}W_t, \quad Y_0 = f(X_0).$$

Since $y \mapsto f' \circ f^{-1}(y)$ is global Lipschitz, there is a unique solution to the above SDE, so does the original equation.

- Let f^{-1} be the inverse function of $x \mapsto f(x)$. Then $Y_t = f(X_t)$ solves the following SDE:

$$\mathrm{d}Y_t = f' \circ f^{-1}(Y_t)\mathrm{d}W_t, \quad Y_0 = f(X_0).$$

  Since $y \mapsto f' \circ f^{-1}(y)$ is global Lipschitz, there is a unique solution to the above SDE, so does the original equation.

- In particular, the Brownian motion has regularizing effect.

- Let f^{-1} be the inverse function of $x \mapsto f(x)$. Then $Y_t = f(X_t)$ solves the following SDE:

$$\mathrm{d}Y_t = f' \circ f^{-1}(Y_t)\mathrm{d}W_t, \quad Y_0 = f(X_0).$$

  Since $y \mapsto f' \circ f^{-1}(y)$ is global Lipschitz, there is a unique solution to the above SDE, so does the original equation.

- In particular, the Brownian motion has regularizing effect.

- This is called Zvonkin's transformation of phase space.

Maximal L^p-solutions of kinetic Fokker-Planck equations

We first introduce some necessary spaces.

- For $\alpha, \beta \geqslant 0$, we define the Bessel potential space $H_p^{\alpha,\beta} := H_p^{\alpha,\beta}(\mathbb{R}^{2d})$ as the completion of $C_c^\infty(\mathbb{R}^{2d})$ with respect to norm:

$$\|f\|_{\alpha,\beta;p} := \|(\mathbb{I} - \Delta_x)^{\frac{\alpha}{2}} f\|_p + \|(\mathbb{I} - \Delta_{\rm v})^{\frac{\beta}{2}} f\|_p.$$

Maximal L^p-solutions of kinetic Fokker-Planck equations

We first introduce some necessary spaces.

- For $\alpha, \beta \geqslant 0$, we define the Bessel potential space $H_p^{\alpha,\beta} := H_p^{\alpha,\beta}(\mathbb{R}^{2d})$ as the completion of $C_c^\infty(\mathbb{R}^{2d})$ with respect to norm:

$$\|f\|_{\alpha,\beta;p} := \|(\mathbb{I} - \Delta_x)^{\frac{\alpha}{2}} f\|_p + \|(\mathbb{I} - \Delta_{\rm v})^{\frac{\beta}{2}} f\|_p.$$

- For $t \in [0, T]$, we introduce the following Banach spaces with natural norms:

$$\mathbb{L}^p(t, T) := L^p([t, T]; L^p(\mathbb{R}^{2d})), \quad \mathbb{H}_p^{\alpha,\beta}(t, T) := L^p([t, T]; H_p^{\alpha,\beta}(\mathbb{R}^{2d})).$$

  We simply write

$$\mathbb{L}^p(T) := \mathbb{L}^p(0, T), \quad \mathbb{H}_p^{\alpha,\beta}(T) := \mathbb{H}_p^{\alpha,\beta}(0, T).$$

Consider the following backward kinetic Fokker-Planck equation

$$\partial_t u + \mathscr{L}_t^{a,b} u + f = 0, u_T = 0, \tag{4.1}$$

where $f_t(x, \mathrm{v}) : [0, T] \times \mathbb{R}^{2d} \to \mathbb{R}$ is a Borel function.

Consider the following backward kinetic Fokker-Planck equation

$$\partial_t u + \mathscr{L}_t^{a,b} u + f = 0, u_T = 0, \tag{4.1}$$

where $f_t(x, \mathrm{v}) : [0, T] \times \mathbb{R}^{2d} \to \mathbb{R}$ is a Borel function.

Definition of solutions

Let $p \in (1, \infty)$ and $f \in \mathbb{L}^p(T)$. A Borel function $u \in \mathbb{H}_p^{0,2}(T)$ is called a solution of equation (4.1) if for any $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and all $t \in [0, T]$,

$$\langle u_t, \varphi \rangle = \int_t^T \langle \mathrm{tr}(a_s \cdot \nabla_{\mathrm{v}}^2 u_s), \varphi \rangle \mathrm{d}s - \int_t^T \langle \mathrm{v} \cdot \nabla_x \varphi, u_s \rangle \mathrm{d}s + \int_t^T \langle b_s \cdot \nabla_{\mathrm{v}} u_s, \varphi \rangle \mathrm{d}s + \int_t^T \langle f_s, \varphi \rangle \mathrm{d}s,$$

where $\langle u_t, \varphi \rangle := \int_{\mathbb{R}^{2d}} u_t(z) \varphi(z) \mathrm{d}z$.

Theorem (Existence and Uniqueness of Solutions)

Let $\alpha \in [0, \frac{2}{3})$, $\beta \in (1,2)$ and $p > \frac{2}{(2-3\alpha)\wedge(2-\beta)}$ not equal to $\frac{d(\alpha+\beta)}{\alpha(\beta-1)}$. Suppose that σ satisfies **($\mathbf{H}_K^\delta$)**, and for some $q \in [p \vee \frac{d(\alpha+\beta)}{\alpha(\beta-1)}, \infty]$,

$$b \in L^p([0,T]; L^q(\mathbb{R}^{2d})).$$

For any $f \in \mathbb{L}^p(T)$, there exists a unique solution u to (4.1) with

$$\|u\|_{\mathbb{H}_p^{2/3,2}(T)} \leqslant C\|f\|_{\mathbb{L}^p(T)}, \tag{4.2}$$

and for all $t \in [0,T]$,

$$\|u_t\|_{\alpha,\beta;p} \leqslant C(T-t)^{1-\frac{1}{p}-\frac{(3\alpha)\vee\beta}{2}}\|f\|_{\mathbb{L}^p(t,T)}, \tag{4.3}$$

where the constant C only depends on $d, \delta, K, \alpha, \beta, p, q, T$ and $\|b\|_{L^p([0,T];L^q(\mathbb{R}^{2d}))}$.

Theorem (Continuing)

If in addition, we also assume that $p > \frac{d(3\beta-1)}{2(\beta-1)}$ and

$$\kappa_1 := \sup_{t,\mathrm{v}} \|\Delta_x^{\frac{1}{3}} \sigma_t(\cdot, \mathrm{v})\|_p + \|b\|_{\mathbb{H}_p^{2/3,0}(T)} < \infty,$$

then for any $f \in \mathbb{H}_p^{2/3,0}(T)$, the unique solution u also satisfies

$$\|\nabla_x \nabla_{\mathrm{v}} u\|_{\mathbb{L}^p(T)} + \|\Delta_x^{\frac{1}{3}} \nabla_{\mathrm{v}}^2 u\|_{\mathbb{L}^p(T)} \leqslant C\|f\|_{\mathbb{H}_p^{2/3,0}(T)}, \tag{4.4}$$

and for all $t \in [0, T]$,

$$\|\Delta_x^{\frac{1}{3}} u_t\|_{\alpha,\beta;p} \leqslant C(T-t)^{1-\frac{1}{p}-\frac{(3\alpha)\vee\beta}{2}} \|f\|_{\mathbb{H}_p^{2/3,0}(t,T)}, \tag{4.5}$$

where the constant C only depends on $d, \delta, K, \alpha, \beta, p, T$ and κ_1.

Basic tools

Let $\sigma_t(x,\mathrm{v}) = \sigma_t$ be independent of (x,v). Define for $t < s$,

$$P_{t,s}f(x,\mathrm{v}) = \mathbb{E}f(x + (s-t)\mathrm{v} + X_{t,s}, \mathrm{v} + V_{t,s}),$$

where

$$(X_{t,s}, V_{t,s}) = \left(\int_t^s \int_t^r \sigma_{r'} \mathrm{d}W_{r'} \mathrm{d}r, \int_t^s \sigma_r \mathrm{d}W_r\right).$$

Fractional order derivative estimate. (F.Y. Wang and Z. (2015))

Let $T > 0$. For any $\alpha, \beta \geqslant 0$, $p > 1$, there exists $C = C(T, p, d, \alpha, \beta) > 0$ such that for all $f \in L^p(\mathbb{R}^{2d})$ and $0 \leqslant s < t \leqslant T$,

$$\|P_{t,s}f\|_{\alpha,0;p} \leqslant C(s-t)^{-\frac{3\alpha}{2}} \|f\|_p,$$
$$\|P_{t,s}f\|_{0,\beta;p} \leqslant C(s-t)^{-\frac{\beta}{2}} \|f\|_p.$$

Theorem (L^p-hypoelliptic regularity estimates, Bramanti, Cupini, Lanconelli and Priola (Math. Nachr. [2012]), Bouchut (JMPA, 2002))

Let $T > 0$. For any $p > 1$, there exists a constant $C = C(d, p) > 0$ such that for all $f \in \mathbb{L}^p(T)$,

$$\|\nabla^2_{\mathrm{v}} u\|_{\mathbb{L}^p(T)} + \|\Delta_x^{1/3} u\|_{\mathbb{L}^p(T)} \leqslant C\|f\|_{\mathbb{L}^p(T)},$$

where $u_t(x, \mathrm{v}) = \int_t^T P_{t,s} f_s(x, \mathrm{v}) \mathrm{d}s$ satisfies $\partial_t u + \mathscr{L}^a u + f = 0$.

Basic tools

Theorem (L^p-hypoelliptic regularity estimates, Bramanti, Cupini, Lanconelli and Priola (Math. Nachr. [2012]), Bouchut (JMPA, 2002))

Let $T > 0$. For any $p > 1$, there exists a constant $C = C(d, p) > 0$ such that for all $f \in \mathbb{L}^p(T)$,

$$\|\nabla^2_{\mathrm{v}} u\|_{\mathbb{L}^p(T)} + \|\Delta^{1/3}_x u\|_{\mathbb{L}^p(T)} \leqslant C\|f\|_{\mathbb{L}^p(T)},$$

where $u_t(x, \mathrm{v}) = \int_t^T P_{t,s} f_s(x, \mathrm{v}) \mathrm{d}s$ satisfies $\partial_t u + \mathscr{L}^a u + f = 0$.

Remark: The above estimate was recently extended to nonlocal kinetic operator by Z.Q. Chen and Z. (2015).

Krylov's estimate

Theorem (Krylov's type estimate)

Let $T_0 > 0$. Suppose that σ satisfies $(\mathbf{H}_K^\delta)$ for some $\delta \in (0,1)$, and

$$b \in \mathbb{L}^q(T_0), \quad q \in (4d+2, \infty].$$

Let $Z_t = (X_t, \dot{X}_t)$ solve SDE (1.1). For any $p > 2d+1$, there is a positive constant C depending on T_0, K, δ, p, q, d and $\|b\|_{\mathbb{L}^q(T_0)}$ such that for all $0 \leqslant S < T \leqslant T_0$ and $f \in \mathbb{L}^p(T_0)$,

$$\mathbb{E}\left(\int_S^T f_s(Z_s)\mathrm{d}s \Big| \mathscr{F}_S\right) \leqslant C(T-S)^{\frac{1}{2d+1}-\frac{1}{p}}\|f\|_{\mathbb{L}^p(S,T)}.$$

Theorem (Characterization of Sobolev differentiability of random fields)

Let $U \subset \mathbb{R}^d$ be a bounded C^1-domain and $f \in L^q\big(U; L^p(\Omega; L^r(T))\big)$ for some $p \in (1, \infty)$ and $q, r \in (1, \infty]$. Then $f \in \mathbb{W}^{1,q}\big(U; L^p(\Omega; L^r(T))\big)$ if and only if there exists a nonnegative measurable function $g \in L^q(U)$ such that for Lebesgue-almost all $x, y \in U$,

$$\|f(x, \cdot) - f(y, \cdot)\|_{L^p(\Omega; L^r(T))} \leqslant |x - y|\big(g(x) + g(y)\big). \tag{5.1}$$

Moreover, if (5.1) holds, then for Lebesgue-almost all $x \in U$,

$$\|\partial_i f(x, \cdot)\|_{L^p(\Omega; L^r(T))} \leqslant 2g(x), \quad i = 1, \cdots, d,$$

where $\partial_i f$ is the weak partial derivative of f with respect to the i-th spacial variable.

Remark: The deterministic case is due to Hajesk (POA, 1995) and random case is due to L. Xie and Z. (AOP, [2015]).

Stochastic Gronwall's inequality

Lemma 5.1

For given $T > 0$, *let* $(\xi_t)_{t\in[0,T]}$ *and* $(\beta_t)_{t\in[0,T]}$ *(resp.* $(\alpha_t)_{t\in[0,T]}$*) be two real-valued (resp.* $\mathbb{R}^d$*-valued) measurable* $\mathscr{F}_t$*-adapted processes. Let* ζ_t *be an Itô process with the form:*

$$\zeta_t = \zeta_0 + \int_0^t \zeta_s^{(1)} \mathrm{d}s + \int_0^t \zeta_s^{(2)} \mathrm{d}W_s.$$

Suppose that for any $\gamma > 0$,

$$\kappa_\gamma := \mathbb{E} \exp\left\{\gamma \int_0^T \left(|\beta_s| + |\alpha_s|^2\right) \mathrm{d}s\right\} < \infty, \tag{5.2}$$

Stochastic Gronwall's inequality

Continuing

and

$$0 \leqslant \xi_t \leqslant \zeta_t + \int_0^t \xi_s \beta_s \mathrm{d}s + \int_0^t \xi_s \alpha_s \mathrm{d}W_s. \tag{5.3}$$

Then for any $q_0 \in [1, \infty)$ and $q_1, q_2, q_3 > q_0$, there is a constant $C > 0$ only depending on $q_i, \kappa_\gamma, i = 0, 1, 2, 3$ such that

$$\|\xi_T^*\|_{q_0} \leqslant C \left(\|\zeta_0\|_{q_1} + \left\| \int_0^T |\zeta_s^{(1)}| \mathrm{d}s \right\|_{q_2} + \left\| \int_0^T |\zeta_s^{(2)}|^2 \mathrm{d}s \right\|_{q_3/2}^{1/2} \right), \tag{5.4}$$

where $\xi_T^* := \sup_{t \in [0,T]} \xi_t$ and $\| \cdot \|_{q_i}$ denotes the norm in $L^{q_i}(\Omega)$.

Proof of Main Theorem

Assume that σ satisfies **(UE)** and for some $p > 2(2d+1)$,

$$\kappa_0 := \|b\|_{L^p(\mathbb{R}_+;H_p^{2/3,0})} + \|\nabla\sigma\|_{L^\infty(\mathbb{R}_+;L^p)} < \infty.$$

For $n \in \mathbb{N}$, let ϱ_n be a family of mollifiers and define

$$b_t^n := b_t * \varrho_n, \ \sigma_t^n := \sigma_t * \varrho_n, \quad b_t^\infty := b_t, \ \sigma_t^\infty := \sigma_t$$

By the property of convolution, we have

$$\|\sigma_s^n - \sigma_s\|_\infty \leqslant C\|\nabla\sigma_s\|_p n^{\frac{2d}{p}-1}, \quad \sup_{\mathrm{v}} \|\Delta_x^{\frac{1}{3}}\sigma_s^n(\cdot,\mathrm{v})\|_p \leqslant C\|\nabla\sigma_s\|_p.$$

Proof of Main Theorem

Let $T \in (0,1)$ be fixed. For $n \in \mathbb{N} \cup \{\infty\}$, let $\mathbf{u}^n \in \mathbb{H}_p^{4/3,2}(T)$ uniquely solve the following PDE:

$$\partial_t \mathbf{u}^n + \mathscr{L}_t^{\sigma^n, b^n} \mathbf{u}^n + b^n = 0, \quad \mathbf{u}_T^n = 0.$$

By (4.2) and (4.3), there is a constant $C = C(d, p, \kappa_0, K) > 0$ such that for all $n \in \mathbb{N} \cup \{\infty\}$,

$$\|\nabla\nabla_{\rm v} \mathbf{u}^n\|_{\mathbb{L}^p(T)} \leqslant C\|b\|_{\mathbb{H}_p^{2/3,0}(T)}, \ \|\nabla_{\rm v} \mathbf{u}_t^n\|_\infty \leqslant CT^{\frac{1}{2(2d+1)} - \frac{1}{p}}\|b\|_{\mathbb{L}^p(T)},$$

and

$$\|\mathbf{u}^n - \mathbf{u}^m\|_{\mathbb{L}^\infty(T)} + \|\nabla_{\rm v}\mathbf{u}^n - \nabla_{\rm v}\mathbf{u}^m\|_{\mathbb{L}^\infty(T)} \leqslant C\Big(\|b^n - b^m\|_{\mathbb{L}^p(T)} + (n \wedge m)^{\frac{2d}{p} - 1}\Big).$$

Let

$$H^n_t(x,\mathrm{v}) := \mathrm{v} + \mathbf{u}^n_t(x,\mathrm{v})$$

By the above estimate, one can choose T_0 small enough so that

$$\|\nabla \mathbf{u}^n\|_{\mathbb{L}^\infty(T_0)} \leqslant \tfrac{1}{2},$$

and thus,

$$\tfrac{1}{2}|\mathrm{v}-\mathrm{v}'| \leqslant |H^n_t(x,\mathrm{v}) - H^n_t(x,\mathrm{v}')| \leqslant \tfrac{3}{2}|\mathrm{v}-\mathrm{v}'|.$$

Observing that

$$\partial_t H^n + \mathscr{L}^{\sigma^n,b^n}_t H^n = 0,$$

by Itô's formula, we have

$$H^n_t(Z^n_t) = H^n_0(Z^n_0) + \int_0^t \Theta^n_s(Z^n_s)\mathrm{d}W_s,$$

where $\Theta^n_s(z) := (\nabla_{\mathrm{v}} H^n_s \cdot \sigma^n_s)(z)$.

Proof of Main Theorem

let $Z_t^n = (X_t^n, \dot{X}_t^n)$ uniquely solve the following SDE:

$$\mathrm{d}Z_t^n = (\dot{X}_t^n, b_t^n(Z_t^n))\mathrm{d}t + (0, \sigma_t^n(Z_t^n)\mathrm{d}W_t), \quad Z_0^n = z = (x, \mathrm{v}) \in \mathbb{R}^{2d}.$$

Proof of Main Theorem

let $Z^n_t = (X^n_t, \dot{X}^n_t)$ uniquely solve the following SDE:

$$\mathrm{d}Z^n_t = (\dot{X}^n_t, b^n_t(Z^n_t))\mathrm{d}t + (0, \sigma^n_t(Z^n_t)\mathrm{d}W_t), \quad Z^n_0 = z = (x, \mathrm{v}) \in \mathbb{R}^{2d}.$$

We have

Lemma

For $T_0 > 0$ being small, there is a constant $C > 0$ such that for all $n, m \in \mathbb{N}$,

$$\mathbb{E}\left(\sup_{t\in[0,T_0]} |Z^n_t - Z^m_t|^2\right) \leqslant C\left(\|b^n - b^m\|^2_{\mathbb{L}^p(T_0)} + (n \wedge m)^{\frac{4d}{p}-2}\right).$$

Proof of Main Theorem

Proof of the Lemma: By Itô's formula, we have

$$|H_t^n(Z_t^n) - H_t^m(Z_t^m)|^2 = |H_0^n(z) - H_0^m(z)|^2 + \int_0^t \|\Theta_s^n(Z_s^n) - \Theta_s^m(Z_s^m)\|^2 \mathrm{d}s$$
$$+ 2\int_0^t \langle H_s^n(Z_s^n) - H_s^m(Z_s^m), (\Theta_s^n(Z_s^n) - \Theta_s^m(Z_s^m))\mathrm{d}W_s\rangle,$$

and also,

$$|X_t^n - X_t^m|^2 = 2\int_0^t \langle X_s^n - X_s^m, \dot{X}_s^n - \dot{X}_s^m\rangle \mathrm{d}s.$$

Proof of Main Theorem

Proof of the Lemma: By Itô's formula, we have

$$|H_t^n(Z_t^n) - H_t^m(Z_t^m)|^2 = |H_0^n(z) - H_0^m(z)|^2 + \int_0^t \|\Theta_s^n(Z_s^n) - \Theta_s^m(Z_s^m)\|^2 \mathrm{d}s$$
$$+ 2\int_0^t \langle H_s^n(Z_s^n) - H_s^m(Z_s^m), (\Theta_s^n(Z_s^n) - \Theta_s^m(Z_s^m))\mathrm{d}W_s\rangle,$$

and also,

$$|X_t^n - X_t^m|^2 = 2\int_0^t \langle X_s^n - X_s^m, \dot{X}_s^n - \dot{X}_s^m\rangle \mathrm{d}s.$$

Key point:

$$|H_t^n(Z_t^n) - H_t^n(Z_t^m)|^2 + |X_t^n - X_t^m|^2 \asymp |Z_t^n - Z_t^m|^2 = |X_t^n - X_t^m|^2 + |\dot{X}_t^n - \dot{X}_t^m|^2.$$

Thank you very much for your kind attention!