# Some Properties of NSFDEs

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# Outline

- Existence and Uniqueness
- Numerical Solutions
- Large Deviations

- Y. Ji, Q. Song, C. Yuan, Neutral stochastic differential delay equations with locally monotone coefficients, arXiv 2016
- J. Bao, Y. Ji, C. Yuan, Convergence of EM scheme for neutral stochastic differential delay equations, arXiv 2016
- J. Bao, C. Yuan, Large deviations for neutral functional SDEs with jumps, Stochastics 87 (2015) 48-70.

Let $\mathscr{C} = C([-\tau, 0]; \mathbb{R}^n)$ which is equipped with the uniform norm $\|\zeta\|_\infty := \sup_{-\tau \le \theta \le 0} |\zeta(\theta)|$ for $\zeta \in \mathscr{C}$. Let $D : \mathscr{C} \to \mathbb{R}^n$, and

$$f : \mathscr{C} \times [0, T] \to \mathbb{R}^n, \qquad g : \mathscr{C} \times [0, T] \to \mathbb{R}^{n \times m}$$

all Borel measurable. Consider the $n-$dimensional NSFDE

$$\mathrm{d}[X(t) - D(X_t)] = f(X_t, t)\mathrm{d}t + g(X_t, t)\mathrm{d}W(t) \quad \text{on } 0 \le t \le T, \tag{1}$$

with initial data $X_0 = \xi = \{\xi(\theta) : -\tau \le \theta \le 0\} \in L^2_{\mathscr{F}_0}([-\tau, 0] : \mathbb{R}^n)$, is an $\mathscr{F}_0-$measurable $\mathscr{C}-$valued random variable such that $\mathbb{E}\|\xi\|^2 < \infty$.

We have the following existence and uniqueness theorem.

### Theorem

(Mao's book, Theorem 6.2.5 ) *For any* $T > 0$ *and any integer* $n \geq 1$, *that there exist positive constants* $K_{T,n}$ *and* $K$ *such that*

(i) *for all* $t \in [0, T]$ *and* $\varphi, \phi \in \mathscr{C}$ *with* $\|\varphi\| \vee \|\phi\| \leq n$,

$$|f(\varphi, t) - f(\phi, t)|^2 \vee |g(\varphi, t) - g(\phi, t)|^2 \leq K_{n,T}\|\varphi - \phi\|^2; \quad (2)$$

(ii) *for all* $(\varphi, t) \in \mathbb{R}^n \times [0, T]$

$$|f(\varphi, t)|^2 \vee |g(\phi, t)|^2 \leq K(1 + \|\varphi\|^2). \quad (3)$$

*Assume also there exists* $\kappa \in (0, 1)$ *such that for all* $\varphi, \phi \in \mathscr{C}$

$$|D(\varphi) - D(\phi)| \leq \kappa\|\varphi - \phi\|. \quad (4)$$

*Then there exists a unique solution* $X(t)$ *to equation (1).*

A special class of NSFDEs is neutral stochastic differential delay equations, that is

$$\mathrm{d}[X(t)-G(X(t-\tau))] = b(X(t), X(t-\delta(t)), t)\mathrm{d}t + \sigma(X(t), X(t-\delta(t)), t)\mathrm{d}W(t) \tag{5}$$

on $t \in [0, T]$ with initial data $\xi$ where

$$G : \mathbb{R}^n \to \mathbb{R}^n,\ b : \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \to \mathbb{R}^n,\ \sigma : \mathbb{R}^n \times \mathbb{R}^n \times \times[0, T] \to \mathbb{R}^n \otimes \mathbb{R}^m.$$

## Theorem

(Mao book Theorem 6.2.5) *Assume that there exist positive constants* $K_{T,n}$ *and* $K$ *such that*

(i) *for all* $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$ *with* $|x| \vee |y| \vee |\overline{x}| \vee |\overline{y}| \leq n$ *and* $t \in [0, T]$

$$|b(x,y,t) - b(\overline{x},\overline{y},t)|^2 \vee |\sigma(x,y,t) - \sigma(\overline{x},\overline{y},t)|^2 \leq K_{T,n}(|x-\overline{x}|^2 + |y-\overline{y}|^2); \tag{6}$$

(iin) *for all* $(\varphi, t) \in \mathbb{R}^n \times [0, T]$

$$|b(x,y,t)|^2 \vee |\sigma(x,y,t)|^2 \leq K(1 + |x|^2 + |y|^2). \tag{7}$$

*Assume also there exists* $\kappa \in (0,1)$ *such that for all* $x, y \in \mathbb{R}^n$

$$|G(x) - G(y)| \leq \kappa |x - y|. \tag{8}$$

*Then there exists a unique solution* $X(t)$ *to equation (5) with initial data* $\xi$.

**Theorem (Ji, Song, Y.)** Assume $G$, $b$ and $\sigma$ satisfy the following assumptions for all $T, R \in [0, \infty)$:

(C0) For every $R > 0, T > 0$

$$\int_0^T \sup_{|x| \le R, |y| \le R} \{|b(x,y,t)| + \|\sigma(x,y,t)\|^2\} dt < \infty, \tag{9}$$

(C1) The functions $b(x,y,t), \sigma(x,y,t)$ are continuous in both $x$ and $y$ for all $t \in [0,T]$.

(C2) There exist two $\mathbb{R}_+-$ valued functions $K_1(t)$, $\widetilde{K}_1(t)$ and a positive constant $C_1(\tau)$ such that for all $t \in [0,T]$, $K_1(t) \le C_1(\tau) K_1(t-\tau)$, $K_1(t) \ge \widetilde{K}_1(t)$ and

$$2\langle x - G(y), b(x,y,t)\rangle + \|\sigma(x,y,t)\|^2 \le K_1(t)(1+|x|^2) + \widetilde{K}_1(t-\tau)(1+|y|^2), \tag{10}$$

for $\forall\ x, y \in \mathbb{R}^n$, $t \in [0,T]$.

## Theorem

(C3) *There exist two* $\mathbb{R}_+-$ *valued functions* $K_R(t)$, $\widetilde{K}_R(t)$ *and a positive constant* $C_R(\tau)$*such that for all* $t \in [0, T]$, $K_R(t) \leq C_R(\tau) K_R(t-\tau)$, $K_R(t) \geq \widetilde{K}_R(t)$ *and*

$$2\langle x - G(y) - \overline{x} + G(\overline{y}), b(x, y, t) - b(\overline{x}, \overline{y}, t)\rangle + \|\sigma(x, y, t) - \sigma(\overline{x}, \overline{y}, t)\|^2 \leq K_R(t)|x - \overline{x}|^2 + \widetilde{K}_R(t-\tau)|y - \overline{y}|^2 \quad (11)$$

*for all* $|x| \vee |y| \vee |\overline{x}| \vee |\overline{y}| \leq R$, $t \in [0, T]$.

(C4) *Assume* $G(0) = 0$ *and that there is a constant* $\kappa \in (0, 1)$ *such that*

$$|G(x) - G(y)| \leq \kappa |x - y|,$$

*holds for all* $x, y \in \mathbb{R}^n$.

**Theorem**

*Moreover, we assume* $C_1(\tau) \vee C_R(\tau) \leq \frac{1}{\kappa}$ *and* $K_1(t), \widetilde{K}_1(t), K_R(t), \widetilde{K}_R(t) \in \mathcal{L}^1([-\tau, T]; \mathbb{R}_+)$. *Then there exists a unique process* $\{X(t)\}_{t \in [0,T]}$ *that satisfies equation* $(5)$ *with the initial data* $\xi$. *Moreover, the mean square of the solution is finite.*

# Numerical Solutions

We assume that there exist constants $L > 0$ and $q \geq 1$ such that, for $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$,

(A1) $|G(y) - G(\overline{y})| \leq L(1 + |y|^q + |\overline{y}|^q)|y - \overline{y}|$;

(A2) $|b(x, y) - b(\overline{x}, \overline{y})| + \|\sigma(x, y) - \sigma(\overline{x}, \overline{y})\| \leq L|x - \overline{x}| + L(1 + |y|^q + |\overline{y}|^q)|y - \overline{y}|$, where $\|\cdot\|$ stands for the Hilbert-Schmidt norm;

(A3) $|\xi(t) - \xi(s)| \leq L|t - s|$ for any $s, t \in [-\tau, 0]$.

**Remark**: There are some examples such that **(A1)** and **(A2)** hold. For instance, if $G(y) = y^2, b(x,y) = \sigma(x,y) = ax + y^3$ for any $x, y \in \mathbb{R}$ and some $a \in \mathbb{R}$, Then both **(A1)** and **(A2)** hold by taking $V(y, \overline{y}) = 1 + \frac{3}{2}y^2 + \frac{3}{2}\overline{y}^2$ for arbitrary $y, \overline{y} \in \mathbb{R}$. We can show that Eq.(5) has a unique solution $\{X(t)\}$ under (**A1**) and (**A2**).

We now introduce the EM scheme associated with (5). Without loss of generality, we assume that $h = T/M = \tau/m \in (0,1)$ for some integers $M, m > 1$. For every integer $k = -m, \cdots, 0$, set $Y_h^{(k)} := \xi(kh)$, and for each integer $k = 1, \cdots, M-1$, we define

$$\begin{aligned} Y_h^{(k+1)} - G(Y_h^{(k+1-m)}) = Y_h^{(k)} - G(Y_h^{(k-m)}) + b(Y_h^{(k)}, Y_h^{(k-m)})h \\ + \sigma(Y_h^{(k)}, Y_h^{(k-m)})\Delta W_h^{(k)}, \end{aligned} \tag{12}$$

where $\Delta W_h^{(k)} := W((k+1)h) - W(kh)$.

For any $t \in [kh,(k+1)h)$, set $\overline{Y}(t) := Y_h^{(k)}$. Define the continuous-time EM approximation solution $Y(t)$ as below: For any $\theta \in [-\tau, 0]$, $Y(\theta) = \xi(\theta)$, and

$$\begin{aligned} Y(t) =& G(\overline{Y}(t-\tau)) + \xi(0) - G(\xi(-\tau)) + \int_0^t b(\overline{Y}(s), \overline{Y}(s-\tau))\mathrm{d}s \\ &+ \int_0^t \sigma(\overline{Y}(s), \overline{Y}(s-\tau))\mathrm{d}W(s), \qquad t \in [0,T]. \end{aligned} \tag{13}$$

The first main result in this paper is stated as below.

### Theorem

Under the assumptions **(A1)-(A3)**, one has

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X(t)-Y(t)|^p\right)\lesssim h^{p/2},\quad p\geq 2. \tag{14}$$

We have the following lemma.

**Lemma**

Under (**A1**) and (**A2**), for any $p \geq 2$ there exists a constant $C_T > 0$ such that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|X(t)|^p\Big) \vee \mathbb{E}\Big(\sup_{0\leq t\leq T}|Y(t)|^p\Big) \leq C_T, \tag{15}$$

and

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|\Gamma(t)|^p\Big) \lesssim h^{p/2}, \tag{16}$$

where $\Gamma(t) := Y(t) - \overline{Y}(t)$.

**Proof of Theorem .** We follow the Yamada-Watanabe approach to complete the proof of Theorem. For fixed $\kappa > 1$ and arbitrary $\varepsilon \in (0,1)$, there exists a continuous non-negative function $\varphi_{\kappa\varepsilon}(\cdot)$ with the support $[\varepsilon/\kappa, \varepsilon]$ such that

$$\int_{\varepsilon/\kappa}^{\varepsilon} \varphi_{\kappa\varepsilon}(x)\mathrm{d}x = 1 \quad \text{and} \quad \varphi_{\kappa\varepsilon}(x) \leq \frac{2}{x \ln \kappa}, \quad x > 0.$$

Set

$$\phi_{\kappa\varepsilon}(x) := \int_0^x \int_0^y \varphi_{\kappa\varepsilon}(z)\mathrm{d}z\mathrm{d}y, \quad x > 0.$$

Let

$$V_{\kappa\varepsilon}(x) = \phi_{\kappa\varepsilon}(|x|), \quad x \in \mathbb{R}^n. \tag{17}$$

By a straightforward calculation, it holds

$$(\nabla V_{\kappa\varepsilon})(x) = \phi'_{\kappa\varepsilon}(|x|)|x|^{-1}x, \qquad x \in \mathbb{R}^n$$

and

$$(\nabla^2 V_{\kappa\varepsilon})(x) = \phi'_{\kappa\varepsilon}(|x|)(|x|^2\mathbf{I} - x\otimes x)|x|^{-3} + |x|^{-2}\phi''_{\kappa\varepsilon}(|x|)x\otimes x, \qquad x \in \mathbb{R}^n,$$

where $\nabla$ and $\nabla^2$ stand for the gradient and Hessian operators, respectively, $x \otimes x = xx^*$ with $x^*$ being the transpose of $x \in \mathbb{R}^n$. Moreover, we have

$$|(\nabla V_{\kappa\varepsilon})(x)| \leq 1 \quad \text{and} \quad \|(\nabla^2 V_{\kappa\varepsilon})(x)\| \leq 2n\Big(1 + \frac{1}{\ln\kappa}\Big)\frac{1}{|x|}\mathbf{1}_{[\varepsilon/\kappa,\varepsilon]}(|x|), \tag{18}$$

For notation simplicity, set

$$Z(t) := X(t) - Y(t) \quad \text{and} \quad \Lambda(t) := Z(t) - G(X(t-\tau)) + G(\overline{Y}(t-\tau)). \tag{19}$$

In the sequel, let $t \in [0, T]$ be arbitrary and fix $p \geq 2$. Due to $\Lambda(0) = \mathbf{0} \in \mathbb{R}^n$ and $V_{\kappa\varepsilon}(\mathbf{0}) = 0$, an application of Itô's formula gives

$$V_{\kappa\varepsilon}(\Lambda(t)) = \int_0^t \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_1(s) \rangle \mathrm{d}s$$
$$+ \frac{1}{2} \int_0^t \text{trace}\{(\Gamma_2(s))^* (\nabla^2 V_{\kappa\varepsilon})(\Lambda(s)) \Gamma_2(s)\} \mathrm{d}s + \int_0^t \langle \nabla(V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_2(s) \mathrm{d}W$$
$$=: I_1(t) + I_2(t) + I_3(t),$$

where

$$\Gamma_1(t) := b(X(t), X(t-\tau)) - b(\overline{Y}(t), \overline{Y}(t-\tau)) \tag{20}$$

and

$$\Gamma_2(t) := \sigma(X(t), X(t-\tau)) - \sigma(\overline{Y}(t), \overline{Y}(t-\tau)).$$

Set

$$V(x,y) := 1 + |x|^q + |y|^q, x, y \in \mathbb{R}^n. \tag{21}$$

According to (15), for any $q \geq 2$ there exists a constant $C_T > 0$ such that

$$\mathbb{E}\Big( \sup_{0\leq t\leq T} V(X(t-\tau), \overline{Y}(t-\tau))^q \Big) \leq C_T. \tag{22}$$

Noting that

$$X(t) - \overline{Y}(t) = \Lambda(t) + \Gamma(t) + G(X(t-\tau)) - G(\overline{Y}(t-\tau)), \tag{23}$$

and using Hölder's inequality and B-D-G's inequality, we get from (18) and (**A1**)-(**A2**) that

$$\begin{aligned}\Theta(t) :&= \mathbb{E}\Big( \sup_{0\leq s\leq t} |I_1(s)|^p \Big) + \mathbb{E}\Big( \sup_{0\leq s\leq t} |I_3(s)|^p \Big) \\ &\lesssim \int_0^t \{\mathbb{E}|\Gamma_1(s)|^p + \mathbb{E}\|\Gamma_2(s)\|^p\}\mathrm{d}s \\ &\lesssim \int_0^t \mathbb{E}\{|\Lambda(s)|^p + |\Gamma(s)|^p\}\mathrm{d}s + + \int_{-\tau}^{t-\tau} \mathbb{E}(V(X(s), \overline{Y}(s))^p |X(s) - \overline{Y}(s)|^p)\mathrm{d}s\end{aligned} \tag{24}$$

In the light of (18)-(23), we derive from (**A1**) that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le s\le t}|I_2(s)|^p\Big)\\
&\lesssim \mathbb{E}\int_0^t \|(\nabla^2 V_{\kappa\varepsilon})(\Lambda(s))\|^p\|\Gamma_2(s)\|^{2p}\mathrm{d}s\\
&\lesssim \mathbb{E}\int_0^t \frac{1}{|\Lambda(s)|^p}\{|X(s)-\overline{Y}(s)|^{2p}+V(X(s-\tau),\overline{Y}(s-\tau))^{2p}\\
&\quad\times(|X(s-\tau)-\overline{Y}(s-\tau)|^{2p})\}\mathbf{I}_{[\varepsilon/\kappa,\varepsilon]}(|\Lambda(s)|)\mathrm{d}s\\
&\lesssim \mathbb{E}\int_0^t \frac{1}{|\Lambda(s)|^p}\{|\Lambda(s)|^{2p}+|\Gamma(s)|^{2p}+|G(X(s-\tau))-G(\overline{Y}(s-\tau))|^{2p}\\
&\quad+V(X(s-\tau),\overline{Y}(s-\tau))^{2p}(|X(s-\tau)-\overline{Y}(s-\tau)|^{2p})\}\mathbf{I}_{[\varepsilon/\kappa,\varepsilon]}(|\Lambda(s)|)\mathrm{d}s\\
&\lesssim \mathbb{E}\int_0^t\{|\Lambda(s)|^p+\varepsilon^{-p}|\Gamma(s)|^{2p}\\
&\quad+\varepsilon^{-p}V^{2p}(X(s-\tau),\overline{Y}(s-\tau))(|X(s-\tau)-\overline{Y}(s-\tau)|^{2p})\}\mathrm{d}s\\
&\lesssim \int_0^t\{\varepsilon^{-p}h^p+\mathbb{E}|\Lambda(s)|^p+\varepsilon^{-p}(\mathbb{E}(|Z(s-\tau)|^{4p}))^{1/2}\}\mathrm{d}s,
\end{aligned}
\tag{25}
$$

Thus, we can have

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t}|\Lambda(s)|^p\Big) &\lesssim \varepsilon^p + h^{p/2} + \varepsilon^{-p}h^p \\ &\quad + \int_0^{(t-\tau)\vee 0}\{(\mathbb{E}(|Z(s)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(s)|^{4p}))^{1/2}\}\mathrm{d}s \\ &\le h^{p/2} + \int_0^{(t-\tau)\vee 0}\{(\mathbb{E}(|Z(s)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(s)|^{4p}))^{1/2}\}\mathrm{d}s,\end{aligned} \tag{26}$$

it follows from Hölder's inequality that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}|Z(t)|^p\Big) &\lesssim \mathbb{E}\Big(\sup_{0\le t\le T}|\Lambda(t)|^p\Big) + \mathbb{E}\Big(\sup_{-\tau\le t\le T-\tau}|G(X(t)) - G(\overline{Y}(t))|^p\Big) \\ &\lesssim \mathbb{E}\Big(\sup_{0\le t\le T}|\Lambda(t)|^p\Big) + \mathbb{E}\Big(\sup_{-\tau\le t\le T-\tau}(V(X(t),\overline{Y}(t))^p|X(t)-\overline{Y}(t)|^p)\Big) \\ &\lesssim \mathbb{E}\Big(\sup_{0\le t\le T}|\Lambda(t)|^p\Big) + h^{p/2} + \Big(\mathbb{E}\Big(\sup_{0\le t\le (T-\tau)\vee 0}|Z(t)|^{2p}\Big)\Big)^{1/2}.\end{aligned} \tag{27}$$

Substituting (34) into (36) yields that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|Z(t)|^p\Big) \lesssim h^{p/2} + \Big(\mathbb{E}\Big(\sup_{0\le t\le (T-\tau)\vee 0}|Z(t)|^{2p}\Big)\Big)^{1/2} + \int_0^{(T-\tau)\vee 0}\{(\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(t)|^{4p}))^{1/2}\}\mathrm{d}t. \tag{28}$$

Hence, we have

$$\mathbb{E}\Big(\sup_{0\le t\le \tau}|Z(t)|^p\Big) \lesssim h^{p/2},$$

and

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le 2\tau}|Z(t)|^p\Big) &\lesssim h^{p/2} + \Big(\mathbb{E}\Big(\sup_{0\le t\le \tau}|Z(t)|^{2p}\Big)\Big)^{1/2} \\ &\quad + \int_0^{\tau}\{(\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(t)|^{4p}))^{1/2}\}\mathrm{d}t \\ &\lesssim h^{p/2}\end{aligned}$$

by taking $\varepsilon = h^{1/2}$.

# Large deviation principle (LDP)

Consider a neutral FSDE

$$\mathrm{d}\{X(t) - G(X_t)\} = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t), \quad t > 0 \tag{29}$$

with the initial data $X_0 = \xi \in \mathscr{C}$. We assume

(H1) There exists $\kappa \in (0, 1)$ such that

$$|G(\xi) - G(\eta)| \leq \kappa\|\xi - \eta\|_\infty, \quad \xi, \eta \in \mathscr{C}.$$

(H2) The mapping $b$ satisfies a local Lipschitz condition and there exists $\lambda > 0$ such that

$$\langle(\xi(0) - \eta(0)) - (G(\xi) - G(\eta)), b(\xi) - b(\eta)\rangle \vee \|\sigma(\xi) - \sigma(\eta)\|_{HS}^2 \leq \lambda\|\xi - \eta\|_\infty^2$$

and

$$\langle \xi(0) - G(\xi), b(\xi)\rangle \leq \lambda(1 + \|\xi\|_\infty^2)$$

for arbitrary $\xi, \eta \in \mathscr{C}$.

Let $\mathbb{S}$ be a Polish space (i.e., a separable completely metrizable topological space), and $\{Y^\epsilon\}_{\epsilon\in(0,1)}$ a family of $\mathbb{S}$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

### Definition

A function $I : \mathbb{S} \mapsto [0, \infty]$ is called a rate function if it is lower semicontinuous. A rate function $I$ is called a good rate function if the level set $\{f \in \mathbb{S} : I(f) \leq a\}$ is compact for each $a < \infty$.

### Definition

The sequence $\{Y^\epsilon\}_{\epsilon\in(0,1)}$ is said to satisfy the LDP with rate function $I$ if, for each $A \in \mathscr{B}(\mathbb{S})$ (Borel $\sigma$-algebra generated by all open sets in $\mathbb{S}$),

$$-\inf_{f\in A^\circ} I(f) \le \liminf_{\epsilon\to 0} \epsilon \log \mu^\epsilon(A) \le \limsup_{\epsilon\to 0} \epsilon \log \mu^\epsilon(A) \le -\inf_{f\in \overline{A}} I(f),$$

where $\mu^\epsilon$ is the law of $\{Y^\epsilon\}_{\epsilon\in(0,1)}$, and the interior $A^\circ$ and closure $\overline{A}$ are taken in $\mathbb{S}$.

### Definition

The sequence $\{Y^\epsilon\}_{\epsilon\in(0,1)}$ is said to satisfy the Laplace principle (LP) on $\mathbb{S}$ with rate function $I$ if, for each bounded continuous mapping $g : \mathbb{S} \mapsto \mathbb{R}$,

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\Big(\exp\Big[-\frac{g(Y^\epsilon)}{\epsilon}\Big]\Big) = -\inf_{f\in\mathbb{S}}\{g(f) + I(f)\}.$$

Define the Cameron-Martin space $\mathbb{H}$ by

$$\mathbb{H} := \Big\{h : [0,T] \mapsto \mathbb{R}^m \Big| h(t) = \int_0^t \dot{h}(s)\mathrm{d}s,\ t \in [0,T],\ \text{and}\ \int_0^T |\dot{h}(s)|^2 \mathrm{d}s < \infty\Big\}, \tag{30}$$

where the dot denotes the generalized derivative. Note that $(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}}, \|\cdot\|_{\mathbb{H}})$ is a Hilbert space equipped with the norm $\|f\|_{\mathbb{H}} := (\int_0^T |\dot{f}(s)|^2 \mathrm{d}s)^{\frac{1}{2}}, f \in \mathbb{H}$. For each $N > 0$, let

$$S_N := \{h \in \mathbb{H} : \|h\|_{\mathbb{H}} \le N\} \tag{31}$$

be the ball in $\mathbb{H}$ with radius $N$, and

$\mathcal{A}_N := \{h : [0,T] \mapsto \mathbb{R}^m,$ an $\mathcal{F}_t-$predictable process such that $h(\cdot,\omega) \in S_N, \mathbb{P}-$a.s.

For $\mathcal{C} := C([0,T];\mathbb{R}^m)$ and a measurable mapping $\mathcal{G}^\epsilon : \mathcal{C} \mapsto \mathbb{S}, \epsilon \in (0,1)$, let

$$Z^\epsilon := \mathcal{G}^\epsilon(\sqrt{\epsilon}W). \tag{32}$$

Assume that there exists a measurable mapping $Z^0 : \mathbb{H} \mapsto \mathbb{S}$ such that, for any $N > 0$,

(i) If the family $\{h^\epsilon\}_{\epsilon\in(0,1)} \subset \mathcal{A}_N$ converge in distribution to an $h \in \mathcal{A}_N$, then $\mathcal{G}^\epsilon(\sqrt{\epsilon}W + h^\epsilon) \to Z^0(h)$ in distribution in $\mathbb{S}$ as $\epsilon \to 0$.

(ii) The set $\mathcal{K}_N := \{Z^0(h) : h \in S_N\}$ is a compact subset of $\mathbb{S}$.

**Lemma (Budhiraja and Dupuis,2000):** Let $\{Z^\epsilon\}_{\epsilon\in(0,1)}$ be defined by (32) and assume that $\{\mathcal{G}^\epsilon\}_{\epsilon\in(0,1)}$ satisfy (i) and (ii). Then the family $\{Z^\epsilon\}_{\epsilon\in(0,1)}$ satisfy the LP (hence LDP) on $\mathbb{S}$ with the good rate function defined by

$$I(f) := \frac{1}{2} \inf_{\{h\in\mathbb{H}, f=Z^0(h)\}} \|h\|_{\mathbb{H}}^2, \quad f \in \mathbb{S}. \tag{33}$$

For $\epsilon \in (0,1)$, consider the small perturbation of (29) in the form

$$\mathrm{d}\{X^\epsilon(t) - G(X_t^\epsilon)\} = b(X_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(X_t^\epsilon)\mathrm{d}W(t), \quad t \in [0,T] \tag{34}$$

with the initial value $X_0^\epsilon = \xi \in \mathscr{C}$.

By Lemma 10, to establish the LDP for the law of $\{X^\epsilon(\cdot)\}_{\epsilon\in(0,1)}$, it is sufficient to (a) choose the Polish space $\mathbb{S}$, (b) construct measurable mappings $\mathcal{G}^\epsilon : \mathcal{C} \mapsto \mathbb{S}$ and $Z^0 : \mathbb{H} \mapsto \mathbb{S}$ respectively, and then (c) show that (i) and (ii) are satisfied for the measurable mapping $\mathcal{G}^\epsilon$.

In the sequel, we take $\mathbb{S} = C([0,T];\mathbb{R}^n)$, the family of continuous functions $f : [0,T] \mapsto \mathbb{R}^n$, which is a Polish space under the uniform topology. On the other hand, by the Yamada-Watanabe theorem there exists a unique measurable functional $\mathcal{G}^\epsilon : \mathcal{C} \mapsto \mathbb{S}$ such that

$$X^\epsilon(t) = \mathcal{G}^\epsilon(\sqrt{\epsilon}W)(t), \quad t \in [0,T]. \tag{35}$$

For $h^\epsilon \in \mathcal{A}_N$, by the Girsanov theorem, we conclude from (34) and (35) that

$$X^{\epsilon,h^\epsilon}(t) := \mathcal{G}^\epsilon(\sqrt{\epsilon}W + h^\epsilon)(t), \quad t \in [0,T]$$

solves the following equation

$$\mathrm{d}\{X^{\epsilon,h^\epsilon}(t) - G(X_t^{\epsilon,h^\epsilon})\} = b(X_t^{\epsilon,h^\epsilon})\mathrm{d}t + \sigma(X_t^{\epsilon,h^\epsilon})\dot{h}^\epsilon(t)\mathrm{d}t + \sqrt{\epsilon}\sigma(X_t^{\epsilon,h^\epsilon})\mathrm{d}W(t), \tag{36}$$

with initial datum $X_0^{\epsilon,h^\epsilon} = \xi \in \mathscr{C}$. For $h \in \mathbb{H}$, consider a deterministic equation

$$\mathrm{d}\{X^h(t) - G(X_t^h)\} = \{b(X_t^h) + \sigma(X_t^h)\dot{h}(t)\}\mathrm{d}t \tag{37}$$

with initial value $X_0^h = \xi \in \mathscr{C}$ and define

$$X^0(h) := X^h. \tag{38}$$

## Lemma

Let (H1) and (H2) hold. For $p \geq 2$, $h \in S_N$ and $h^\epsilon \in \mathcal{A}_N$,

$$\sup_{-\tau \leq t \leq T} |X^h(t)|^p \vee \mathbb{E}\Big( \sup_{-\tau \leq t \leq T} |X^{\epsilon,h^\epsilon}(t)|^p \Big) \leq C.$$

## Lemma

Under (H1) and (H2), $\mathcal{K}_N = \{X^0(h) : h \in S_N\}$ is a compact subset of $\mathbb{S}$.

## Lemma

Let (H1) and (H2) hold and assume further that the family $\{h^\epsilon\}_{\epsilon \in (0,1)} \subset \mathcal{A}_N$ converge almost surely in $\mathbb{H}$ to $h \in \mathcal{A}_N$. Then $X^{\epsilon,h^\epsilon} \to X^h$ converges in distribution in $\mathbb{S}$ as $\epsilon \to 0$.

**Theorem**

Under (H1) and (H2), $X^\epsilon$ satisfies the LDP on $\mathbb{S}$ with the good rate function $I(f)$ defined by (33), where $X^0(h)$ solves Eq. (16).

# LDP for Neutral SDDEs

Consider a neutral SDDE on $\mathbb{R}^n$

$$\mathrm{d}\{Y(t) - G(Y(t-\tau))\} = b(Y(t), Y(t-\tau))\mathrm{d}t + \sigma(Y(t), Y(t-\tau))\mathrm{d}W(t) \tag{39}$$

with the initial data $Y_0 = \xi$. Assume that there exist $\lambda_3, \lambda_4 > 0$ such that

(A1) $|G(x) - G(y)| \leq \lambda_3 V_1(x,y)|x-y|, \ x, y \in \mathbb{R}^n$.

(A2) $|b(x_1,y_1) - b(x_2,y_2)| \vee \|\sigma(x_1,y_1) - \sigma(x_2,y_2)\|_{HS} \leq \lambda_4(|x_1 - x_2| + V_2(y_1,y_2)|y_1 - y_2|), \ x_i, y_i \in \mathbb{R}^n, i = 1, 2$, where $V_i : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}_+$ such that

$$V_i(x,y) \leq \lambda_i(1 + |x|^{q_i} + |y|^{q_i}), \quad x, y \in \mathbb{R}^n$$

for some $\lambda_i > 0$ and $q_i \geq 1, i = 1, 2$.

For $\epsilon \in (0,1)$, consider the small perturbation of (39)

$$\mathrm{d}\{Y^\epsilon(t)-G(Y^\epsilon(t-\tau))\} = b(Y^\epsilon(t),Y^\epsilon(t-\tau))\mathrm{d}t+\sqrt{\epsilon}\sigma(Y^\epsilon(t),Y^\epsilon(t-\tau))\mathrm{d}W(t) \tag{40}$$

By the Yamada-Watanabe theorem there exists a unique measurable functional $\mathcal{G}^\epsilon : \mathcal{C} \mapsto \mathbb{S}$ such that

$$Y^\epsilon(t) = \mathcal{G}^\epsilon(\sqrt{\epsilon}W)(t), \quad t \in [0,T]. \tag{41}$$

Then, for $h^\epsilon \in \mathcal{A}_N$, by the Girsanov theorem, (40) and (41),

$$Y^{\epsilon,h^\epsilon}(t) := \mathcal{G}^\epsilon(\sqrt{\epsilon}W + h^\epsilon)(t), \quad t \in [0,T]$$

solves the following equation

$$\begin{aligned} &\mathrm{d}\{Y^{\epsilon,h^\epsilon}(t) - G(Y^{\epsilon,h^\epsilon}(t-\tau))\} \\ &= b(Y^{\epsilon,h^\epsilon}(t), Y^{\epsilon,h^\epsilon}(t-\tau))\mathrm{d}t \\ &\quad + \sigma(Y^{\epsilon,h^\epsilon}(t), Y^{\epsilon,h^\epsilon}(t-\tau))\dot{h}^\epsilon(t)\mathrm{d}t \\ &\quad + \sqrt{\epsilon}\sigma(Y^{\epsilon,h^\epsilon}(t), Y^{\epsilon,h^\epsilon}(t-\tau))\mathrm{d}W(t) \end{aligned} \tag{42}$$

For any $h \in \mathbb{H}$, we introduce the skeleton equation associated with (39)

$$\mathrm{d}\{Y^h(t) - G(Y^h(t-\tau))\} = \{b(Y^h(t), Y^h(t-\tau)) + \sigma(Y^h(t), Y^h(t-\tau))\dot{h}(t)\}\mathrm{d}t \tag{43}$$

with $Y_0^h = \xi$. Define

$$Y^0(h) := Y^h, \quad h \in \mathbb{H}. \tag{44}$$

**Lemma**

Let (A1) and (A2) hold. For any $p \geq 2$, $h \in S_N$ and $h^\epsilon \in \mathcal{A}_N$,

$$\sup_{-\tau \leq s \leq T} |Y^h(s)|^p \vee \mathbb{E}\Big( \sup_{-\tau \leq s \leq T} |Y^{\epsilon, h^\epsilon}(s)|^p \Big) \leq C. \tag{45}$$

**Lemma**

Under (A1) and (A2), $\mathcal{K}_N = \{Y^0(h) : h \in S_N\}$ is a compact subset of $\mathbb{S}$.

**Lemma**

Assume that the family $\{h^\epsilon\}_{\epsilon\in(0,1)} \subset \mathcal{A}_N$ converge almost surely in $\mathbb{H}$ to $h \in \mathcal{A}_N$. Then $Y^{\epsilon,h^\epsilon} \to Y^h$ converges in distribution in $\mathbb{S}$.

Our second main result is:

**Theorem**

Under (A1) and (A2), $Y^\epsilon = \{Y^\epsilon(t)\}_{t\in[0,T]}$, the solution of (40), satisfies the LDP on $\mathbb{S}$ with the good rate function $I(f)$ defined by (33), where $Y^0(h)$ solves (43).

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