The Seneta-Heyde Scaling for a Stable Branching Random Walk

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July 13, 2016 @ JSNU

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Branching random walk

- It starts with an initial particle located at the origin.
- At time 1, the particle dies, producing some new particles, positioned according to Θ.
- At time 2, these particles die, each giving birth to new particles positioned (with respect to the birth place) according to Θ.
- The process goes on with the same mechanism. We assume the particles produce new particles independently.

For each vertex x on the branching tree, We denote the position by $V(x)$. The family of random variables $(V(x))$ is referred as a branching random walk (Biggins (2010)).

Throughout the paper, we assume (Biggins and Kyprianou (2005)):

$$\mathbf{E}(\sum_{|x|=1} 1) > 1, \quad \text{(supercritical)}$$

$$\mathbf{E}(\sum_{|x|=1} e^{-V(x)}) = 1, \quad \mathbf{E}(\sum_{|x|=1} V(x)e^{-V(x)}) = 0, \quad \text{(boundary case)} \tag{1}$$

where $|x|$ denotes the generation of x. Every branching random walk satisfying certain integrability assumptions can be reduced to this case by some renormalization, if Θ is not bounded from below. (see Jaffuel (2012)).

Additive martingale

$$W_n := \sum_{|x|=n} e^{-V(x)}, \quad \text{(additive martingale)}$$

- $V(x) \equiv 0$: (W_n) degenerates to a supercritical GW process (Z_n). $c_n^{-1} Z_n \to W$ with $P(W=0) < 1$ (Seneta (1968),Heyde (1970), Kesten and Stigum (1966)).
- $V(x) \neq 0$: (W_n) converges almost surely to W. Under (1), $W = 0$, a.s. (Biggins (1977), Lyon (1997)).

It is natural to ask

At which rate W_n goes to 0?

Related work:

- Galton-Watson processes: Seneta (1968), Heyde (1970).
- Branching random walk:
  - General case: Biggins and Kyprianou (1996, 1997).
  - Boundary case: Liu (2000), Biggins and Kyprianou (2005), Hu and Shi (2009), Aidekon and Shi (2014) (under weaker integrability assumption than Hu and Shi (2009)).
    (finite $1+\delta$ order moment for $\sum_{|x|=1} 1$ and exponential moment condition for $V(x) \Rightarrow$ certain 2-order moment condition)

Derivative martingale

$$D_n := \sum_{|x|=n} V(x)e^{-V(x)}, \quad \text{(derivative martingale)}$$

Related work: Barral (2000), Biggins (1991,1992) for non-boundary cases, Kyprianou (1998), Biggins and Kyprianou (2004) for the boundary case. Chen (2015) gave the necessary and sufficient condition for the non-trival limit of (D_n).

Related results on derivative martingale

Suppose that

$$\mathbf{E}\left(\sum_{|x|=1} V^2(x)e^{-V(x)}\right) < \infty. (\Leftrightarrow E(S_1^2) < \infty) \qquad (2)$$

Theorem A (Biggins and Kyprianou (2004))

Assume (1) and (2) hold. Then there exists a nonnegative random variable D_∞ such that

$$D_n \to D_\infty, \ \ \mathbf{P}-a.s.$$

Theorem B (Chen (2015))

Assume (1) and (2) hold. Then $\mathbf{P}(D_\infty > 0) > 0$ if and only if the following condition holds:

$$\mathbf{E}[X \log_+^2 X + \widetilde{X} \log_+ \widetilde{X}] < \infty, \tag{3}$$

where $\log_+ y := \max\{0, \log y\}$, $\log_+^2 y := (\log_+ y)^2$ for any $y > 0$, and

$$X := \sum_{|x|=1} e^{-V(x)}, \quad \widetilde{X} := \sum_{|x|=1} V(x)_+ e^{-V(x)}, \tag{4}$$

with $V(x)_+ := \max\{V(x), 0\}$.

When D_∞ is nontrivial, $\mathbf{P}(D_\infty > 0)$ equals to the non-extinction probability of the branching tree. Define

$$\mathbf{P}^*(\cdot) := \mathbf{P}(\cdot\,|\text{non extinction})$$

Obviously $W_n \to 0,\ \mathbf{P}^*\text{–}a.s.$

Theorem C (Aidekon and Shi (2014))

Assume (1), (2) and (3) hold. Under $\mathbf{P}^*$, we have

$$\lim_{n\to\infty} n^{1/2} W_n = \Big(\frac{2}{\pi\sigma^2}\Big)^{1/2} D_\infty \ \ in\ probability,$$

where $D_\infty > 0$ is the random variable in Theorem A, and

$$\sigma^2 := \mathbf{E}\Big(\sum_{|x|=1} V(x)^2 e^{-V(x)}\Big) < \infty.$$

In this paper, instead of

$$\mathbf{E}\Big(\sum_{|x|=1} V^2(x)e^{-V(x)}\Big) < \infty, \ \mathbf{E}[X \log_+^2 X + \widetilde{X} \log_+ \widetilde{X}] < \infty,$$

we shall study W_n under (1) and ($\alpha \in (1,2)$):

$$(i)\ \ \mathbf{E}\Big(\sum_{|x|=1} I_{\{V(x)\le -y\}} e^{-V(x)}\Big) = o(y^{-\alpha}), \quad y \to +\infty, \tag{5}$$

$$(ii)\ \ \mathbf{E}\Big(\sum_{|x|=1} I_{\{V(x)\ge y\}} e^{-V(x)}\Big) \sim C y^{-\alpha}, \quad y \to +\infty, \tag{6}$$

$$(iii)\ \ \mathbf{E}(X(\log_+ X)^\alpha + \widetilde{X}(\log_+ \widetilde{X})^{\alpha-1}) < \infty. \tag{7}$$

Under (5) and (6), the step of the one-dimensional random walk (S_n) associated with $(V(x))$ belongs to the domain of attraction of a stable law.

(The many-to-one formula)

$$\mathbf{E}\Big[\sum_{|x|=n} g(V(x_1),\ldots,V(x_n))\Big] = \mathbf{E}\big[e^{S_n} g(S_1,\ldots,S_n)\big],$$

where (S_n) is a random walk, and S_1 belongs to the domain of attraction of a stable law with characteristic function (spectrally positive). see Biggins and Kyprianou (1997), Lyons (1997), Lyons et al (1995).

$$G_{\alpha,-1}(t) := \exp\Big\{-c|t|^{\alpha}\Big(1 - i\frac{t}{|t|}\tan\frac{\pi\alpha}{2}\Big)\Big\},\ c>0.$$

(5) and (6)

$$\Leftrightarrow P(S_1 > y) \sim Cy^{-\alpha},\ P(S_1 < -y) = o(y^{-\alpha}), \quad y \to +\infty$$

Main results – derivative martingale

Theorem 1 Assume (1), (5), (6). Then there exists a nonnegative random variable D_∞ such that

$$D_n \to D_\infty, \quad \mathbf{P}-a.s.$$

Moreover, if condition (7) holds, then $\mathbf{P}^*(D_\infty > 0) = 1$.

Main results – additive martingale

Theorem 2 Assume (1), (5), (6) and (7). We have, under $\mathbf{P}^*$

$$\lim_{n\to\infty} n^{\frac{1}{\alpha}} W_n = \frac{\theta}{\Gamma(1-1/\alpha)} D_\infty, \text{ in probability.}$$

where $D_\infty > 0$ is given in Theorem 1, and θ is a positive constant.

Theorem 3 Assume (1), (5), (6) and (7). We have

$$\overline{\lim}_{n\to\infty} n^{\frac{1}{\alpha}} W_n = \infty \quad \mathbf{P}^* - a.s. \tag{8}$$

Sketch of proofs – estimates for (S_n)

In the proofs, we depend heavily on the probability estimations for (S_n). For example,

$$\sum_{l\geq 0} \mathbf{P}_z(S_l \leq x,\, \underline{S}_l \geq 0) \leq c\,(1+x)^{\alpha-1}(1+\min(x,z));$$

and some properties for (S_n) conditioned to stay in $[-x,\infty)$:

$$E\Big(f\big(\frac{S_n+x}{n^{1/\alpha}}\big)\mathbf{1}_{\{\underline{S}_n\geq -x\}}\Big) = \frac{R(x)}{\Gamma(1-\frac{1}{\alpha})n^{1/\alpha}}\Big(\int_0^\infty f(t)p_\alpha(t)dt + o_n(1)\Big).$$

uniformly in $x \in [0,d_n]$ with $d_n = o(n^{1/\alpha})$. And $\mathbf{E}\,(M_\alpha) = \frac{\Gamma(1-\frac{1}{\alpha})}{\theta}$ $(M_\alpha \leftrightarrow p_\alpha)$.

Sketch of proofs – truncating argument

It originated from Harris (1999), was formalized for BBM by Kyprianou (2004), and later be used for BRW by Biggins and Kyprianou (2004) and then Aidekon and Shi (2014).

Define $\underline{V}(x) := \min_{y \in \langle \varnothing, x \rangle} V(y)$. We use the renewal function $R(u)$ of (S_n) to introduce the truncated processes $(\beta > 0)$:

$$W_n^\beta := \sum_{|x|=n} e^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \geq -\beta\}}, \qquad (\sim W_n)$$

$$D_n^\beta := \sum_{|x|=n} R(V(x)+\beta) e^{-V(x)} \mathbf{1}_{\{\underline{V}(x) \geq -\beta\}}. \qquad (\sim \theta D_n)$$

Note that $\lim_{u \to \infty} \frac{R(u)}{u} = \theta \in (0, \infty)$ for $S_1 \in D(\alpha, -1)$.

Sketch of proofs – change of probabilities

$$\left.\frac{d\,\mathbf{P}^{\beta}}{d\,\mathbf{P}}\right|_{\mathcal{F}_n} := \frac{D_n^{\beta}}{R(\beta)}, \quad \text{(change of probabilities)}$$

We consider the random walk $(V(x))$ under $\mathbf{P}^{\beta}$. Now $(V(x), |x| = 1)$ are distributed as another point process Θ^{a} under $\mathbf{P}^{\beta}$. And there is a "spine" in the branching tree. For each generation n, there is a w_n^{β} which takes the branching $\mathbf{P}^{\beta}$, see Biggins and Kyprianou (2004).

Kanhane and Peyrieye (1976), Lyons et al (1995)

(1) Proof of Theorem 1:

- prove $D_n^\beta \to D_\infty^\beta$ (truncated martingale convergence)
- prove $D_n \to D_\infty$.
- prove $\mathbf{P}(D_\infty^\beta > 0) > 0$. ($(D_n^\beta)$ is uniformly integrable)
- prove $\mathbf{P}^*(D_\infty > 0) = 1$. ($D_\infty^\beta \leq cD_\infty$, a.s.)

(2) Proof of Theorem 2: We first have

$$\mathbf{E}^{\beta}\left(\frac{W_n^{\beta}}{D_n^{\beta}}\right) \sim \frac{1}{\Gamma(1-1/\alpha)n^{\frac{1}{\alpha}}},$$

$$\mathbf{E}^{\beta}\left(\left(\frac{W_n^{\beta}}{D_n^{\beta}}\right)^2\right) \sim \frac{1}{\big(\Gamma(1-1/\alpha)\big)^2 n^{\frac{2}{\alpha}}}.$$

Therefore $\lim_{n\to\infty} n^{\frac{1}{\alpha}}\left(\frac{W_n^{\beta}}{D_n^{\beta}}\right) = \Gamma(1-\frac{1}{\alpha}), \quad \textit{in probability} \quad (\mathbf{P}^{\beta})$

Finally, we manage to change the setting from $\mathbf{P}^{\beta}$ to $\mathbf{P}$.

(3) Proof of Theorem 3:

- $$\mathbf{P}\left\{\exists x : |x| \in [n, \alpha n], V(x) \in \left[\frac{1}{\alpha}\log n, \frac{1}{\alpha}\log n + C\right]\right\} \geq c_0.$$

- $$\underline{\lim}_{n\to\infty}\left(\min_{|x|=n} V(x) - \frac{1}{\alpha}\log n\right) = -\infty, \quad \mathbf{P}^* - a.s.$$

(For $\alpha = 2$, $\min_{|x|=n} V(x) \sim \frac{3}{2}\log n$, Hu and Shi (2009)).

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Thank you !