Gaussian estimates of the density for systems of non-linear stochastic heat equations

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1. Introduction and Preliminaries

Consider systems of non-linear stochastic heat equations:

$$\frac{\partial u_i}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u_i}{\partial x^2}(t,x) + b_i(u(t,x)) + \sum_{j=1}^{q} \sigma_{ij}(u(t,x))\dot{W}^j(t,x), \tag{1}$$

with vanishing initial conditions, $x \in \mathbb{R}^d$, $u = (u_1, \ldots, u_m)$, $\sigma_{ij}, b_i : \mathbb{R}^m \to \mathbb{R}$ are globally Lipschitz functions.

- $\dot{W}$ is a centered Gaussian noise with covariance

  $$\mathbb{E}[\dot{W}^i(t,x)\dot{W}^j(s,y)] = \delta(t-s)f(x-y)\delta_{ij},$$

  where f is a non-negative and non-negative definite continuous function on $\mathbb{R}^d \backslash \{0\}$ such that it is the Fourier transform of a non-negative measure μ on $\mathbb{R}^d$.

- $W = \{W^j(\varphi), j = 1, \ldots, q, \varphi \in \mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d)\}$, is a zero mean Gaussian random variables with covariance

$$\mathbb{E}(W^i(\varphi)W^j(\psi)) = \delta_{ij} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x) f(x-y) \psi(t,y) dxdydt \,.$$

- W can be extended to $\mathcal{H}_T^q = L^2([0,T]; \mathcal{H}^q)$, where $\mathcal{H}^q$ is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^d; \mathbb{R}^q)$ under the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}^q} = \sum_{l=1}^q \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \varphi_l(x) f(x-y) \psi_l(y)$$

Then $W_t(g) := \sum_{i=1}^q W(1_{[0,t]} g_i)$ is a cylindrical Wiener process in $\mathcal{H}^q$.

- For any predictable process $g \in L^2(\Omega \times [0,\infty); \mathcal{H}^q)$

$$\int_0^T g(t) dW_t = \int_0^T \int_{\mathbb{R}^d} g(t,x) W(dt,dx)$$

and we have the isometry property

$$\mathbb{E} \left| \int_0^T g(t) dW_t \right|^2 = \mathbb{E} \int_0^T \|g(t)\|^2_{\mathcal{H}^q} dt.$$

See Da Prato-Zabczyk 1992, Dalang-Quer Sardanyons 2011

Definition

A $\mathbb{R}^m$-valued adapted stochastic process $\{u(t,x)=(u_1(t,x),\ldots,u_m(t,x)),(t,x)\in[0,T]\times\mathbb{R}^d\}$ is a mild solution of Eq.(1) if for all $0\le t\le T$, $x\in\mathbb{R}^d$, $i=1,\ldots,m$,

$$\begin{aligned} u_i(t,x) &= \sum_{j=1}^{q}\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)\sigma_{ij}(u(s,y))W^j(ds,dy) \\ &\quad+\int_0^t\int_{\mathbb{R}^d} b_i(u(s,y))\Gamma(t-s,x-y)dyds, \qquad \mathbb{P}-a.s., \end{aligned}$$

where $\Gamma(t,x)=(2\pi t)^{-\frac{d}{2}}\exp\{-\frac{|x|^2}{2t}\}$ is the fundamental solution to $\frac{\partial u}{\partial t}-\frac{1}{2}\frac{\partial^2 u}{\partial x^2}=0$.

The main assumptions are the following:

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(t)(\xi)|^2 \mu(d\xi)dt < \infty \,, \tag{2}$$

where $\mathcal{F}\varphi$ the Fourier transform of φ, given by $\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} \varphi(x)e^{-i\xi\cdot x}dx$. Condition (2) is equivalent to

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty \,. \tag{3}$$

However, we need a slightly stronger condition than (3) to prove our main result.

($\mathbf{H}_\eta$) For some $\eta \in (0,1)$, it holds:

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^\eta}\mu(d\xi) < \infty \,.$$

Refer to E. Nualart, 2013.

Theorem

Assume condition (2) is satisfied, then there exists a unique mild solution u to Eq.(1) such that for all $p \geq 1$ *and* $T > 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|u(t,x)|^p < +\infty. \tag{4}$$

Furthermore, if condition $(\mathbf{H}_\eta)$ *holds, then for all* $\gamma_1 \in (0, \frac{1-\eta}{2})$, $s, t \in [0, T]$, $x \in \mathbb{R}^d$ *and* $p > 1$,

$$\mathbb{E}|u(t,x) - u(s,x)|^p \leq C_{p,T}|t-s|^{\gamma_1 p} \tag{5}$$

and for all $\gamma_2 \in (0, 1-\eta)$, $t \in [0, T]$, $x, y \in \mathbb{R}^d$ *and* $p > 1$,

$$\mathbb{E}|u(t,x) - u(t,y)|^p \leq C_{p,T}|x-y|^{\gamma_2 p}. \tag{6}$$

- Notice that $\mathbb{E}(W(h_1)W(h_2)) = \langle h_1, h_2\rangle_{\mathcal{H}_T^q}, h_1, h_2 \in \mathcal{H}_T^q$. Then we can develop Malliavin calculus (see, for instance, Nualart 2006). The Malliavin derivative is denoted by D and for any $N \geq 1$ and any real number $p \geq 2$, the domain of the iterated derivative D^N in $L^p(\Omega; \mathcal{H}_0^{\otimes N})$ is denoted by $\mathbb{D}^{N,p}$, $\mathbb{D}^\infty := \cap_{p\geq 1} \cap_{k\geq 1} \mathbb{D}^{k,p}$.
- Set $\mathcal{H}_{s,t}^q = L^2([s,t]; \mathcal{H}^q)$ and $\|\cdot\|_{s,t} := \|\cdot\|_{\mathcal{H}_{s,t}^q}$. $\mathbb{D}_{s,t}^{k,p}$ is the completion of the space of smooth functionals with respect to this following seminorm:

$$\|F\|_{k,p}^{s,t} = \left\{ \mathbb{E}_s[|F|^p] + \sum_{j=1}^{k} \mathbb{E}_s[\|D^j F\|_{(\mathcal{H}_{s,t}^q)^{\otimes j}}] \right\}^{\frac{1}{p}},$$

where $\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_s]$. We say that $F \in \overline{\mathbb{D}}_{s,t}^{k,p}$ if $F \in \mathbb{D}_{s,t}^{k,p}$ and $\|F\|_{k,p}^{s,t} \in \cap_{q\geq 1} L^q(\Omega)$, and we set $\overline{\mathbb{D}}_{s,t}^{\infty} := \cap_{k\geq 1} \cap_{p\geq 1} \overline{\mathbb{D}}_{s,t}^{k,p}$.

Lemma

Assume σ, b are smooth functions with bounded partial derivatives of all orders. Then, for any $(t,x) \in [0,T] \times \mathbb{R}^d$, $u_i(t,x) \in \mathbb{D}^\infty$, for all $i = 1, \dots, m$. Moreover, for $0 \le a < b \le T$ and $p \ge 1$, there exits a positive constant $C = C(a,b)$ such that for all $\delta \in (0, b-a]$:

$$\sup_{(t,x)\in[b-\delta,b]\times\mathbb{R}^d} \mathbb{E}_a \|D^m u_i(t,x)\|^{2p}_{(\mathcal{H}^q_{b-\delta,b})^{\otimes m}} \le C(\Phi(\delta))^{mp}, \quad a.s., \quad (7)$$

where $\Phi(\delta) = \int_0^\delta \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt$.

In order to prove the existence of the smooth density of $u(t,x)$, we need the following conditions:

(**H1**) There exists $\beta > 0$ such that for all $\varepsilon \in (0,1]$,

$$C\varepsilon^{\beta} \le \int_0^{\varepsilon} \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(r)(\xi)|^2 \mu(d\xi) dr.$$

(**H2**) Let β, γ_1 and γ_2 be given in (**H1**), (5) and (6) respectively.
(i) There exists $\beta_1 > \gamma_2 \vee \beta$ such that

$$\int_0^{\varepsilon} \langle |\cdot|^{\gamma_2}\Gamma(r,\cdot), \Gamma(r,*)\rangle_{\mathcal{H}} dr \le C\varepsilon^{\beta_1}, \tag{8}$$

(ii) There exists $\beta_2 > \gamma_1 \vee \beta$ such that,

$$\int_0^{\varepsilon} r^{\gamma_1} \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(r)(\xi)|^2 \mu(d\xi) dr \le C\varepsilon^{\beta_2}. \tag{9}$$

Refer to E. Nualart, 2013.

Theorem

Assume conditions $(\mathbf{H}_\eta)$, $(\mathbf{H1})$ *and* $(\mathbf{H2})$ *hold,* σ, *b are smooth functions with bounded partial derivatives of all orders . Then for all* $(t, x) \in (0, T] \times \mathbb{R}^d$, $u(t, x)$ *admits a smooth density* $p_{t,x}(\cdot)$ *on* $\Sigma := \{y \in \mathbb{R}^m : \sigma_1(y), \ldots, \sigma_q(y) \text{ span } \mathbb{R}^m\}$.

2. Lower and upper bound for the density

- S. Kusuoka and D. Stroock, J. Fac, Sci, Univ. Tokyo, 1987, diffusion process with uniform ellipticity
- A. Kohatsu-Higa, PFRF, 2003, stochastic heat equation with space-time white noise
- V. Bally, AOP, 2006, locally elliptic Itô processes
- H. Guérin, S. Méléard and E. Nualart, JFA, 2006, Landau stochastic differential equation.
- E. Nualart and L. Quer-Sardanyons, SPA, 2013, stochastic heat equation with white in time color in space noise

  ...

We intend to consider the system of stochastic heat equation with white in time color in space noise.

For heat equation, condition (2) implies

$$C_1(t-s) \le \int_s^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(r)(\xi)|^2 \mu(d\xi)dr. \tag{10}$$

Furthermore, condition $(\mathbf{H}_\eta)$ implies

$$\int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(r)(\xi)|^2 \mu(d\xi)dr \le C_2 t^{1-\eta}. \tag{11}$$

- Estimate (10) will play an important role in the proof of the lower bound. This has prevented us from considering the other type of SPDEs, such as stochastic wave equation.

In order to obtain the lower bound, we need more conditions on the coefficients σ and b:
(**H3**) Assume that b_i are bounded, for any $i = 1, \dots, m$, and there exist positive constants C_1 and C_2, such that for all $\xi \in \mathbb{R}^m$,

$$C_1|\xi|^2 \leq \inf_{x,y\in\mathbb{R}^d} \sum_{i,j=1}^{m} \sum_{k=1}^{q} \sigma_{ik}(x)\sigma_{jk}(y)\xi_i\xi_j \tag{12}$$

and

$$\sup_{x,y\in\mathbb{R}^d} \sum_{i,j=1}^{m} \sum_{k=1}^{q} \sigma_{ik}(x)\sigma_{jk}(y)\xi_i\xi_j \leq C_2|\xi|^2. \tag{13}$$

Theorem (Main result)

Assume that conditions $(\mathbf{H}_\eta)$, $(\mathbf{H1})$-$(\mathbf{H3})$ *hold, and* σ, b *are* $\mathcal{C}^\infty$ *with bounded derivatives of all orders. Then for all* $(t,x) \in (0,T] \times \mathbb{R}^d$, $u(t,x)$ *has a smooth density* $p_{t,x}(y)$,

$$C_1 \Phi(t)^{-\frac{m}{2}} e^{-\frac{|y|^2}{C_2 \Phi(t)}} \leq p_{t,x}(y) \leq C_3 \Phi(t)^{-\frac{m}{2}} e^{-\frac{(|y|-C_4 T)^2}{C_5 \Phi(t)}}, \qquad (14)$$

where $\Phi(t) = \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds$.

The idea of the proof of lower bound:

Definition (See Kohatsu-Higa, 2003 or E. Nualart and L. Quer-Sardanyons, 2013)

Let F be a non-degenerate m-dimensional $\mathcal{F}_t$-measurable random vector. F is called uniformly elliptic if for any partition $\pi_N = \{0 = t_0 < t_1 < \cdots < t_N = t\}$ whose norm $\|\pi_N\| := \max\{|t_{i+1} - t_i|; i = 0, \ldots, N-1\}$ is smaller than some $\varepsilon > 0$ and $\|\pi_N\| \to 0$ as $N \to \infty$, there exist smooth $\mathcal{F}_{t_n}$-measurable random vectors $F_n \in (\overline{\mathbb{D}}^{\infty}_{t_{n-1},t_n})^m (n = 0, \ldots, N)$ such that $F_N = F$ and F_n can be written in the following form:

$$F_n = F_{n-1} + I_n(h) + G_n, \quad n = 1, \ldots, N, \tag{15}$$

where the random vectors $I_n(h)$ and G_n satisfy the following conditions:

Definition (Cont.)

(A1) $\mathcal{F}_{t_n}$-measurable $G_n \in (\overline{\mathbb{D}}^{\infty}_{t_{n-1},t_n})^m$, and $\exists g \in \mathcal{H}_T$ with $\|g(s)\|_{\mathcal{H}} > 0$ such that, for all $k \in \mathbb{N}$ and $p \geq 1$,

$$\|G_n\|_{k,p}^{t_{n-1},t_n} \leq C\Delta_{n-1}(g)^{1/2+\gamma} \quad a.s., \tag{16}$$

for some $\gamma > 0$, where

$$0 < \Delta_{n-1}(g) := \int_{t_{n-1}}^{t_n} \|g(s)\|^2_{\mathcal{H}} ds < \infty, \quad n = 1, \ldots, N.$$

(A2) Random vector $I_n(h)$ with the component:

$$I_n^i(h) = \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} h_i(s,y) W(ds,dy), \quad i = 1, \ldots, m,$$

Definition (Cont.)

where $h_i \in \mathcal{F}_{t_{n-1}}$ and

$$\|F_n^i\|_{k,p} + \sup_{\omega\in\Omega} \|h_i\|_{t_{n-1},t_n}(\omega) \le C.$$

(A3) Let $A = (a_{i,j})$ denote the $m \times m$ matrix defined by

$$a_{i,j} = \Delta_{n-1}(g)^{-1} \int_{t_{n-1}}^{t_n} \langle h_i(s), h_j(s)\rangle_{\mathcal{H}^q} ds,$$

and for all $\xi \in \mathbb{R}^m$, $C_1|\xi|^2 \le \xi^T A\xi \le C_2|\xi|^2$.
(A4) There is a constant C such that, for $p > 1$ and $\rho \in (0, 1]$,

$$\mathbb{E}_{t_{n-1}}\left[\det(M^{t_{n-1},t_n}_{I_n(h)+\rho G_n})^{-p}\right] \le C\Delta_{n-1}(g)^{-mp} \quad a.s..$$

Theorem (E. Nualart and L. Quer-Sardanyons, 2013)

Let F be a m-dimensional uniformly elliptic random vector and denote by p_F its density. Then $\exists M > 0$ such that for all $y \in \mathbb{R}^m$,

$$p_F(y) \geq M \|g\|_{\mathcal{H}_t}^{-m/2} \exp\left(-\frac{|y|^2}{M\|g\|_{\mathcal{H}_t}^2} \right). \tag{17}$$

It suffices to check that $u(t,x)$ is a m-dimensional uniformly elliptic random vector with $g(\cdot) := \Gamma(t-\cdot)$ in (A1).
We consider a partition $0 = t_0 < t_1 \cdots < t_N = t$ with $\sup_{1\le i\le N}(t_i - t_{i-1}) \to 0$ as $N \to \infty$, and define, for $i = 1, \dots, m$,

$$
\begin{aligned}
F_n^i &= \int_0^{t_n}\int_{\mathbb{R}^d}\sum_{j=1}^q \Gamma(t-s,x-y)\sigma_{ij}(u(s,y))W^j(ds,dy) \\
&\quad + \int_0^{t_n}\int_{\mathbb{R}^d} b_i(u(s,y))\Gamma(t-s,x-y)dyds.
\end{aligned}
$$

Then we can obtain a decomposition of F_n:

$$
F_n = F_{n-1} + I_n(h) + G_n,
$$

where $I_n(h) := (I_n^1(h), \dots, I_n^m(h))$ with

$$I_n^i(h) := \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \sum_{j=1}^{q} h_{ij}(s,y) W^j(ds,dy)$$

and

$$h_{ij}(s,y) := \Gamma(t-s, x-y)\sigma_{ij}(u_{n-1}(s,y)),$$

and $G_n := (G_n^1, \dots, G_n^m)$ with

$$G_n^i := \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} b_i(u(s,y))\Gamma(t-s, x-y)ds$$
$$+ \int_{t_{n-1}}^{t_n} \int_{\mathbb{R}^d} \sum_{j=1}^{q} \Gamma(t-s, x-y)[\sigma_{ij}(u(s,y)) - \sigma_{ij}(u_{n-1}(s,y))]W^j(ds,dy)$$

The idea of the proof of upper bound:

Using integration by part formula of Malliavin calculus, the joint density $p_{t,x}(\cdot)$ of $u(t,x)$ has the expression:

$$p_{t,x}(y) = (-1)^{m-\mathrm{card}(\mathbb{S})}\mathbb{E}[\mathbf{1}_{\{u_i(t,x)>y_i, i\in\mathbb{S};\ u_i(t,x)<y_i,\ i\notin\mathbb{S}, i=1,\dots,m\}} H_{(1,2,\dots,m)}(u(t,x),1)]$$

where $\mathbb{S}$ be a subset of $\{1,\ldots,m\}$, card$(\mathbb{S})$ denotes the cardinality of $\mathbb{S}$, $H_\alpha(F,G)$ are recursively given by

$$H_{(i)}(F,G) := \sum_{j=1}^{m} \delta(G(M_F^{-1})_{ij}DF^j),$$

$$H_\alpha(F,G) := H_{(\alpha_m)}(F, H_{(\alpha_1,\alpha_2,\ldots,\alpha_{m-1})}(F,G))$$

for any $F \in (\mathbb{D}^\infty)^m$, $G \in \mathbb{D}^\infty$, $\alpha = (\alpha_1,\ldots,\alpha_m) \in \{1,\ldots,m\}^m$.

Then

$$p_{t,x}(y) \leq \mathbb{P}\left\{|u(t,x)| > |y|\right\}^{1/2} \left\{\mathbb{E}[H_{(1,2,\ldots,m)}(u(t,x),1)]^2\right\}^{1/2}$$

Applying the following two estimates:

(i) $\left\|\mathbb{D}^k(u_i(t,x))\right\|_{L^p(\Omega,\mathcal{H}_t^k)} \leq C\Phi(t)^{1/2}$, i=1, . . . , m,

(ii) $\left\|\det(M_{u(t,x)})^{-1}\right\|_{L^p(\Omega)} \leq C\Phi(t)^{-m}$.

(iii) $\mathbb{P}\{|u(t,x)| > |y|\} \leq 2\exp\left\{-\frac{(|y|-CT)^2}{2C_1\Phi(t)}\right\}$.

we obtain the upper bound finally.

3. Examples

(A) *Riesz kernel*:

$$f(x) = |x|^{-\gamma}, \quad 0 < \gamma < 2 \wedge d.$$

- ($\mathbf{H}_\eta$) holds with $\eta > \frac{\gamma}{2}$.
- (**H1**) holds with $\beta = \frac{2-\gamma}{2}$.
- (**H2**) holds with $\beta_1 = \frac{2-\gamma}{2} + \frac{\gamma_2}{2}$, $\beta_2 = \frac{2-\gamma}{2} + \gamma_1$.

(B) *Bessel kernel*:

$$f(x) = \int_0^\infty u^{\frac{\alpha-d-2}{2}} e^{-u} e^{-\frac{|x|^2}{4u}} du, \quad d-2 < \alpha < d.$$

- ($\mathbf{H}_\eta$) holds with $\eta > \frac{d-\alpha}{2}$.
- (**H1**) holds with $\beta = \frac{\alpha-d}{2} + 1$.
- (**H2**) holds with $\beta_1 = \frac{2+\alpha-d}{2} + \frac{\gamma_2}{2}$, $\beta_2 = \frac{2+\alpha-d}{2} + \gamma_1$.

(C) *Fractional kernel*:

$$f(x) = \prod_{j=1}^{d} |x_j|^{2H_j-2}, \quad \frac{1}{2} < H_j < 1, \sum_{j=1}^{d} H_j > d-1.$$

- ($\mathbf{H}_\eta$) holds with $\eta > d - \sum_{j=1}^{d} H_j$.
- (**H1**) holds with $\beta = \sum_{j=1}^{d} H_j - d + 1$.
- (**H2**) holds with $\beta_1 = \frac{\gamma_2}{2} + \sum_{j=1}^{d} H_j - d + 1$, $\beta_2 = \sum_{j=1}^{d} H_j - d + 1 + \gamma_1$.

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Thank you very much!