Stochastic SIR Models

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Outline

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- We consider epidemic and ecological models subject to noise perturbations.
- By analyzing the dynamics near the boundary and ergodicity of the dynamics on the boundary, we find sufficient and almost necessary conditions for extinction and permanence of the populations.
- We study the ergodicity of the models, establish the convergence, and estimate the rates of convergence.

A Stochastic SIR Model

Ronald Ross, winner of Nobel Prize in Medicine

Figure 1: Sir Ronald Ross, May 13, 1857-Sept. 16, 1932: a British medical doctor who received the Nobel Prize for Physiology or Medicine in 1902 for his work on the transmission of malaria, becoming the first British Nobel laureate, and the first born outside of Europe. (“mathematical models of malaria epidemiology”) [Giovanni Grassi’s work was more directly relevant to human health (correctly identified the mosquito species as Anopheles claviger and established the complete life cycle of the first human malarial parasite).]

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- One of the classic epidemic models is the SIR (Susceptible-Infected-Removed) model that is suitable for modeling some diseases with permanent immunity such as rubella, whooping cough, measles, smallpox, etc.

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- (S): The susceptible class, the class of those individuals who are capable of contracting the disease and becoming infected,
- (I): the infected class, the class of those individuals who are capable of transmitting the disease to others,
- (R): the removed class, the class of infected individuals who have recovered, and are permanently immune, or are isolated.

Deterministic SIR Model

The spread of infection can be formulated by the following deterministic system of differential equations:

$$\begin{cases} dS(t) = (\alpha - \beta S(t)I(t) - \mu S(t))dt \\ dI(t) = (\beta S(t)I(t) - (\mu + \rho + \gamma)I(t))dt \\ dR(t) = (\gamma I(t) - \mu R(t))dt, \end{cases} \tag{2.1}$$

where

- α is the per capita birth rate of the population,
- β is the effective per capita contact rate
- μ is the per capita disease-free death rate and
- ρ is the excess per capita death rate of infected class,
- γ is per capita recovery rate of the infected individuals.

Deterministic Model: Threshold λ_d

For the above deterministic model (2.1),

- if $\lambda_d = \frac{\beta\alpha}{\mu} - (\mu + \rho + \gamma) \leq 0$, then the population tends to the disease-free equilibrium $(\frac{\alpha}{\mu}, 0, 0)$;
- if $\lambda_d > 0$, the population approaches an endemic equilibrium.

Thus, using the critical threshold value λ_d, the asymptotic behavior of the system has been completely classified.

Stochastic Nondegenerate Model

- It is well recognized that random effect is often not avoidable and a population is always subject to random disturbances. Thus, it is important to investigate stochastic epidemic models.
- A typical stochastic SIR model is

$$\begin{cases} dS(t) = (\alpha - \beta S(t)I(t) - \mu S(t))dt + \sigma_1 S(t)dB_1(t) \\ dI(t) = (\beta S(t)I(t) - (\mu + \rho + \gamma)I(t))dt + \sigma_2 I(t)dB_2(t) \\ dR(t) = (\gamma I(t) - \mu R(t))dt + \sigma_3 R(t)dB_3(t), \end{cases} \tag{2.2}$$

where $B_1(t)$, $B_2(t)$, and $B_3(t)$ are mutually independent Brownian motions, σ_1, σ_2, and σ_3 are the intensities of the white noises.

Stochastic Degenerate Model

- Moreover, in reality, the classes (S), (I), and (R) are usually subject to the same random factors such as temperature, humidity, pollution and other extrinsic influences. As a result, it is more plausible to assume that the random noises perturbing the three classes are from the same source.

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- If we assume that the Brownian motions $B_1(t)$, $B_2(t)$, and $B_3(t)$ are the same, we obtain the following model

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- One of the important questions is whether or not the disease will survive permanently. Our main goal is to provide a classification for survival (permanence) and extinction of the disease.

- We shall derive a sufficient and almost necessary condition for permanence (as well as ergodicity) and extinction of the disease for the stochastic SIR model by using a value λ, which is similar to λ_d in the deterministic model.

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- Because the dynamics of the recovered class has no effect on the disease transmission dynamics, we only consider the following system:

$$\begin{cases} dS(t) = [\alpha - \beta S(t)I(t) - \mu S(t)]dt + \sigma_1 S(t)dB(t), \\ dI(t) = [\beta S(t)I(t) - (\mu + \rho + \gamma)I(t)]dt + \sigma_2 I(t)dB(t). \end{cases} \tag{2.4}$$

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- Using standard arguments, it can be easily shown that for any positive initial value $(u, v) \in \mathbb{R}^{2,\circ}_+$, there exists a unique global solution $(S_{u,v}(t), I_{u,v}(t)), t \geq 0$ that remains in $\mathbb{R}^{2,\circ}_+$ w.p.1.

- To obtain further properties of the solution, we first consider the equation on the boundary (setting $I(t) = 0$ in the first eq.),

$$d\widehat{S}(t) = (\alpha - \mu\widehat{S}(t))dt + \sigma_1\widehat{S}(t)dB(t). \tag{2.5}$$

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- Let $\widehat{S}_u(t)$ be the solution to (2.5) with initial value u. The process $\widehat{S}_u(t)$ has a unique invariant probability measure (inverse Gamma distribution) with density

$$f^*(x) = \frac{b^a}{\Gamma(a)}x^{-(a+1)}e^{\frac{-b}{x}}, x > 0 \tag{2.6}$$

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- By the strong law of large numbers

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \widehat{S}_u(s)ds = \int_0^\infty xf^*(x)dx := \frac{\alpha}{\mu} \text{ a.s.} \tag{2.7}$$

Growth Rate or Lyapunov Exponent λ

To proceed, we claim the threshold is follows:

$$\lambda := \frac{\alpha\beta}{\mu} - \left(\mu + \rho + \gamma + \frac{\sigma_2^2}{2}\right). \tag{2.8}$$

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- To determine whether or not $I(t)$ converges to 0, we consider the Lyapunov exponent of $I(t)$ when $I(t)$ is small for a sufficiently long time. Hence, we look at the following equation derived from Itô's formula:

$$\frac{\ln I(T)}{T} = \frac{\ln I(0)}{T} + \frac{\sigma_2}{T} B(T) + \frac{1}{T} \int_0^T \left(\beta S(t) - (\mu + \rho + \gamma + \frac{\sigma_2^2}{2})\right) dt.$$

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- When T is large, the first and the second terms on the right-hand side of the above equation are small. Intuitively, if $I(t)$ is small for $t \in [0, T]$, $S(t)$ is close to $\widehat{S}(t)$. Using the ergodic mean of $\widehat{S}(t)$,

$$\begin{aligned}
&\frac{1}{T}\int_0^T \left(\beta S(t) - (\mu + \rho + \gamma + \frac{\sigma_2^2}{2})\right) dt \\
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- Thus, $\frac{\ln I(T)}{T}$ is close to λ.
- As a result, if $\lambda < 0$, $I(t)$ is likely to decay exponentially.
- if $\lambda > 0$, $I(t)$ cannot be small for a long time, that is the disease will survive permanently.

Extinction

Theorem 2.1

If $\lambda < 0$*, then for any initial value* $(S(0), I(0)) = (u, v) \in \mathbb{R}^{2,\circ}_+$ *we have* $\lim_{t\to\infty} \frac{\ln I_{u,v}(t)}{t} = \lambda$ *a.s. and the distribution of* $S_{u,v}(t)$ *converges weakly to the unique invariant probability measure* μ^* *with the density* f^**. As a result, the disease will go to extinction in the sense that*

$$\mathbb{P}\left\{\lim_{t\to\infty} I_{u,v}(t) = 0\right\} = 1 \text{ for any initial value} (u, v) \in \mathbb{R}^{2,\circ}_+$$

This theorem is obtained mainly by comparing $S(t)$ and $\widehat{S}(t)$.

Example 1

- Consider (2.4) with parameters $\alpha = 5$, $\beta = 5$, $\mu = 4$, $\rho = 1$, $\gamma = 1$, $\sigma_1 = 2$, and $\sigma_2 = -1$.
- It can be shown that $\lambda = -1.75 < 0$
- Our Theorem indicates $I_{u,v}(t) \to 0$ a.s. as $t \to \infty$. This claim is supported by Figures 2. That is, the population will eventually have no disease. The distribution of $S_{u,v}(t)$ convergence to $f^*(x)$ as $t \to \infty$. The graphs of $f^*(x)$ and empirical density of $S_{u,v}(t)$ at $t = 50$ are illustrated by Figure 3.

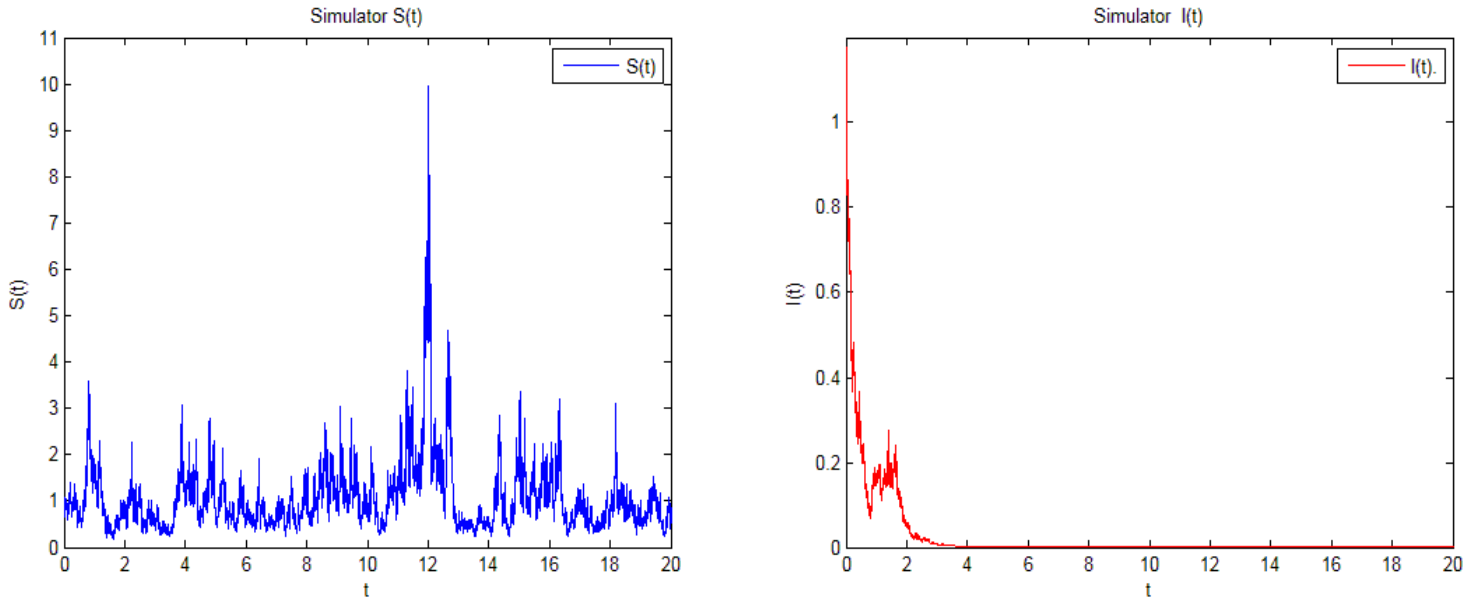

Figure 2: Trajectories of $S_{u,v}(t), I_{u,v}(t)$ in Example 1.

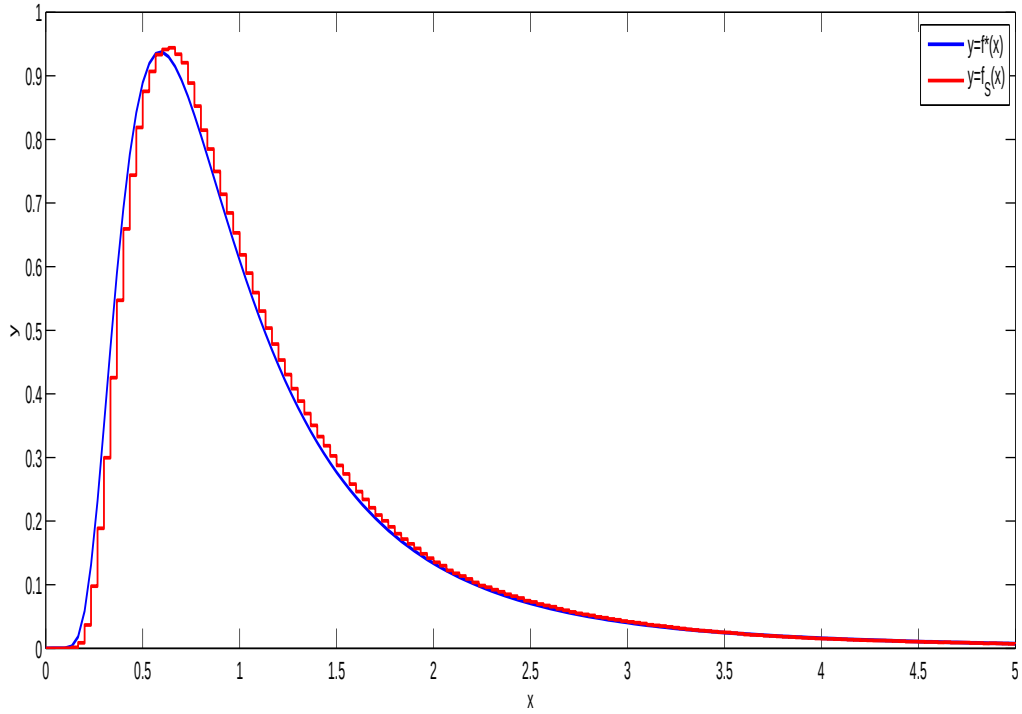

Figure 3: The graph of the stationary density f^* (in blue) and the graph of the empirical density of $S(t)$ (in red) in Example 1.

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- Because the diffusion is degenerate, we need to verify certain condition holds.
  - Hörmander's condition is a property of vector fields (in differential geometry) that we need here.
  - In the following development, we will use some idea from geometric control theory...

Las Hörmander

Figure 4: Lars Valter Hörmander, Jan. 24, 1931-Nov. 25, 2012: A Swedish mathematician who has been called “the foremost contributor to the modern theory of linear partial differential equations”. He was awarded the Fields Medal in 1962, the Wolf Prize in 1988, and the Leroy P. Steel Prize in 2006.

- Consider

$$A(x,y) = \begin{pmatrix} \alpha - c_1 x - \beta xy \\ -c_2 y + \beta xy \end{pmatrix} \text{ and } B(x,y) = \begin{pmatrix} \sigma_1 x \\ \sigma_2 y \end{pmatrix}.$$

- If $\Phi(x,y) = (\Phi_1, \Phi_2)^\top$ and $\Psi(x,y) = (\Psi_1, \Psi_2)^\top$ are vector fields on $\mathbb{R}^2$ then the Lie bracket $[\Phi, \Psi]$ is a vector field given by

$$[\Phi,\Psi]_j(x,y) = \left(\Phi_1 \frac{\partial \Psi_j}{\partial x}(x,y) - \Psi_1 \frac{\partial \Phi_j}{\partial x}(x,y)\right) + \left(\Phi_2 \frac{\partial \Psi_j}{\partial y}(x,y) - \Psi_2 \frac{\partial \Phi_j}{\partial y}(x,y)\right)$$

- The diffusion satisfies Hörmander's condition if the set of vectors $B, [A,B], [A,[A,B]], [B,[A,B]], \ldots$ spans $\mathbb{R}^2$ at every $(x,y) \in \mathbb{R}^{2,\circ}_+$

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- If $d^* \leq 0$, the support is the whole space $\mathbb{R}_+^{2,\circ}$.
- If $d^* > 0$, the support is a subset of $\mathbb{R}_+^{2,\circ}$.

Estimating convergence rate

- To obtain convergence rate for the process $(S(t), I(t))$, we first work with its "skeleton", that is, the Markov chain $(S(nT), I(nT))_{n\in\mathbb{N}}$ where T is a sufficiently large number so that the ergodicity takes effects.
- We can obtain that

$$\mathbb{E}\ln I(T) > \ln I(0) + \frac{\lambda T}{2} \text{ if } I(0) \text{ is sufficiently small}$$

If we could somehow interchange expectation and exponentiation, we would obtain

$$\mathbb{E}\frac{1}{I(T)} < \frac{k}{I(0)}, k \in (0,1) \text{ if } I(0) \text{ is sufficiently small} \tag{2.9}$$

- This would show an exponential rate of convergence. Unfortunately, we are unable to obtain (2.9).

- We are looking for a slower rate: polynomial one.
- We use a Lyapunov-type result saying that if there is

$$\mathbb{E}V\big(S_{u,v}(T), I_{u,v}(T)\big) \leq V(u,v) - \kappa V^{\gamma}(u,v) \tag{2.10}$$

for some $\gamma \in (0,1)$ then

$$\lim_{n\to\infty} n^{\frac{\gamma}{1-\gamma}} \| P(nT, (u,v), \cdot - \mu^* \|_{TV} \to 0$$

- With some technical arguments, we are able to show that, for $p \in (0, 1]$

$$\mathbb{E}|\ln I(T)|^{1+p} < |\ln I(0)|^{1+p} - k|\ln I(0)| \tag{2.11}$$

  when $I(0)$ is sufficiently small.

- Then we obtain the polynomial convergence rate with degree $\frac{1}{p}$.
- Since we can take any p in $(0, 1]$, we show that the convergence rate is bounded above by any polynomial rate.

Main theorem for permanence of the disease

Theorem 2.2

Let $\lambda > 0$. There exists an invariant probability measure π^* such that

$$\lim_{t\to\infty} t^q \|P(t,(u,v),\cdot) - \pi^*(\cdot)\|_{TV} = 0 \;\forall (u,v) \in \mathbb{R}^{2,\circ}_+, \tag{2.12}$$

where $\|\cdot\|$ is the total variation norm and q is any positive number. The support of π^* is $\{(u,v) \in \mathbb{R}^{2,\circ}_+ : u^r v \geq d^*\}$. Moreover, for any initial value $(u,v) \in \mathbb{R}^{2,\circ}_+$ and a π^*-integrable function f we have

$$\mathbb{P}\Big\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f\big(S_{u,v}(t), I_{u,v}(t)\big)\,dt = \int_{\mathbb{R}^{2,\circ}_+} f(u',v')\pi^*(du',dv') \Big\} = 1. \tag{2.13}$$

Example 2

- Consider (2.4) with parameters $\alpha = 20$, $\beta = 4$, $\mu = 1$, $\rho = 10$, $\gamma = 1$, $\sigma_1 = 1$, and $\sigma_2 = -1$.
- Direct calculation shows that $\lambda = 67.5 > 0$, $d^* = 1.9375$. By virtue of Theorem 2.2, (2.4) has a unique invariant probability measure π^* whose support is $\{(u, v) : u \geq \frac{1.9375}{v}\}$.
- Consequently, the strong law of large numbers and the convergence in total variation norm of the transition probability hold.

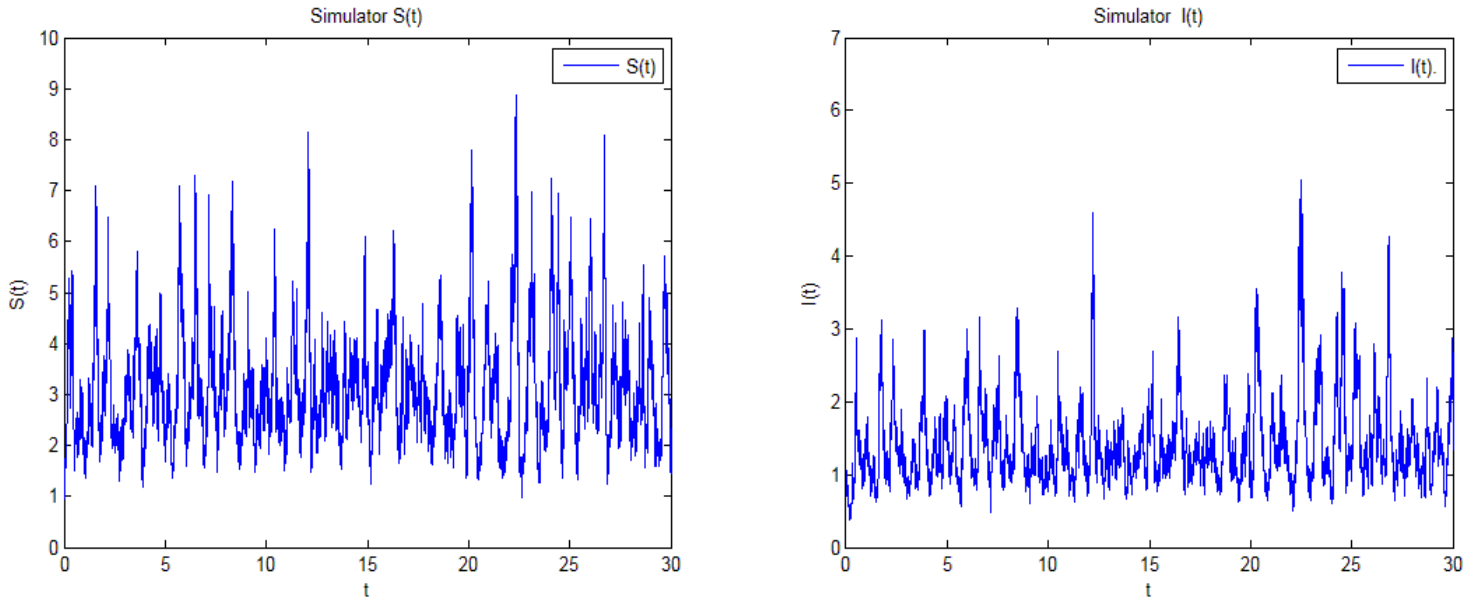
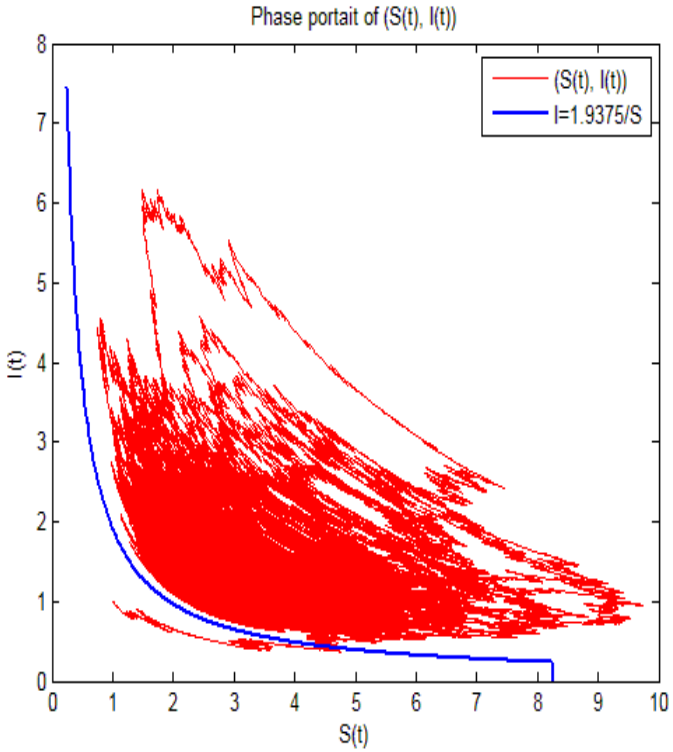


Figure 5: Trajectories of $S_{u,v}(t), I_{u,v}(t)$



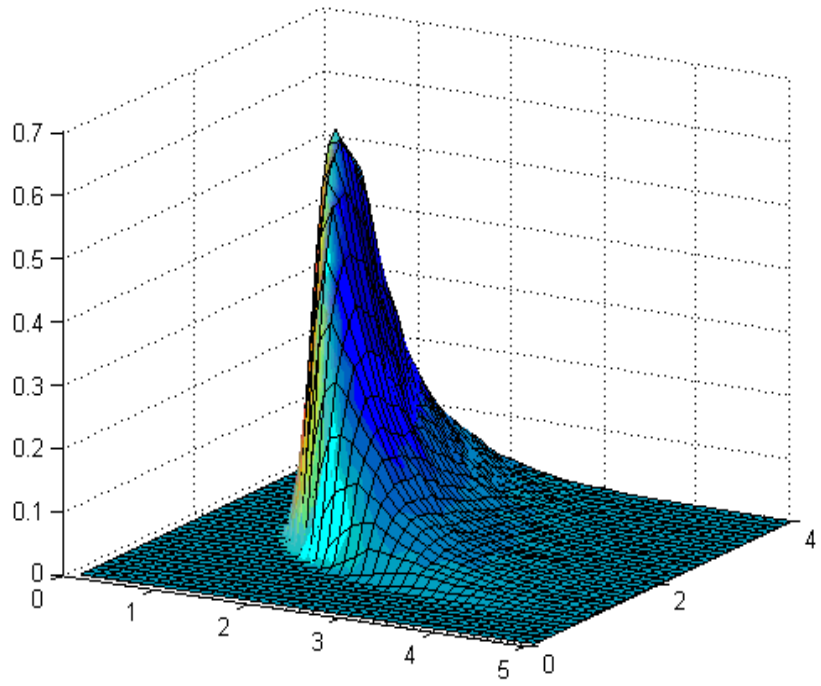


Figure 6: Phase portrait of (2.4); the boundary $s = \frac{1.9375}{i}$ of the support of π^* and the empirical density of $(S(t), I(t))$, which is approximate to the density of π^* in Example 2.

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- The main results can also be obtained if both white noise and colored noise are included. The colored noise is modeled by a Markov switching process.
- In this setting, some interesting properties are:
  - The switching could help to remove the degeneracy, so that the process has properties similar to the nondegenerate case. For instance, the support of the invariant measure could be the whole space
  - Moreover, the extinction at each single state may be avoided by the switching. Nevertheless, the switching could lead to extinction even if each state is permanent.

Thank you