Some recent results for stochastic scalar conservation laws with boundary conditions

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Outline

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[1] Renormalized entropy solutions of stochastic scalar conservation laws with boundary conditions, J. Funct. Anal., http://dx.doi.org/10.1016/j.jfa.2016.06.012, in press.
[2] On a first order stochastic scalar conservation law with non-homogeneous Dirichlet boundary condition, submitted.
[3] Uniqueness of stochastic entropy solutions for stochastic scalar conservation law with non-homogeneous Dirichlet boundary conditions, submitted.
[4] On heterogeneous stochastic scalar conservation laws with non-homogeneous Dirichlet boundary conditions, submitted.

Let D be a bounded open set in $\mathbb{R}^N$ with boundary ∂D in which we assume the boundary ∂D is Lipschitz in case the space dimension $N > 1$. Let $T > 0$ be arbitrarily fixed. Set $Q = (0, T) \times D$ and $\Sigma = (0, T) \times \partial D$. Let $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t\in[0,T]})$ be a given probability set-up. We are concerned with the first order stochastic conservation laws driven by a multiplicative noise of the following type

$$du - div(f(u))dt = h(u)dw(t), \quad \text{in } \Omega \times Q, \tag{1}$$

with initial condition

$$u(0, \cdot) = u_0(\cdot), \quad \text{in } D, \tag{2}$$

and non-homogeneous Dirichlet boundary condition

$$u = a, \quad \text{on } \Sigma, \tag{3}$$

for a scalar random field
$u : (\omega, t, x) \in \Omega \times [0, T] \times D \mapsto u(\omega, t, x) =: u(t, x) \in \mathbb{R}$,

where $f = (f_1, ..., f_N) : \mathbb{R} \to \mathbb{R}^N$ is a differentiable vector field standing for the flux, $h : \mathbb{R} \to \mathbb{R}$ is measurable and $w = \{w(t)\}_{0 \le t \le T}$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t \in [0,T]})$. The initial data $u_0 : D \subset \mathbb{R}^N \to \mathbb{R}$ will be specified later and the boundary data $a : \Sigma \to \mathbb{R}$ is supposed to be measurable.

Problem (1)-(3) was studied recently by K. Kobayasi and D. Noboriguchi (*Acta Math Vietnamica, 2015*) via kinetic solution approach. By introducing a notion of kinetic formulations in which the kinetic defect measures on the boundary of domain are truncated, they obtained the well-posedness of (1)-(3).

When $h = 0$, the deterministic problem (1)-(3) is well studied in the PDEs literature, see e.g. *K. Ammar,P. Wittbold, J. Carrillo,JDE 2006* and references therein. The notion of entropy solutions for the deterministic problem (1)-(3) in the L^∞ framework was initiated by F. Otto in *C.R. Acad. Sci. Paris 1996*. Furthermore,*A. Porretta and J. Vovelle, CPDE 2003* studied the problem (1)-(3) with $h = 0$ in the L^1-setting, that is, the solutions are allowed to be unbounded. In order to deal with unbounded solutions, they have defined a notion of renormalized entropy solutions which generalizes Otto's original definition of entropy solutions. They have proved existence and uniqueness of such generalized solution in the case when f is locally Lipschitz and the boundary data a verifies the following condition: $f_{max}(a) \in L^1(\Sigma)$, where f_{max} is the “maximal effective flux” defined by

$$f_{max}(a) = \{\sup |f(u)|, \quad u \in [-a^-, a^+]\}.$$

They gave an example to illustrate that the assumption $a \in L^1(\Sigma)$ is not enough in order to prove a priori estimates in $L^1(Q)$, and that the assumption should be $f_{max}(a) \in L^1(\Sigma)$. Moreover, the paper *K. Ammar,P. Wittbold, J. Carrillo,JDE 2006* revisited the problem (1)-(3) with $h = 0$ and introduced the following notion of entropy solutions to the problem (1)-(3)

An entropy solution of (1)-(3) is a function $u \in L^\infty(Q)$ satisfying

$$
\begin{aligned}
-\int_\Sigma \xi\omega^+(x,k,a(t,x)) &\leq \int_Q \left[(u-k)^+\xi_t - \chi_{u>k}(f(u)-f(k))\cdot\nabla\xi\right] \\
&\quad + \int_D (u_0-k)^+\xi(0,\cdot) \quad \text{and} \qquad (4) \\
-\int_\Sigma \xi\omega^-(x,k,a(t,x)) &\leq \int_Q \left[(k-u)^+\xi_t - \chi_{k>u}(f(k)-f(u))\cdot\nabla\xi\right] \\
&\quad + \int_D (k-u_0)^+\xi(0,\cdot) \qquad (5)
\end{aligned}
$$

for any $\xi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$, $\xi \geq 0$ and for all $k \in \mathbb{R}$, where

$$
\omega^+(x,k,a) := \max_{k\leq r,s\leq a\vee k} |(f(r)-f(s))\cdot\vec{n}(x)|
$$

$$
\omega^-(x,k,a) := \max_{a\wedge k\leq r,s\leq k} |(f(r)-f(s))\cdot\vec{n}(x)|
$$

for any $k \in \mathbb{R}$, a.e. $x \in \partial D$, and $\vec{n}$ denoting the unit outer normal to ∂D.

The above definition of entropy solution is a natural extension of the definition of Otto. Having a stochastic forcing term $h(u)dw(t)$ in Equation (1) is very natural for problem modeling arising in a wide variety of fields in physics, engineering, biology, jut mention a few. The Cauchy problem of equation (1) with additive noise has been studied by J.V. Kim in *Indiana Univ Math J 2003* wherein the author proposed a method of compensated compactness to prove, via vanishing viscosity approximation, the existence of a stochastic weak entropy solution. Moreover, a Kruzhkov-type method was used there to prove the uniqueness. Further, *G. Vallet and P. Wittbold, Infin Dimens Anal Quantum Probab 2009* extended the results of Kim to the multi-dimensional (i.e., vector-valued) Dirichlet problem with additive noise. By utilising the vanishing viscosity method, Young measure techniques and Kruzhkov doubling variables technique, they managed to show the existence and uniqueness of the stochastic entropy solutions.

Concerning the case of multiplicative noise, for Cauchy problem over the whole spatial space, *J. Feng and D. Nualart, J Funct Anal 2008* introduced a notion of strong entropy solutions in order to prove the uniqueness for the entropy solution. Using the vanishing viscosity and compensated compactness arguments, they established the existence of stochastic strong entropy solutions only in $1D$ case. On the other hand, *G. Chen, Q. Ding, K.H. Karlsen, Arch Ration Mech Anal 2012* considered higher space dimensional problem and they proved the well-posedness of the multi-dimensional stochastic problem, by using a uniform spatial BV-bound.

Furthermore, *C. Bauzet, G. Vallet, P. Wittbold, J. Hyperbolic Diff Eqs 2012* proved a result of existence and uniqueness of the weak measure-valued entropy solution to the multi-dimensional Cauchy problem. *C. Bauzet, G. Vallet, P. Wittbold, J Funct Anal 2014* studied the problem (1)-(3) with $a = 0$ (i.e., the homogeneous boundary condition). Under the assumptions that the flux function f and h satisfy the global Lipschitz condition, they obtained the existence and uniqueness of measure-valued solution to problem (1)-(3) with $a = 0$. *G. Lv, J. Duan, H. Gao, J.-L. Wu, Bull. Sci. Math. 140 (2016), pp718-746* extended the results of *C. Bauzet, G. Vallet, P. Wittbold, J Funct Anal 2014* to the stochastic nonlocal conservation laws in bounded domains.

Using a kinetic formulation, *A. Debussche, J. Vovelle, J Funct Anal 2012* obtained a result of existence and uniqueness of the entropy solution to the problem posed in a d-dimensional torus. *M. Hofmanova, T. Zhang, arXiv:1501.00548* discussed the degenerate case.
G Lv, J Duan, H Gao, Discret Contin Dyn Syst 2016 considered the Cauchy problem of stochastic nonlocal conservation laws on a while spatial space. As a natural generalisation, we propose to establish the existence and uniqueness of stochastic entropy solution to the initial boundary value problem (1)-(3).

Our strategy is as follows: First, a method of artificial viscosity is introduced for verifying the existence of a solution. Second, the compactness properties used are based on the theory of Young measures and on measure-valued solutions which were considered in *Ch. Castaning, P. Raynaud de Fitte and M. Valadier,* Young Measures on Topological Spaces with Applications in Control Theory and Probability Theory*, Math. Apple., vol. 571, Kluwer Academic Publishers, Dordrecht, 2004*. Finally, an approximation adaptation of the Kruzhkov's doubling variables is then proposed to prove the uniqueness of the measure-valued entropy solution.
It is worth noting that the results of *C. Bauzet, G. Vallet, P. Wittbold, J Funct Anal 2014* are a special case to ours.

<u>Notations</u> In general, if $G \subset \mathbb{R}^N$, $\mathcal{D}(G)$ denotes the restriction of functions $u \in \mathcal{D}(\mathbb{R}^N)$ to G such that $support(u) \cap G$ is compact. The notation $\mathcal{D}^+(G)$ stands for the subset of non-negative elements of $\mathcal{D}(G)$.
For a given separable Banach space X, we denote by $N_w^2(0, T, X)$ the space of the predictable X-valued processes. This space is the space $L^2((0, T) \times \Omega, X)$ for the product measure $dt \otimes dP$ on $\mathcal{P}_T$, the predictable σ-field (i.e. the σ-field generated by the sets $\{0\} \times \mathcal{F}_0$ and the rectangles $(s, t) \times A$ for any $A \in \mathcal{F}_s$, for $t > s > 0$).

Denote $\mathcal{E}^+$ the totality of non-negative convex functions η in $C^{2,1}(\mathbb{R})$, approximating the semi-Kruzhkov entropies $x \to x^+$ such that $\eta(x) = 0$ if $x \le 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ if $x > \delta$. Then η'' has a compact support and η and η' are Lipschitz-continuous functions. $\mathcal{E}^-$ denotes the set $\{\breve{\eta} := \eta(-\cdot), \eta \in \mathcal{E}^+\}$ and $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^-$. Then, for convenience, denote

$$
\begin{aligned}
& sgn_0^+(x) = 1 \ \text{ if } x > 0 \text{ and } 0 \text{ else;} \\
& sgn_0^-(x) = -sgn_0^+(-x) \ \ sgn_0 = sgn_0^+ + sgn_0^-, \\
& F(a,b) = sgn_0(a-b)[f(a) - f(b)]; \\
& F^{+(-)}(a,b) = sgn_0^{+(-)}(a-b)[f(a) - f(b)], \\
& \text{and for any } \eta \in \mathcal{E}, \\
& F^\eta(a,b) = \int_b^a \eta'(\sigma - b) f'(\sigma) d\sigma.
\end{aligned}
$$

In order to propose an entropy formula, let us analyse the viscous parabolic case. For this, let us assume that for any $\varepsilon > 0$, u_ε is the solution of the stochastic nonlinear parabolic problem

$$\begin{cases} du_\varepsilon - [\varepsilon \Delta u_\varepsilon + div(f(u_\varepsilon))]dt = h(u_\varepsilon)dw(t) & \text{in } Q, \\ u_\varepsilon(0,x) = u_{0\varepsilon}(x) & \text{in } D, \\ u_\varepsilon = a_\varepsilon & \text{on } \Sigma. \end{cases} \tag{6}$$

Let $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^N)$, k a real number, and $\eta \in \mathcal{E}$.
Since $\eta(u_\varepsilon - k)\varphi \in L^2(0,T;H^1(D))$ a.s., it is possible to apply the Itô formula to the operator $\Psi(t,u_\varepsilon) := \int_D \eta(u_\varepsilon - k)\varphi dx$ and thus we get

$$
\begin{aligned}
&0 \le \int_D \eta(u_\varepsilon(T) - k)\varphi(T)dx = \\
&\int_D \eta(u_{0\varepsilon} - k)\varphi(0)dx + \int_Q \eta(u_\varepsilon - k)\partial_t\varphi dxdt \\
&+ \int_Q \eta'(u_\varepsilon - k)h(u_\varepsilon)\varphi dxdw(t) + \frac{1}{2}\int_Q \eta''(u_\varepsilon - k)h^2(u_\varepsilon)\varphi dxdt \\
&-\varepsilon \int_Q \eta'(u_\varepsilon - k)\nabla u_\varepsilon \cdot \nabla\varphi dxdt - \int_Q \eta'(u_\varepsilon - k)f(u_\varepsilon)\cdot\nabla\varphi dxdt \\
&-\varepsilon \int_Q \eta''(u_\varepsilon - k)\varphi|\nabla u_\varepsilon|^2 dxdt - \int_Q \eta''(u_\varepsilon - k)\varphi f(u_\varepsilon)\cdot\nabla u_\varepsilon dxdt \\
&+\varepsilon \int_\Sigma \eta'(a_\varepsilon - k)\varphi\nabla a_\varepsilon \cdot \vec{n}(x)dxdt + \int_\Sigma \eta'(a_\varepsilon - k)\varphi f(a_\varepsilon)\cdot\vec{n}(x)dxdt.
\end{aligned} \tag{7}
$$

Since the support of η'' is compact, for any $i = 1, \cdots, N$, $\mathbb{R} \ni r \mapsto \eta''(r-k)f_i(r)$ is a bounded continuous function (Here we assume that f_i is a continuous function and $f_i(0) = 0$). Then, by using the chain-rule Sobolev functions and integrating by part, we have

$$\begin{aligned}
& -\int_Q \eta'(u_\varepsilon - k)f(u_\varepsilon)\cdot\nabla\varphi dxdt - \int_Q \eta''(u_\varepsilon - k)\varphi f(u_\varepsilon)\cdot\nabla u_\varepsilon dxdt \\
= & -\int_Q \eta'(u_\varepsilon - k)f(u_\varepsilon)\cdot\nabla\varphi dxdt \\
& \qquad -\int_Q \varphi div\left(\int_0^{u_\varepsilon}\eta''(\sigma - k)f(\sigma)d\sigma\right)dxdt \\
= & -\int_Q F^\eta(u_\varepsilon, k)\nabla\varphi dxdt - \int_\Sigma \varphi\eta'(u_\varepsilon - k)f(u_\varepsilon)\cdot\vec{n}(x)dSdt \\
& + \int_\Sigma \varphi\int_0^{u_\varepsilon}\eta'(\sigma - k)f'(\sigma)d\sigma\cdot\vec{n}(x)dSdt,
\end{aligned} \tag{8}$$

where we have used $\eta'(\sigma - k) = 0$ if $0 < \sigma < k$. And thus we get

$$\begin{aligned} 0 \quad \leq \quad & \int_D \eta(u_{0\varepsilon} - k)\varphi(0)dx + \int_Q \eta(u_\varepsilon - k)\partial_t \varphi dxdt \\ & + \int_Q \eta'(u_\varepsilon - k)h(u_\varepsilon)\varphi dxdw(t) + \frac{1}{2}\int_Q \eta''(u_\varepsilon - k)h^2(u_\varepsilon)\varphi dxdt \\ & -\varepsilon \int_Q \eta'(u_\varepsilon - k)\nabla u_\varepsilon \cdot \nabla\varphi dxdt - \int_Q F^\eta(u_\varepsilon, k)\nabla\varphi dxdt \\ & +\varepsilon \int_\Sigma \eta'(a_\varepsilon - k)\varphi\nabla a_\varepsilon \cdot \vec{n}(x)dxdt + \int_\Sigma \varphi \int_0^{u_\varepsilon} \eta'(\sigma - k)f'(\sigma)d\sigma \cdot \end{aligned}$$

Now, let us assume that as ε tends to 0, the approximation solution u_ε converges in an appropriate sense to a function $u \in N_w^2(0, T; L^2(D))$ such that for any dP-measurable set A

$$\varepsilon \mathbb{E} \int_Q 1_A \eta'(u_\varepsilon - k) \nabla u_\varepsilon \cdot \nabla \varphi dxdt \to 0, \quad \text{as } \varepsilon \to 0,$$
$$\varepsilon \mathbb{E} \int_\Sigma \eta'(a_\varepsilon - k) \varphi \nabla a_\varepsilon \cdot \vec{n}(x) dxdt \to 0, \quad \text{as } \varepsilon \to 0,$$

where a_ε is an approximate function of a such that $a_\varepsilon \in C^1(\Sigma)$, $\|a_\varepsilon\|_{C^1} \leq \|a\|_{L^\infty}$. Since $\eta'(u) = 1$ if $u > \delta$ and $\eta'(u) = 0$ if $u \leq 0$, and $f \in C^2$, we can assume that f' keeps sign in $(k, k + \delta)$ for any $k \in \mathbb{R}$.

Note that $\eta'' \geq 0$. If $f' \geq 0$ in $(k, k+\delta)$, we have

$$\begin{aligned}
& \int_0^{u_\varepsilon} \eta'(\sigma - k) f'(\sigma) d\sigma \\
= & \int_{k+\delta}^{u_\varepsilon \vee (k+\delta)} f'(\sigma) d\sigma + \int_k^{k+\delta} \eta'(\sigma - k) f'(\sigma) d\sigma \\
\leq & \int_{k+\delta}^{u_\varepsilon \vee (k+\delta)} f'(\sigma) d\sigma + \eta'(\delta) \int_k^{k+\delta} f'(\sigma) d\sigma \\
= & (u_\varepsilon - (k+\delta))^+ [f(u_\varepsilon) - f(k+\delta)] + \eta'(\delta)[f(k+\delta) - f(k)] \\
\leq & \eta'(u_\varepsilon - k)[f(u_\varepsilon) - f(k)]. \qquad (10)
\end{aligned}$$

If $f' \leq 0$ in $(k, k+\delta)$, we have

$$\begin{aligned}
& \int_0^{u_\varepsilon} \eta'(\sigma - k) f'(\sigma) d\sigma \\
= & \int_{k+\delta}^{u_\varepsilon \vee (k+\delta)} f'(\sigma) d\sigma + \int_k^{k+\delta} \eta'(\sigma - k) f'(\sigma) d\sigma \\
\leq & \int_{k+\delta}^{u_\varepsilon \vee (k+\delta)} f'(\sigma) d\sigma \\
\leq & (u_\varepsilon - (k+\delta))^+ [f(u_\varepsilon) - f(k+\delta)] \\
\leq & \eta'(u_\varepsilon - k)[f(u_\varepsilon) - f(k+\delta)]. \qquad (11)
\end{aligned}$$

In order to have the same estimate for (10) and (11), we will take maximum. Combining the above discussion, we get

$$\begin{aligned} & \left| \int_{\Sigma} \varphi \int_0^{u_\varepsilon} \eta'(\sigma - k) f'(\sigma) d\sigma \cdot \vec{n}(x) dSdt \right| \\ \leq & \int_{\Sigma} \varphi \int_0^{u_\varepsilon} \eta'(\sigma - k) |f'(\sigma) d\sigma \cdot \vec{n}(x)| dSdt \\ \leq & \int_{\Sigma} \eta'(u_\varepsilon - k) \varphi \omega^+(x, k, u_\varepsilon) dxdt. \end{aligned}$$

Here we can see why we can define the boundary effect. Then we may pass to the limit in (9) and obtain a family of entropy inequalities satisfied by the limit of u. This observation motivates the definition of entropy solution for the stochastic conservation law (1)-(3). For convenience, for $u \in N_w^2(0, T; L^2(D))$, for any real k and any regular function $\eta \in \mathcal{E}^+$, denote dP-a.s. in Ω by $\mu_{\eta,k}$, the distribution in D defined by

$$\varphi \mapsto \mu_{\eta,k}(\varphi) =$$
$$\int_D \eta(u_0 - k)\varphi(0)dx + \int_Q \eta(u-k)\partial_t\varphi - F^\eta(u,k)\nabla\varphi dxdt$$
$$+ \int_Q \eta'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q \eta''(u-k)h^2(u)\varphi dxdt$$
$$+ \int_\Sigma \eta'(a-k)\varphi\omega^+(x,k,a(t,x))dSdt;$$
$$\varphi \mapsto \mu_{\breve{\eta},k}(\varphi) =$$
$$\int_D \breve{\eta}(u_0 - k)\varphi(0)dx + \int_Q \breve{\eta}(u-k)\partial_t\varphi - F^{\breve{\eta}}(u,k)\nabla\varphi dxdt$$
$$+ \int_Q \breve{\eta}'(u-k)h(u)\varphi dxdw(t) + \frac{1}{2}\int_Q \breve{\eta}''(u-k)h^2(u)\varphi dxdt$$
$$+ \int_\Sigma \breve{\eta}'(a-k)\varphi\omega^-(x,k,a(t,x))dSdt,$$

Now we propose the following definition of entropy solution of (1)-(3).

Definition

A function u of $N_w^2(0,T;L^2(D))$ *is an entropy solution of stochastic conservation law (1) with the initial condition* $u_0 \in L^p(D)$ *and boundary condition* $a \in L^\infty(\Sigma)$*, if* $u \in L^2(0,T;L^2(\Omega;L^p(D)))$*,* $p = 2, 3, \cdots$ *and*

$$\mu_{\eta,k}(\varphi) \geq 0, \qquad \mu_{\breve{\eta},k}(\varphi) \geq 0 \quad dP - a.s.,$$

where $\varphi \in \mathcal{D}^+((0,T \times \mathbb{R}^N))$*,* $k \in \mathbb{R}$*,* $\eta \in \mathcal{E}^+$ *and* $\breve{\eta} \in \mathcal{E}^-$*.*

For technical reasons, we need to consider a generalized notion of entropy solution. In fact, in the first step, we will only prove the existence of a Young measure-valued solution. Then, thanks to a result of uniqueness, we will be able to deduce the existence of an entropy solution in the sense of Definition 3.1.

Definition

A function u of $N_w^2(0,T;L^2(D\times(0,1)))\cap L^\infty(0,T;L^p(\Omega\times D\times(0,1)))$ *is a Young measure-valued solution of stochastic conservation law (1) with the initial condition* $u_0\in L^p(D)$ *and boundary condition* $a\in L^\infty(\Sigma)$, $p=2,3,\cdots$, *if*

$$\int_0^1 \mu_{\eta,k}(\varphi)d\alpha \geq 0, \qquad \int_0^1 \mu_{\breve{\eta},k}(\varphi)d\alpha \geq 0 \quad dP-a.s.,$$

where $\varphi\in\mathcal{D}^+((0,T\times\mathbb{R}^N))$, $k\in\mathbb{R}$, $\eta\in\mathcal{E}^+$ *and* $\breve{\eta}\in\mathcal{E}^-$.

Remark

Note that an entropy solution of (1)-(3) is a.s. a weak solution. In fact, choosing $\varphi \in \mathcal{D}(Q)$, $\varphi \geq 0$, *and letting* $k \to -\infty$ *in* $\mu_{\eta,k}(\varphi) \geq 0$ *and* $k \to +\infty$ *in* $\mu_{\breve{\eta},k}(\varphi) \geq 0$, *we find that*

$$\partial_t \left[u - \int_0^t h(u)dw(s) \right] - divf(u) = 0 \quad \text{in } \mathcal{D}'(Q).$$

Moreover, u satisfies the initial condition in the following sense: $ess \lim_{t \to 0+} \mathbb{E} \int_D |u - u_0| ds = 0.$

Remark

Let $a = 0$, then we find $\mu_{\eta,k}(\varphi)$ will become the "$\mu_{\eta,k}(\varphi)$" in Definition 1 of C. Bauzet, G. Vallet, P. Wittbold, J Funct Anal 2014. Let $h = 0$, then $\mu_{\eta,k}(\varphi) \geq 0$ and $\mu_{\breve{\eta},k}(\varphi) \geq 0$ will coincide with (4) and (5), respectively. That is, Definition 3.1 is a natural extension of the definition of entropy solution given by K. Ammar,P. Wittbold, J. Carrillo,JDE 2006 and C. Bauzet, G. Vallet, P. Wittbold, J Funct Anal 2014.

We assume the following
(H_1): The flux function $f : \mathbb{R} \mapsto \mathbb{R}^N$ is of class C^2, its derivatives have at most polynomial growth, $f(0) = 0_{\mathbb{R}^N}$, and f'' is bounded in $\mathbb{R}$ if $a \not\equiv 0$;
(H_2): $h : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz-continuous function with $h(0) = 0$;
(H_3): $u_0 \in L^p(D)$, $p \geq 2$ and $a \in L^\infty(\Sigma)$.

Theorem

Under assumptions $H_1 - H_3$ there exists a unique measure-valued entropy solution in sense of Definition 4.1 and this solution is obtained by viscous approximation.
It is unique entropy solution in sense of Definition 3.1.
If u_1, u_2 are entropy solutions of (1) corresponding to initial data $u_{01},\ u_{02} \in L^p(D)$ and the boundary data $a_1,\ a_2 \in L^\infty(\Sigma)$, respectively, then for any $t \in (0, T)$

$$\mathbb{E}\int_D |u_1 - u_2| \leq \int_D |u_{01} - u_{02}| dx + \int_\Sigma \max_{min(a_1,a_2) \leq r,s \leq max(a_1,a_2)} |(f(r) - f(s)) \cdot \vec{n}(x)|.$$

Theorem

Under assumptions H_1-H_3 there exists a unique measure-valued entropy solution.

For a continuous flux function $f : \mathbb{R} \mapsto \mathbb{R}^N$ and for any measurable boundary data $a : \Sigma \mapsto \mathbb{R}$ with $\overline{f}(a, x) \in L^1(\Sigma)$ where $\overline{f} : \mathbb{R} \times \partial D \mapsto \mathbb{R}$ is defined by $\overline{f}(s, x) := \sup\{|f(r) \cdot \vec{n}(x)|,\ r \in [-s^-, s^+]\}$. Now we give the definition of renormalized stochastic entropy solution.

Definition

Let $a \in \mathcal{M}(\Sigma)$ *with* $\bar{f}(a,x) \in L^1(\Sigma)$ *and* $u_0 \in L^1(D)$. *We call* $u \in L^1(\Omega; L^1(Q))$ *a renormalized stochastic entropy solution of the conservation law (1)-(3) if there exist some families of non-negative random measures* $\mu_l := \mu_l(\omega; t, x)$ *and* $\nu_l := \nu_l(\omega; t, x)$ *on* $[0, T] \times \bar{D}$ *such that*

$$\mathbb{E}\mu_l(\cdot; [0, T] \times \bar{D}) \to 0, \ \mathbb{E}\nu_{-l}(\cdot; [0, T] \times \bar{D}) \to 0, \ \text{as } l \to +\infty,$$

Definition

and the following entropy inequalities hold: for all $k \in \mathbb{R}$, *for all* $l \geq k$, *for any* $\xi \in \mathcal{D}^+([0,T) \times \mathbb{R}^N)$,

$$\int_Q (u \wedge l - k)^+ \xi_t - \int_Q sgn_0^+(u \wedge l - k)[f(u \wedge l) - f(k)] \cdot \nabla \xi$$
$$+ \int_Q sgn_0^+(u \wedge l - k) h(u \wedge l) \xi dx dw(t)$$
$$+ \frac{1}{2} \int_Q [1 - sgn_0^+(k - u \wedge l)] h^2(k) \xi$$
$$+ \int_D (u_0 \wedge l - k)^+ \xi + \int_\Sigma sgn_0^+(a \wedge l - k) \xi \omega^+(x, k, a \wedge l)$$
$$\geq -\langle \mu_l, \xi \rangle, \quad dP - a.s.,$$

Definition

and for all $k \in \mathbb{R}$, *for all* $l \leq k$, *for any* $\xi \in \mathcal{D}^+([0,T) \times \mathbb{R}^N)$,

$$\int_Q (k - u \vee l)^+ \xi_t - \int_Q sgn_0^+(k - u \vee l)[f(k) - f(u \vee l)] \cdot \nabla \xi$$
$$+ \int_Q sgn_0^+(k - u \vee l) h(u \vee l) \xi dx dw(t)$$
$$+ \frac{1}{2} \int_Q [1 - sgn_0^+(u \vee l - k)] h^2(k) \xi$$
$$+ \int_D (k - u_0 \wedge l)^+ \xi + \int_\Sigma sgn_0^+(k - a \vee l) \xi \omega^-(x, k, a \vee l)$$
$$\geq -\langle \nu_l, \xi \rangle, \quad dP - a.s..$$

By using the facts $\lim_{\delta\to 0}\eta_\delta(x) = x^+$, $\lim_{\delta\to 0}\eta'_\delta(x) = sgn_0^+(x)$ and $\lim_{\delta\to 0}\eta''_\delta(x-k) = \delta_x(k)$, we have

$$\lim_{\delta\to 0}\mu_{\eta_\delta,k}(\xi) =$$
$$\int_Q (u-k)^+\xi_t - \int_Q sgn_0^+(u-k)[f(u)-f(k)]\cdot\nabla\xi$$
$$+\int_Q sgn_0^+(u-k)h(u)\xi dxdw(t)$$
$$+\frac{1}{2}\int_Q [1-sgn_0^+(k-u)]h^2(k)\xi$$
$$+\int_D (u_0-k)^+\xi + \int_\Sigma sgn_0^+(a-k)\xi\omega^+(x,k,a)$$
$$=: -\tilde{\mu}_k(\xi).$$

In addition, we can also have the following

Definition

Let $a \in \mathcal{M}(\Sigma)$ *with* $\bar{f}(a, x) \in L^1(\Sigma)$ *and* $u_0 \in L^1(D)$*. A function* u *of* $L^1(\Omega; L^1(Q))$ *is said to be a renormalized stochastic entropy solution of conservation law (1)-(3) if for all* $k, l \in \mathbb{R}$*, for any* $\xi \in \mathcal{D}^+([0, T) \times \mathbb{R}^N)$*, the functionals*

Definition

$$\mu_{k,l}(\xi) = -\int_Q (u \wedge l - k)^+ \xi_t$$
$$+\int_Q sgn_0^+(u \wedge l - k)[f(u \wedge l) - f(k)] \cdot \nabla \xi$$
$$-\int_Q sgn_0^+(u \wedge l - k)h(u \wedge l)\xi dxdw(t)$$
$$-\frac{1}{2}\int_Q [1 - sgn_0^+(k - u \wedge l)]h^2(k)\xi$$
$$-\int_D (u_0 \wedge l - k)^+ \xi - \int_\Sigma sgn_0^+(a \wedge l - k)\xi\omega^+(x, k, a \wedge l) \ \ a.s.,$$

Definition

$$\nu_{k,l}(\xi) = -\int_Q (k - u \vee l)^+ \xi_t$$
$$+ \int_Q sgn_0^+(k - u \vee l)[f(k) - f(u \vee l)] \cdot \nabla \xi$$
$$- \int_Q sgn_0^+(k - u \vee l) h(u \vee l) \xi dx dw(t)$$
$$-\frac{1}{2} \int_Q [1 - sgn_0^+(u \vee l - k)] h^2(k) \xi$$
$$- \int_D (k - u_0 \wedge l)^+ \xi - \int_\Sigma sgn_0^+(k - a \vee l) \xi \omega^-(x, k, a \vee l) \;\; a.s.$$

are random measure on $[0, T] \times \bar{D}$ *satisfying*

Definition

$$\lim_{l \to +\infty} \mathbb{E}\mu_{k,l}^{+}(\cdot; [0, T] \times \bar{D}) = 0$$

$$\lim_{l \to -\infty} \mathbb{E}\nu_{k,l}^{+}(\cdot; [0, T] \times \bar{D}) = 0$$

$\forall k \in \mathbb{R}$, *where* $\mu_{k,l}^{+}$ *denotes the positive part of the random measure* $\mu_{k,l}$.

It is not difficult to show the equivalence of the above two definitions by using the following decomposition

$$\mu_{k,l}(\xi) = \tilde{\mu}_k(\xi) - \tilde{\mu}_l(\xi) - \int_Q sgn_0^+(u-l)h(l)\xi dxdw(t) - \frac{1}{2}\int_Q [1 - sgn_0^+(l-u)]h^2(l)dxdt - \int_\Sigma [\omega^+(x,k,a\wedge l) - \omega^+(x,k,a) + \omega^+(x,l,a)]\xi, \quad dP-a.s.,$$

where we used the facts that for $l > k$, $(u\wedge l - k)^+ = (u-k)^+ - (u-l)^+$ and $sgn_0^+(u\wedge l - k)[f(u\wedge l) - f(k)] = sgn_0^+(u-k)[f(u)-f(k)] - sgn_0^+(u-l)[f(u)-f(l)]$. In other words, μ_l in the first definition is $\mu_{k,l}^+$ of the second definition.

Next, we consider the equivalence between the renormalized stochastic entropy solutions and stochastic entropy solutions.

Proposition

If u is a stochastic entropy solution in sense of Definition 3.1, then u is a renormalized stochastic entropy solution in Definition 4.1.

The main result is the following

Theorem

Let $a \in \mathcal{M}(\Sigma)$ with $\bar{f}(a, x) \in L^1(\Sigma)$ and $u_0 \in L^1(D)$. Under assumptions $H_1 - H_2$ there exists a unique renormalized stochastic entropy solution.

Remark *C. Bauzet, G. Vallet, P. Wittbold, J Funct Anal 2014* posed an open problem: whether there exists a renormalized stochastic entropy solution to the problem (1)-(3) with $a = 0$? Our result clearly provides a positive answer to this open problem.

Thank You!