Singular Stochastic differential equations driven by Markov processes

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Overview

Motivation

Generally, a Markov process X in $\mathbb{R}^d$ has the generator of the form

$$\mathcal{A}\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x) + b(x)\cdot\nabla\varphi(x) + c(x)\varphi(x)$$
$$+ \int_{\mathbb{R}^d}\left[\varphi(x+z) - \varphi(x) - 1_{\{|z|\leqslant 1\}} z\cdot\nabla\varphi(x)\right]\mu(x,\mathrm{d}z),$$

where for each $x\in\mathbb{R}^d$,

- $a(x) = (a_{i,j}(x))$ is a non-negative real symmetric $d\times d$-matrix,
- $b(x)$ is a real vector-valued function,
- $c(x)\geqslant 0$,
- the measure $\mu(x,\mathrm{d}z)$ satisfies

$$\int_{\mathbb{R}^d}(1\wedge|z|^2)\mu(x,\mathrm{d}z) < \infty.$$

Motivation

Consider the following non-local and non-symmetric operator:

$$\mathscr{L}_\mu \varphi(x) := \int_{\mathbb{R}^d} \left[\varphi(x+z) - \varphi(x) - 1_{\{|z|\leqslant 1\}} z \cdot \nabla \varphi(x)\right] \mu(x, \mathrm{d}z).$$

In particular, when

$$\mu(x, \mathrm{d}z) = a(x) \frac{\mathrm{d}z}{|z|^{d+\alpha}}, \quad \alpha \in (0, 2),$$

we have

$$\mathscr{L}_\mu = a(x) \Delta^{\alpha/2}.$$

Motivation

The operator $\mathscr{L}_\mu$ was studied mainly by the theory of PDEs. Under certain assumptions, there exists a Markov process X_t with $\mathscr{L}_\mu$ as its generator. We want to study X_t via Itô's SDE.

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Notice that the usual SDEs driven by Lévy processes does not fit here.

$$\mathrm{d}X_t = \int_{|z|\leqslant 1} \sigma(X_t, z)\tilde{N}(\mathrm{d}z, \mathrm{d}s) + \int_{|z|>1} \sigma(X_t, z)\tilde{N}(\mathrm{d}z, \mathrm{d}s)$$

$$\Updownarrow$$

$$\mathscr{L}\varphi(x) = \int_{\mathbb{R}^d} \big[\varphi(x+\sigma(x,z)) - \varphi(x) - 1_{\{|z|\leqslant 1\}}\sigma(x,z)\cdot\nabla\varphi(x)\big]\nu(\mathrm{d}z)$$

Motivation

Let m the Lebesgue measure, and N be a Poisson random measure on $\mathbb{R}^d \times [0,\infty) \times [0,\infty)$ with mean measure $\nu \times m \times m$, where ν is a Lévy measure. Set for $A \in \mathscr{B}\big(\mathbb{R}^d \times [0,\infty) \times [0,\infty)\big)$,

$$\tilde{N}(A) := N(A) - \nu \times m \times m(A).$$

Consider the following SDE:

$$\begin{aligned} X_t = x &+ \int_0^t \int_0^\infty \int_{|z|\leqslant 1} 1_{[0,\sigma(X_{s-},z)]}(r) z \tilde{N}(\mathrm{d}z \times \mathrm{d}r \times \mathrm{d}s) \\ &+ \int_0^t \int_0^\infty \int_{|z|>1} 1_{[0,\sigma(X_{s-},z)]}(r) z N(\mathrm{d}z \times \mathrm{d}r \times \mathrm{d}s) + \int_0^t b(X_s)\mathrm{d}s. \end{aligned}$$

Motivation

The infinitesimal generator of X_t is given by

$$\mathscr{L}f(x) = \int_{\mathbb{R}^d} \Big[f(x+z) - f(x) - 1_{\{|z|\leqslant 1\}} z \cdot \nabla f(x)\Big] \sigma(x,z)\nu(\mathrm{d}z) + b(x) \cdot \nabla f(x).$$

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Aim: Solve the SDE with singular coefficients which has $\mathscr{L}$ as generator.

Background

◇ T. Kurtz and P. Protter (1996), T. Kurtz (2010):

Conditions: • b is bounded and global Lipschitz continuous;

• σ satisfies

$$\int_{\mathbb{R}^d} |\sigma(x,z) - \sigma(y,z)| \cdot |z| \nu(\mathrm{d}z) \leqslant C_1 |x-y|,$$

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Notice that when $\sigma \equiv 0$, it is an ODE:

$$x'(t) = b\big(x(t)\big), \quad x(0) = x_0,$$

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It is interesting to find that noises may produce some regularization effects.

Background

SDEs driven by Brownian motion:

- N. V. Krylov and M. Röckner (2005, PTRF):

$$\mathrm{d}X_t = \mathrm{d}W_t + b(X_t)\mathrm{d}t, \quad X_0 = x.$$

Condition: $b \in L^p(\mathbb{R}^d)$ with $p > d$.

- X. Zhang (2005, SPA):

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t, \quad X_0 = x.$$

Condition: σ is uniformly continuous in x, bounded and uniformly elliptic and $\nabla\sigma \in L^p(\mathbb{R}^d)$ with $p > d$.

Background

SDEs driven by pure jump Lévy process:

$$\mathrm{d}X_t = \mathrm{d}L_t + b(X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d,$$

where L_t is a symmetric α-stable process with $\alpha \in (0, 2)$.

- Tanaka, Tsuchiya and Watanabe (1974, JMKU):
  When $d = 1$, $\alpha < 1$, b is bounded and β-Hölder continuous with $\alpha + \beta < 1$, SDE may not has pathwise uniqueness strong solutions.

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- Priola (2012, OJM):
  Condition: $\alpha \geqslant 1$, b is bounded and β-Hölder continuous with $\beta > 1 - \alpha/2$.
- Zhang (2013, Poincare):
  Condition: $\alpha > 1$, $b \in L^\infty(\mathbb{R}^d) \cap W^{\beta,p}(\mathbb{R}^d)$ with $p > 2d/\alpha$ and $\beta \in (1 - \alpha/2, 1)$.

Main result

Recall that

$$X_t = x + \int_0^t \int_0^\infty \int_{|z|\leqslant 1} 1_{[0,\sigma(X_{s-},z)]}(r) z \tilde{N}(\mathrm{d}z \times \mathrm{d}r \times \mathrm{d}s)$$
$$+ \int_0^t \int_0^\infty \int_{|z|>1} 1_{[0,\sigma(X_{s-},z)]}(r) z N(\mathrm{d}z \times \mathrm{d}r \times \mathrm{d}s) + \int_0^t b(X_s)\mathrm{d}s. \quad (3.1)$$

Its generator is given by

$$\mathscr{L} f(x) = \int_{\mathbb{R}^d} \Big[f(x+z) - f(x) - 1_{\{|z|\leqslant 1\}} z \cdot \nabla f(x) \Big] \sigma(x,z) \nu(\mathrm{d}z)$$
$$+ b(x) \cdot \nabla f(x)$$
$$=: \mathscr{L}_\nu f(x) + b(x) \cdot \nabla f(x).$$

Main result

Conditions 1.:

◇ For all $x \in \mathbb{R}^d$,

$$\sigma(x,z) = \sigma(x,-z), \quad \forall z \in \mathbb{R}^d, \tag{3.2}$$

and for all $x, y \in B_n$, there exists $\beta \in (0,1)$ such that

$$k_0^n \leqslant \sigma(x,z) \leqslant k_1^n, \ |\sigma(x,z) - \sigma(y,z)| \leqslant C_n|x-y|^\beta, \ \forall z \in \mathbb{R}^d. \tag{3.3}$$

◇ There exists a function κ such that

$$\nu(\mathrm{d}z) = \frac{\kappa(z)}{|z|^{d+\alpha}}\mathrm{d}z, \ \kappa(z) = \kappa(-z), \ \kappa_0 \leqslant \kappa(z) \leqslant \kappa_1, \tag{3.4}$$

with $\alpha \in (1,2)$.

Main result

Conditions 2.:

◇ There exists a function $g \in L^q(B_n)$ with $q > d/\alpha$, such that for almost all $x, y \in B_n$,

$$\int_{\mathbb{R}^d} |\sigma(x,z) - \sigma(y,z)|(|z| \wedge 1)\nu(\mathrm{d}z) \leqslant |x-y|\Big(g(x) + g(y)\Big). \qquad (3.5)$$

◇ $b \in L^\infty(B_n) \cap W^{\theta,p}(B_n)$ with $p > 2d/\alpha$ and $\theta \in (1 - \alpha/2, 1)$.

Main result

Theorem 1

For each $x \in \mathbb{R}^d$, there exists an stopping time $\varsigma(x)$ (called the explosion time) and a unique strong solution $X_t(x)$ to SDE (3.1) such that

$$\lim_{t \uparrow \varsigma(x)} X_t(x) = \infty, \quad a.s.. \tag{3.6}$$

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For a example of σ, we can take

$$\sigma(x,z) = K(z) + \tilde{\sigma}(x)|z|^\gamma \ \text{ for } \ |z| \leqslant 1,$$

with $0 < K_1 \leqslant K(z) \leqslant K_2$, $\gamma > \alpha - 1$ and

$$\nabla \tilde{\sigma} \in L^q_{loc}(\mathbb{R}^d), \quad q > d/\alpha.$$

Since we assume $\alpha > 1$, our theorem can cover the regime $q \in (d/\alpha, d]$.

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  1. The operator $\mathscr{L}_\nu$ is non-local and non-symmetric.
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- Difficulties:
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  2. Proving the Krylov's estimate and solving the resolvent equation of $\mathscr{L}_\nu$ in the framework of Sobolev space.
  3. New challenges appear when dealing with the factor $1_{[0,\sigma(x,z)]}$ — L_1-estimate is essential.

Lemma 1 – Krylov's estimate

Let X_t be a strong solution of SDE (3.1). Then, for any $T > 0$, there exist a constant C_T such that for any $f \in L^p(\mathbb{R}^d)$ with $p > d/\alpha$, we have

$$\mathbb{E}\left(\int_0^T f(X_s)\mathrm{d}s\right) \leqslant C_T\|f\|_p. \tag{3.7}$$

Proof of the main result

Instead of studying the elliptic equation

$$\lambda u - \mathscr{L}_\nu u - b \cdot \nabla u = b,$$

we consider the following integral equation:

$$u(x) = \int_0^\infty \mathrm{e}^{-\lambda t} T_t\big(b \cdot \nabla u + b\big)(x)\mathrm{d}t, \tag{3.8}$$

where T_t is the semigroup corresponding to $\mathscr{L}_\nu$.

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Lemma 2 – Resolvent equation

Let $1 < \gamma < \alpha$. Suppose that for some $p > \frac{d}{\gamma}$ and $0 < \theta \in (1 - \gamma + \frac{d}{p}, 1)$,

$$b \in L^\infty(\mathbb{R}^d) \cap \mathbb{W}_p^\theta.$$

Then, there exists a function $u \in \mathbb{H}_p^{\gamma+\theta}$ satisfying the integral equation (3.8). Moreover, $\|u\|_\infty + \|\nabla u\|_\infty \leqslant \frac{1}{2}$.

Proof of the main result

Define $\Phi(x) := x + u(x)$. Then, $x \to \Phi(x)$ forms a C^1-diffeomorphism.

Lemma 3 – Zvonkin transformation

Let X_t solve SDE (3.1). Then, $Y_t := \Phi(X_t)$ satisfies

$$
\begin{aligned}
Y_t = \Phi(x) &+ \int_0^t \int_0^\infty \int_{|z|\leqslant 1} \tilde{g}(Y_{s-}, z) 1_{[0,\tilde{\sigma}(Y_{s-},z)]}(r) \tilde{N}(\mathrm{d}z \times \mathrm{d}r \times \mathrm{d}s) \\
&+ \int_0^t \int_0^\infty \int_{|z|>1} \tilde{g}(Y_{s-}, z) 1_{[0,\tilde{\sigma}(Y_{s-},z)]}(r) N(\mathrm{d}z \times \mathrm{d}r \times \mathrm{d}s) \\
&+ \int_0^t \tilde{b}(Y_s) \mathrm{d}s. \qquad (3.9)
\end{aligned}
$$

Proof of the main result

The new coefficients:

$$\tilde{g}(x,z) := \Phi\big(\Phi^{-1}(x)+z\big) - x, \quad \tilde{\sigma}(x,z) := \sigma\big(\Phi^{-1}(x), z\big).$$

and

$$\tilde{b}(x) = \lambda u\big(\Phi^{-1}(x)\big) - \int_{|z|>1} \big[u(\Phi^{-1}(x)+z\big) \\ - u\big(\Phi^{-1}(x)\big)\big]\sigma\big(\Phi^{-1}(x), z\big)\nu(\mathrm{d}z).$$

Proof of the main result

Let Y_t and $\hat{Y}_t$ be two strong solutions for SDE (3.9). Define

$$Z_t := Y_t - \hat{Y}_t,$$

Then,

$$Z_t = \int_0^t \int_0^\infty \int_{|z|\leqslant 1} \Big[\tilde{g}(Y_{s-}, z) 1_{[0, \tilde{\sigma}(Y_{s-}, z)]}(r) - \tilde{g}(\hat{Y}_{s-}, z) 1_{[0, \tilde{\sigma}(\hat{Y}_{s-}, z)]}(r) \Big] \tilde{N}(\mathrm{d}z \times \mathrm{d}r \times \mathrm{d}s) + \cdots.$$

Proof of the main result

We arrive at that for any stopping time τ,

$$\mathbb{E}\left[\sup_{t\in[0,\tau]}|Z_t|\right] \leqslant C_0\mathbb{E}\int_0^\tau |Z_s|\mathrm{d}A(s) + C_0\mathbb{E}\left(\int_0^\tau |Z_s|^2\mathrm{d}A(s)\right)^{\frac{1}{2}},$$

where $t \mapsto A(t)$ is a continuous strictly increasing process.

This implies that $Z_t \equiv 0$, *a.e.*.

Reference

- Chen Z. and Zhang X.: Heat kernel and analyticity of non-symmetric jump diffusion semigroups. Prob. Theory and Related Fields.
- Krylov N. V. and Röckner M.: Strong solutions of stochastic equations with singular time dependent drift. Probab. Theory Related Fields.
- Kurtz T. G.: Equivalence of Stochastic Equations and Martingale Problems. Stochastic Analysis.
- Kurtz T. G. and Protter P. E.: Weak convergence of stochastic integrals and differential equations. II. Infinite-dimensional case. Lecture Notes in Math..

Reference

- Priola E.: Pathwise uniqueness for singular SDEs driven by stable processes. Osaka Journal of Mathematics.
- Tanaka H., Tsuchiya M. and Watanabe S.: Perturbation of drift-type for Lévy processes. J. Math. Kyoto Univ..
- Zhang X.: Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients. Stoch. Proc. Appl..
- Zhang X.: Stochastic differential equations with Sobolev drifts and driven by α-stable processes. Ann. Inst. H. Poincare Probab. Statist..

Thank You !