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An introduction to stochastic filtering and optimal control

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Outline

1. Stochastic filtering
2. Optimal control
3. Combined problem

1.1 Basic model

Signal: X_t Stochastic process, i.e. fix t, $X_t(\omega)$ is a r.v.; fix ω, $X_t(\omega)$ a function of t.

Observation:

$$y_t = h_t(X_t) + n_t,$$

where n_t is white noise, i.e., i.i.d. with mean 0.

Note that, fix ω, as a function of t, $n_t(\omega)$ is not an ordinary function, it is a generalized function. This in not nice! What should we do?

Let

$$W_t = \int_0^t n_s ds.$$

W_t is a continuous process, called the Brownian motion.
Observation model:

$$Y_t = \int_0^t h_s(X_s) ds + W_t,$$

where

$$Y_t = \int_0^t y_s ds$$

is accumulated observation process.

Note that Y and y provide the same amount of information.

$$\mathcal{G}_t = \sigma(Y_s : \ s \le t\} = \sigma(y_s : \ s \le t).$$

Filtering problem: Estimate $f(X_t)$ based on information $\mathcal{G}_t$ generated by Y_s up to time t.

1.2. Example in Finance: Model for stock price S_t

$$\frac{dS_t}{S_t} = \mu_t dt + \sum_{i=1}^{n} \sigma_t^i dW_t^i,$$

where

$$\mu_t = \text{appreciation rate}$$

$$\sigma_t^i = \text{volatility due to } i\text{th random factor.}$$

Observation: $\{S_r : r \leq t\}$.

Applying Itô's formula,

$$\begin{aligned} d\log S_t &= \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}\sum_{i=1}^{n}(\sigma_t^i)^2 dt \\ &= (\mu_t - \frac{1}{2}a_t)dt + \sum_{i=1}^{n}\sigma_t^i dW_t^i, \end{aligned}$$

where

$$a_t = \sum_{i=1}^{n}(\sigma_t^i)^2.$$

Calculate quadratic variation

$$\langle \log S \rangle_t = \lim_{m \to \infty} \sum_{j=1}^{m} (\log S_{t_j} - \log S_{t_{j-1}})^2 = \int_0^t a_s ds,$$

where $0 = t_0 < t_1 < \cdots < t_m = t$ is a partition of $[0, t]$. So, a_t is observable.

By Martingale rep. theorem,

$$d\log S_t = (\mu_t - \frac{1}{2}a_t)dt + \sigma_t dW_t,$$

where $\sigma_t = \sqrt{a_t}$ and W is a B.M.

Let

$$Y_t = \int_0^t \sigma_s^{-1} d \log S_s.$$

Then

$$Y_t = \int_0^t h_s(\mu_s) ds + W_t,$$

where

$$h_t(\mu) = \sigma_t^{-1}(\mu - \frac{1}{2} a_t).$$

How to estimate μ_t based on $\mathcal{G}_t = \sigma(Y_s : \ s \le t)$ is a filtering problem.

1.3. Optimal filter

Find a r.v. ξ_t which is $\mathcal{G}_t$-measurable s.t.

$$\mathbb{E}\left(f(X_t) - \xi_t\right)^2$$

is minimized. Then

$$\xi_t = \mathbb{E}(f(X_t)|\mathcal{G}_t).$$

Let

$$\pi_t = P\left(X_t \in \cdot | \mathcal{G}_t\right).$$

Then, π_t is a random prob. meas. and

$$\xi_t = \langle \pi_t, f \rangle .$$

π_t is the optimal filter.

π_t is a prob. meas. valued process. How to characterize it?

- If X_t is indep. of $\mathcal{G}_t$, then

$$\mathbb{E}(f(X_t)|\mathcal{G}_t) = \mathbb{E}f(X_t).$$

- If X_t is measurable w.r.t. $\mathcal{G}_t$, then

$$\mathbb{E}(f(X_t)|\mathcal{G}_t) = f(X_t).$$

Consider model

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dB_t, \\ Y_t &= \int_0^t h(X_s)ds + W_t, \end{cases}$$

where B, W indep. B.M.
Girsanov transform: Under new prob. meas. $\hat{P}$, Y, B indep. B.M., where

$$\frac{dP}{d\hat{P}} = M_t$$

$$dM_t = M_t h(X_t) dY_t.$$

How to calculate conditional expectation using $\hat{P}$?

Kallianpur-Striebel formula:

$$\langle \pi_t, f\rangle = \frac{\hat{\mathbb{E}}(M_t f(X_t)|\mathcal{G}_t)}{\hat{\mathbb{E}}(M_t|\mathcal{G}_t)} \equiv \frac{\langle V_t, f\rangle}{\langle V_t, 1\rangle}. \tag{1.1}$$

V_t meas. valued process. Note that

$$(X_t, M_t) = H(t, B, Y),$$

and hence,

$$\langle V_t, f\rangle = \hat{\mathbb{E}}(\tilde{H}(t, B, Y)|\mathcal{G}_t).$$

How to use this to calculate V_t?

1.4. Monte-Carlo method

$(B^i,\ i=1,2,\cdots)$ be indep. copies of B. Consider

$$\begin{cases} dX_t^i &= b(X_t^i)dt+\sigma(X_t^i)dB_t^i \\ dM_t^i &= M_t^i h(X_t^i)dY_t. \end{cases} \tag{1.2}$$

Then,

$$\langle V_t, f\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n M_t^i f(X_t^i). \tag{1.3}$$

Remark

i) Based on (1.3), we can derive SPDE for V_t (Zakai equation).
ii) Based on (1.1), we can derive SPDE for π_t (Kushner-FKK equation).

Remark

$$\mathbb{E}M_t^i = 1, \qquad Var(M_t^i) \approx e^{Kt}$$

exponential growth. Error of Monte-Carlo method grows expo.

1.5. Branching particle system

Divide time interval into subintervals of length δ each.

Initially, there are $m_0 = n$ number of particles moving as (1.2) $(i = 1, 2, \cdots, m_0)$.

At time δ, the ith particle gives birth to ξ_1^i number of particles,

$$\mathbb{E}(\xi_1^i | \mathcal{F}_{\delta-}) = M_\delta^i,$$

$$Var(\xi_1^i | \mathcal{F}_{\delta-}) \text{ as small as possible.}$$

Take

$$\xi_1^i = \begin{cases} [M_\delta^i] & \text{with prob. } 1 - \{M_\delta^i\} \\ [M_\delta^i] + 1 & \text{with prob. } \{M_\delta^i\}. \end{cases}$$

During $[\delta, 2\delta)$, there are

$$m_1 = \sum_{i=1}^{m_0} \xi_1^i$$

particles.

For $k\delta \le t < (k+1)\delta$, let

$$V_t^n = \frac{1}{n}\sum_{i=1}^{m_k^n} M_t^i f(X_t^i).$$

Then,

$$V_t^n \to V_t, \qquad \text{as } n \to \infty.$$

2.1. BSDE example: European call option

Stock S_t, bond B_t, Portfolio u_t.

$$u_t = \text{ \$ amount invested in stock.}$$

Find portfolio to replicate EC:

$$\xi = (S_T - K)^+.$$

Recall

$$\begin{cases} \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t dW_t, \\ \frac{dB_t}{B_t} &= r_t dt, \end{cases}$$

where

$$\mu_t = \text{ appreciation rate,}$$

$$r_t = \text{ interest rate,}$$

$$\sigma_t = \text{ volatility.}$$

Let

$$V_t = \text{ value of portfolio.}$$

By self-finance condition,

$$dV_t = \frac{u_t}{S_t} dS_t + \frac{V_t - u_t}{B_t} dB_t.$$

Then,

$$\begin{cases} dV_t = (u_t \mu_t + (V_t - u_t) r_t) dt + u_t \sigma_t dW_t, \\ V_T = (S_T - K)^+. \end{cases}$$

Solution consists two variables V_t and u_t.

2.2 Mean-variance

Minimize $\mathbb{E}(V_T - z)^2$ subject to $\mathbb{E}V_T = z$.

For any $\lambda > 0$, minimize cost function

$$J(u) = \mathbb{E}V_T^2 - \lambda \mathbb{E}V_T.$$

Recall state equation

$$dV_t = (u_t\mu_t + (V_t - u_t)r_t)dt + u_t\sigma_t dW_t.$$

2.3. Control problem.

State equation:

$$\begin{cases} dX_t &= b(X_t, u_t)dt + \sigma(X_t, u_t)dW_t, \\ X_0 &= x. \end{cases}$$

Cost functional

$$J(u) = \mathbb{E}\left(\int_0^T \ell(X_s, u_s)ds + h(X_T)\right).$$

Optimal control

$$\bar{u} = \text{argmin } J(u).$$

Recall calculus: minimize $f(u)$ subject to $g(u) = 0$.
Define LeGrange

$$F(u, \lambda) = \lambda g(u) - f(u).$$

λ is adjoint variable.
Take

$$\partial_u F(u, \lambda) = 0.$$

2.4. Stochastic maximum principle

Two adjoint variables (p_t, q_t). Adjoint equation is backward:

$$\begin{cases} dp_t &= -\partial_x H(X_t, u_t, p_t, q_t)dt + q_t dW_t \\ p_T &= -h'(X_T), \end{cases}$$

where

$$H(x, u, p, q) = (p, q) \cdot (b(x, u), \sigma(x, u)) - \ell(x, u).$$

Why is it called Hamiltonian?

$$\begin{cases} dX_t &= \partial_p H(X_t, u_t, p_t, q_t)dt + \partial_q H(X_t, u_t, p_t, q_t)dW_t, \\ dp_t &= -\partial_x H(X_t, u_t, p_t, q_t)dt + q_t dW_t, \\ X_0 &= x, \;\; p_T = -h'(X_T). \end{cases}$$

Theorem

$$\partial_u H(X_t, \bar{u}_t, p_t, q_t).$$

Idea of proof: For any v,

$$\frac{d}{d\epsilon} J(\bar{u} + \epsilon v)\Big|_{\epsilon=0} = 0.$$

3.1. Mean-variance under partial info.

Info.

$$\mathcal{G}_t = \sigma(S_u,\ B_u :\ u \leq t).$$

Note that $r_t = \frac{d}{dt} \log B_t$ observable.
Recall filtering equation

$$Y_t = \int_0^t h_s(\mu_s)ds + W_t,$$

where

$$h_t(\mu) = \sigma_t^{-1}(\mu - \frac{1}{2}a_t)$$

and

$$Y_t = \int_0^t \sigma_s^{-1} d \log S_s.$$

Recall wealth process

$$dV_t = (u_t\mu_t + (V_t - u_t)r_t)dt + u_t\sigma_t dW_t.$$

(μ_t, W_t) not observable.
Define innovation process ν_t by

$$d\nu_t = dY_t - \langle \pi_t, h_t \rangle\, dt.$$

Then, ν_t is observable B.M. and

$$d\nu_t = dY_t - \sigma_t^{-1}(\bar{\mu}_t - \frac{1}{2}a_t)dt,$$

where $\bar{\mu}_t = \mathbb{E}(\mu_t|\mathcal{G}_t)$.

The state equation becomes

$$dV_t = (u_t\bar{\mu}_t + (V_t - u_t)r_t)dt + u_t\sigma_t d\nu_t.$$

This is observable.
Cost functional is

$$J(u) = \mathbb{E}(V_T^2 - \lambda V_T).$$

It is also observable. Thus, we have transformed it to a full info. optimization problem. Such a filtering-control problem is said to satisfy *separation principle*.

3.2. SMP under partial information

State equation (backward):

$$\begin{cases} -dy_t = f(y_t, z_t, \bar{z}_t, u_t)dt - z_t dW_t - \bar{z}_t d\bar{W}_t, \\ y_T = \xi. \end{cases}$$

Cost functional

$$J(u) = \mathbb{E}\left(\int_0^T \ell(y_t, z_t, \bar{z}_t, u_t)dt + h(y_0) \right).$$

We need one adjoint variable p satisfying a forward adjoint equation.

Suppose that $\mathcal{G}_t \subset \mathcal{F}_t$ is info available. Define Hamiltonian

$$H(y, z, \bar{z}, u, p) = f(y, z, \bar{z}, u)p + \ell(y, z, \bar{z}, u).$$

Adjoint eq.

$$\begin{cases} dp_t &= \partial_y H(y_t, z_t, \bar{z}_t, u_t, p_t)dt + \partial_z H(y_t, z_t, \bar{z}_t, u_t, p_t)dW_t \\ & \quad \partial_{\bar{z}} H(y_t, z_t, \bar{z}_t, u_t, p_t)d\bar{W}_t, \\ p_0 &= h'(y_0). \end{cases}$$

Theorem

$$\mathbb{E}\left(\partial_u H(y_t, z_t, \bar{z}_t, \bar{u}_t, p_t) \middle| \mathcal{G}_t\right) = 0.$$

3.3. Application to an LQ problem

Linear state

$$\begin{cases} -dy_t = (A_t y_t + B_t z_t + C_t u_t)dt - z_t dW_t - \bar{z}_t d\bar{W}_t, \\ y_T = \xi. \end{cases}$$

Quadratic cost

$$J(u) = \frac{1}{2}\mathbb{E}\left[\int_0^T (Q_t y_t^2 + R_t u_t^2)dt + H y_0^2\right].$$

Info: $\mathcal{G}_t = \mathcal{F}_t^W$.

Define Hamiltonian

$$H(y, z, u, p) = (Ay + Bz + cu)p + \frac{1}{2}(Qy^2 + Ru^2),$$

and adjoint

$$\begin{cases} dp_t &= (Qy_t + Ap_t)dt + Bp_t dW_t, \\ p_0 &= Hy_0. \end{cases}$$

SMP implies

$$\mathbb{E}(Ru_t + cp_t | \mathcal{G}_t) = 0.$$

Thus

$$u_t = -R^{-1}C\hat{p}_t = -R^{-1}C\mathbb{E}(p_t | \mathcal{G}_t).$$

Filtering for p,

$$\begin{cases} d\hat{p}_t &= (Q\hat{y}_t + A\hat{p}_t)dt + B\hat{p}_t dW_t, \\ \hat{p}_0 &= Hy_0. \end{cases}$$

Combine with $\hat{y}$:

$$\begin{cases} -d\hat{y}_t &= (A_t\hat{y}_t + B_t\hat{z}_t - R_t^{-1}C_t^2\hat{p}_t)dt - \hat{z}_t dW_t, \\ \hat{y}_T &= \mathbb{E}(\xi|\mathcal{G}_T). \end{cases}$$

3.4. More general LQ problem

Linear state

$$\begin{cases} -dy_t &= (A_t y_t + B_t z_t + \bar{B}_t \bar{z}_t + C_t u_t)dt - z_t dW_t - \bar{z}_t d\bar{W}_t, \\ y_T &= \xi. \end{cases}$$

Quadratic cost

$$J(u) = \frac{1}{2}\mathbb{E}\left[\int_0^T (Q_t y_t^2 + R_t u_t^2)dt + H y_0^2\right].$$

Info: $\mathcal{G}_t = \mathcal{F}_t^W$.

Define Hamiltonian

$$H(y, z, \bar{z}, u, p) = (Ay + Bz + \bar{B}\bar{z} + cu)p + \frac{1}{2}(Qy^2 + Ru^2),$$

and adjoint

$$\begin{cases} dp_t &= (Qy_t + Ap_t)dt + Bp_t dW_t + \bar{B}p_t \bar{W}_t, \\ p_0 &= Hy_0. \end{cases}$$

SMP implies

$$\mathbb{E}(Ru_t + cp_t | \mathcal{G}_t) = 0.$$

Thus

$$u_t = -R^{-1}C\hat{p}_t.$$

Filtering for p,

$$\begin{cases} d\hat{p}_t = (Q\hat{y}_t + A\hat{p}_t)dt + B\hat{p}_t dW_t, \\ \hat{p}_0 = H y_0. \end{cases}$$

Then, filtering for y,

$$\begin{cases} -d\hat{y}_t = (A_t\hat{y}_t + B_t\hat{z}_t + \bar{B}_t\hat{\hat{z}}_t + C_t u_t)dt - \hat{z}_t dW_t, \\ \hat{y}_T = \hat{\xi}. \end{cases}$$

Cannot solve for $\hat{\hat{z}}_t$! What to do?

Filtering for p,

$$\begin{cases} d\hat{p}_t &= (Q\hat{y}_t + A\hat{p}_t)dt + B\hat{p}_t dW_t, \\ \hat{p}_0 &= Hy_0. \end{cases}$$

Then, filtering for y,

$$\begin{cases} -d\hat{y}_t &= (A_t\hat{y}_t + B_t\hat{z}_t + \bar{B}_t\hat{\hat{z}}_t + C_t u_t)dt - \hat{z}_t dW_t, \\ \hat{y}_T &= \hat{\xi}. \end{cases}$$

Cannot solve for $\hat{\hat{z}}_t$! What to do? Return to original equations and guess

$$p_t = \alpha_t y_t + \beta_t, \qquad \alpha_0 = H,\ \beta_0 = 0.$$

Applying dp_t and comparing, we get

$$z = -\alpha^{-1} B p, \qquad \bar{z} = -\alpha^{-1} \bar{B} p.$$

So,

$$\bar{z} = B^{-1} \bar{B} z,$$

and hence,

$$\hat{\bar{z}} = B^{-1} \bar{B} \hat{z}.$$

Finally, we solve FBFSDE

$$\left\{\begin{array}{rcl} d\hat{p}_t &=& (Q\hat{y}_t + A\hat{p}_t)dt + B\hat{p}_t dW_t, \\ -d\hat{y}_t &=& (A_t\hat{y}_t + (B_t + B_t^{-1}\bar{B}_t^2)\hat{z}_t - R_t^{-1}C_t^2\hat{p}_t)dt - \hat{z}_t dW_t, \\ \hat{p}_0 &=& Hy_0, \ \ \hat{y}_T = \hat{\xi}. \end{array}\right.$$

Optimal control

$$u_t = -R^{-1}C\hat{p}_t.$$

Thanks for your attention!