Quasi-Stationary Distributions And Their Applications

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Talk in Xuzhou2016, July 13, 2016

Outline of Topics

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QSDs

- What is QSD, and why study it? The first thing is to define Quasi-Stationary Distributions(QSDs).

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- What is QSD, and why study it? The first thing is to define Quasi-Stationary Distributions(QSDs).
- Before we do that, we first recall the concept of Stationary Distribution.
- Suppose that $(X_t)_{t\geq 0}$ is a Markov process, if $(X_t)_{t\geq 0}$ exists a stationary distribution μ. Then,

$$P(X_t \in A) \to \mu(A), \qquad t \to \infty$$

for any Borel set A, where $\mu(A) > 0$ for any nonempty open set A.

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QSDs

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- When we discuss the QSDs, we still assume that $(X_t)_{t \geq 0}$ is a Markov Process and limit distribution exists. i.e.
- $\lim_{t\to\infty} P(X_t \in A) = \mu(A)$ for any Borel set A. But, where

$$\mu(A) = \begin{cases} 1 & 0 \in A, \\ 0 & \text{otherwise.} \end{cases}$$

QSDs

- A Quasi-Stationary Distribution (in short QSD) for X is a probability measure supported on $(0, \infty)$ satisfying for all $t \geq 0$,

$$\mathbf{P}_{\nu}(\mathbf{X}(\mathbf{t}) \in \mathbf{A} | \mathbf{T} > \mathbf{t}) = \nu(\mathbf{A}), \ \forall \ \textbf{borel set } \mathbf{A} \subseteq (\mathbf{0}, \infty).$$

where $T = \inf\{t \geq 0, X_t = 0\}$

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  for some $\mu \in (0,\infty)$.

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- By the definition, a QSD is a fixed point of the conditional evolution.
- It is not hard to show if such ν exist then

$$\mathbf{P}_\nu(\mathbf{T} > \mathbf{t}) = \mathbf{e}^{-\mu\mathbf{t}}$$

  for some $\mu \in (0, \infty)$.
- A QSD must be infinitely divisible (D.Vere-Jones 1969)

QSDs

- A probability measure π supported on $(0, \infty)$ is a LCD If there exists a probability measure ν on $(0, \infty)$ such that the following limit exists in distribution

$$\lim_{t \to \infty} \mathbf{P}_{\nu}(\mathbf{X(t)} \in \bullet \mid \mathbf{T} > \mathbf{t}) = \pi(\bullet).$$

We also say that ν is attracted to π or is in the domain of attraction of π or π is a ν-LCD

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  We also say that ν is attracted to π or is in the domain of attraction of π or π is a ν-LCD
- Trivially, any QSD ν is an ν-LCD.
- The ν-LCD is a QSD (Vere-Jones(1969)).

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QSDs

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- (iii) The rate of convergence of the transition probabilities of the conditioned process to their limiting values.

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QSDs of Markov Chains

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- Both (i) and (ii) are known only for finite Markov processes, and for the subcritical Markov Branching Process(MBP)
- For a Birth and death process, Erik A. VAN DOORN (Adv.Appl.Prob.23, 683-700,1991) obtained
  Proposition (i) if $S = \infty$, then either $\lambda_C = 0$ and there is no QSD, or $\lambda_C > 0$ and there is a one-parameter family of QSDs, Viz,$\{\{q_j(x)\}, 0 < x \leq \lambda_C\}$.
  (ii) If $S < \infty$, then $\lambda_C > 0$ and there is precisely one QSD,Viz,$\{q_j(\lambda_C)\}$

QSDs of Markov Chains

- Our result is
  Theorem 1 For a Birth and death process,
  (i) there is QSD iff $A = \infty$ and $\delta < \infty$ holds;
  When they hold, in addition to $S = \infty$, then there is a one-parameter family of QSDs, Viz, $\{\{q_j(x)\}, 0 < x \leq \lambda_C\}$.
  (ii) If $S < \infty$, then $\lambda_C > 0$ and there is precisely one QSD, Viz, $\{q_j(\lambda_C)\}$. And the unique QSD $\{q_j(\lambda_C)\}$ attracts all initial distributions ν supported in $(0, \infty)$, that is,

$$\lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t} = \mathbf{j} | \mathbf{T} > \mathbf{t}) = \mathbf{q_j}(\lambda_\mathbf{C}).$$

QSDs of Markov Chains

- Let X_t be a continuous-time Markov chain in $I = \{0\} \cup \{1, 2, \ldots\}$ such that 0 is an absorbing state. Let $C \equiv \{1, 2, \ldots\}$. Denote by $Q = (q_{ij})$ the q-matrix (transition rate matrix) and $P(t) = (P_{ij}(t))$ the transition function. X_t is stochastically monotone if and only if $\sum_{j \geq k} P_{ij}(t)$ is a nondecreasing function of i for every fixed $k \in I$ and $t > 0$. We assume that all states other than 0 form an irreducible class and that Q is totally stable, conservative and regular, that is, $q_i = \sum_{i \neq j} q_{ij} < \infty$, and the minimal process $\{X_t\}_{t \geq 0}$corresponding to Q is an honest process. We further define $T = \inf\{t \geq 0 : X_t = 0\}$, the absorption time at 0. So $X_t = 0$ for any $t \geq T$.

QSDs of Markov Chains

For stochastically monotone Markov chains, we discuss the existence, uniqueness and domain of attraction of QSDs.

Theorem 2 Assume X_t is stochastically monotone, then

(i) there exists a QSD if and only if

$$E_i(e^{\theta T}) < \infty$$

for some $\theta > 0$ and some $i \in C$ (and hence for all i).

QSDs of Markov Chains

(ii) there is a unique QSD if and only if

$$\lim_{i \to \infty} E_i(e^{\theta T}) < \infty$$

for some $\theta > 0$. Moreover, the unique QSD $\rho = \{\rho_j, j \in C\}$ attracts all initial distributions that supported in C, that is, for any probability measure $\nu = \{\nu_i, i \in C\}$,
$\rho_j = \lim_{t \to \infty} \mathbf{P}_\nu(\mathbf{X_t = j | T > t}), \quad \mathbf{j} \in \mathbf{C}.$

QSDs of Markov Chains

We consider a generalized Markov branching process Z_t with q-matrix Q given by

$$q_{ij} = \begin{cases} r_i p_{j-i+1}, & \text{if } j \geq i-1 \geq 0, j \neq i, \\ -r_i(1-p_1), & \text{if } j = i \geq 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

where $r_i \geq 0$ and $p_i \geq 0$, $\sum_{i=0}^{\infty} p_k = 1$. For $s \geq 0$, define $P(s) = \sum_{i=0}^{\infty} p_i s^i - s$. If $m_1 := \sum_{k=1}^{\infty} k p_k \leq 1$, then Q is regular, and the unique Q-process is denoted by Z_t.

Proposition

The generalized Markov branching q-matrix of equation (3.1) is monotone if and only if the non-negative sequence r_i is increasing.

QSDs of Markov Chains

Proposition

Assume that the non-negative sequence r_i is increasing. If

$$m_1 < 1 \quad and \quad \sum_{i=1}^{\infty} \frac{1}{r_i} < \infty,$$

then there exists a unique QSD that attracts all initial distributions supported in $C = \{1, 2, \cdots\}$.

QSDs of Markov Chains

- A conservative q-matrix $Q = (q_{ij},\ i, j \in I)$ is called a Power-Law branching q-matrix if it takes the following form:

$$q_{ij} = \begin{cases} i^{\nu} p_{j-i+1} & \text{if } j \geq i-1, j \neq i, i \geq 1, \\ -i^{\nu}(1-p_1) & \text{if } j = i, i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$p_k \geq 0, k \geq 0 \text{ and } \ \nu > 0.$$

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  where

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- From Lemma 2.2 in Chen(2002), we know that

$$\mathbb{E}_i T_0 = \frac{1}{\Gamma(\nu)} \int_0^1 \frac{1-x^i}{P(x)(-\ln x)^{1-\nu}} \mathrm{d}x$$

  for all $i \geq 1$, where $\Gamma(\nu)$ is the gamma function.

QSDs of Markov Chains

Define convergence radius of $P(s)$ $\rho := 1/\limsup_{n\longrightarrow\infty} \sqrt[n]{p_n}$. Clearly, $\rho \geq 1$.

Proposition

For a Power-Law branching process Z_t, suppose the extinction probability of Z_t is equal to 1 (which is equivalent to $m_1 \leq 1$). If
(1) $\nu > 2$, then there exists a unique QSD that attracts all initial distributions supported in $C = \{1, 2, \cdots\}$;
(2) $1 < \nu \leq 2$ and $m_1 < 1$, then there exists a unique QSD that attracts all initial distributions supported in $C = \{1, 2, \cdots\}$;
(3) $0 < \nu \leq 1$, then there exists a QSD if $m_1 < 1$ and $\rho > 1$.

QSDs of Markov Chains

- For continuous-time general Markov chains, P.A.Ferrari, H.Kesten,S.Martinez, and P.Picco (The Annals of Probability 1995,Vol.23, No.2,501-521.) obtained
  Proposition 2 Assume that

$$\lim_{i\to\infty} P_i(T < t) = 0 \quad \text{for any} \quad t \geq 0$$

and that $P_i(T < \infty) = 1$ for some (and hence all) i. Then a necessary and sufficient condition for the existence of a QSD is that

$$E_i(e^{\theta T}) < \infty$$

for some $\theta > 0$ and some $i \in C$ (and hence for all i).

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QSDs of Markov Chains

- Our result is
  Theorem 3 (i) Assume that
  $$\lim_{i\to\infty} E_i T = \infty,$$
  and that $P_i(T<\infty)=1$ for some (and hence all) i. Then a necessary and sufficient condition for the existence of a QSD is that
  $$E_i(e^{\theta T}) < \infty$$
  for some $\theta > 0$ and some $i \in C$ (and hence for all i). When it holds, There exists a family of QSDs.

QSDs of Markov Chains

- (ii) If

$$\limsup_{i\to\infty} E_i T < \infty,$$

then there exists unique QSD;Moreover, the unique QSD $\rho = \{\rho_j, j \in C\}$ attracts all initial distributions that supported in C, that is, for any probability measure $\nu = \{\nu_i, i \in C\}$,

$$\rho_j = \lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t} = \mathbf{j} | \mathbf{T} > \mathbf{t}), \quad \mathbf{j} \in \mathbf{C}.$$

QSDs of One-dimensional Diffusion Processes

- A One-dimensional diffusion process will now be taken to mean a continuous Strong Markov process with values in $[0, \infty)$.

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- Consider the solution of SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x > 0,$$

where B is a standard Brownian motion. $\sigma : (0, \infty) \to (0, \infty)$ and $b : R \to R$. We shall assume that both σ and b are locally bounded and measurable.

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- We also assume that the SDE has a unique solution.

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  where B is a standard Brownian motion. $\sigma : (0, \infty) \to (0, \infty)$ and $b : R \to R$. We shall assume that both σ and b are locally bounded and measurable.
- We also assume that the SDE has a unique solution.
- Let $T_z = \inf\{t > 0 : X_t = z\}$ be the hitting time of z. We are mainly interested in the case $z = 0$ and we denote $T = T_0$.

QSDs of One-dimensional Diffusion Processes

- Let P_t the associated sub-markovian semi-group defined by $P_t f(x) = E_x(f(X_t), T > t)$. whose generator is defined by

$$L = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}.$$

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- Set

$$Q(x) = \int_1^x \frac{2b(z)}{\sigma^2(z)}dz \quad and \quad \Lambda(x) = \int_1^x \exp\{-Q(y)\}dy$$

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$$Q(x) = \int_1^x \frac{2b(z)}{\sigma^2(z)}dz \quad and \quad \Lambda(x) = \int_1^x \exp\{-Q(y)\}dy$$

- $\Lambda(x)$ is called the scale function. $\Lambda(x)$ is a strictly increasing smooth function on $(0, \infty)$ satisfying

$$L\Lambda(x) \equiv 0 \quad on \ (0, \infty).$$

QSDs of One-dimensional Diffusion Processes

- Set

$$\kappa(x) = \int_1^x \exp\{-Q(y)\} \left(\int_1^y \frac{2}{\sigma^2(z)} \exp\{Q(z)\} dz \right) dy.$$

QSDs of One-dimensional Diffusion Processes

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- For all $x > 0$,

$$P_x(T < \infty) = 1$$

if and only if

$$\Lambda(\infty) = \infty \quad and \ \kappa(0^+) < \infty.$$

QSDs of One-dimensional Diffusion Processes

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$$S = \int_1^\infty \exp\{-Q(y)\} \left(\int_y^\infty \frac{2}{\sigma^2(z)} \exp\{Q(z)\} dz \right) dy.$$

QSDs of One-dimensional Diffusion Processes

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QSDs of One-dimensional Diffusion Processes

- It is easy to prove that If $\Lambda(\infty) = \infty$, then $\kappa(\infty) = \infty$.
- ∞ is a natural boundary for X in the sense of Feller boundary if and only if $\kappa(\infty) = \infty, \kappa(0^+) < \infty$ and $S = \infty$, which implies for any $s > 0$

$$\lim_{x \to \infty} P_x(T_0 > s) = 1 \textit{ and } \lim_{M \to \infty} P_x(T_M > s) = 1$$

QSDs of One-dimensional Diffusion Processes

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$$\lim_{x \to \infty} P_x(T_0 > s) = 1 \ and \ \lim_{M \to \infty} P_x(T_M > s) = 1$$

- ∞ is an entrance boundary for X if and only if $\kappa(\infty) = \infty, \kappa(0^+) < \infty$ and $S < \infty$, which is equivalent to the property that there is $y > 0$ and a time $t > 0$ such that

$$\lim_{x \uparrow \infty} P_x(T_y < t) > 0$$

.

QSDs of One-dimensional Diffusion Processes

- In the literature there are some works directly related with the problem mentioned above. The first one was published by S. Martínez, J. San Martín, P. Picco (1994,1998) who consider the B.M. with negative constant drift, that is, $dX_t = dB_t - \alpha dt (\alpha > 0)$. They proved that there exits a one-parameter family of QSDs and solve the domain of attraction problem.

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- The second was published by M. Lladser, J. San Martín (2000) who consider the Ornstein-Uhlenbeck process, that is, $dX_t = dB_t - aX_t dt (a > 0)$. They solved the domain of attraction problem.

QSDs of One-dimensional Diffusion Processes

- The third was published by Cattiaux *et al*. (The Annals of Probability 2009,Vol.37,No.5,1926-1969.) who study the existence and uniqueness of the QSD for one-dimensional diffusions killed at 0 and whose drift is allowed to go to $-\infty$ at 0 and the process is allowed to have an entrance boundary at $+\infty$

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- The last one was was published by M. Kolb, D. Steinsaltz (The Annals of Probability 2012,Vol.40,No.1,162-212.) who consider the one-dimensional diffusions with killing. They proved that this process either converges to QSD or escapes to ∞ from a compactly supported distribution.

QSDs of One-dimensional Diffusion Processes

- Moreover, it is worth mentioning that Prof. Jinwen Chen (Tsinghua University) has also made some contribution on this topic (QSD). He proved the existence and uniqueness of both QSD and mean ratio quasi-stationary distribution (mrqsd) for killed Brownian motion by using an eigenfunction expansion for the transition density. In particular, he gived interpretations of the mrqsd from different points of view not only for killed Brownian motion but also for absorbing Markov processes.

QSDs of One-dimensional Diffusion Processes

- Now our results are

  Theorem 4 For the diffusion process, then

  (i) there is QSD iff $\Lambda(\infty) = \infty, \quad \kappa(0^+) < \infty$, and

$$\sup_{x>0} \int_0^x \exp\left(-Q(y)\right) dy \int_x^\infty \frac{2\exp\left(Q(y)\right)}{\sigma^2(y)} dy < \infty.$$

  holds;

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  holds;

- (ii) If in addition to (i) by
  $S := \int_1^\infty \exp\{-Q(y)\} \left(\int_y^\infty \frac{2}{\sigma^2(z)} \exp\{Q(z)\}dz \right) dy = \infty$,
  then for any $0 < \lambda \leq \lambda_c$, $d\nu_\lambda = 2\lambda\eta_\lambda d\mu$ is a quasi-stationary distribution and all of quasi-stationary distributions for X only have this form.

QSDs of One-dimensional Diffusion Processes

- Theorem 5 For the diffusion process,
  (i) there is exactly one QSD ν_1 iff $S < \infty$.
  (ii) When it holds, the unique quasi-stationary distribution ν_1 attracts all initial distributions ν supported in $(0, \infty)$, that is,

$$\lim_{t\to\infty} \mathbf{P}_{\nu}(\mathbf{X_t} \in \bullet | \mathbf{T} > \mathbf{t}) = \nu_{\mathbf{1}}(\bullet).$$

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$$\lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t} \in \bullet | \mathbf{T} > \mathbf{t}) = \nu_\mathbf{1}(\bullet).$$

- Theorem 6 For the one-dimensional diffusions with killing, if the limit inferior of the killing at infinity is larger than the zero, then there is exactly one quasi-stationary distribution ν_1, and that this distribution attracts all initial distributions ν, that is

$$\lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t} \in \bullet | \mathbf{T} > \mathbf{t}) = \nu_\mathbf{1}(\bullet).$$

Thank you all for your attention!