Heat kernels of non-symmetric jump processes: beyond the stable case

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Joint work with P. Kim and Z. Vondraček: arXiv:1606.02005

Suppose $d \geq 1$, $\alpha \in (0,2)$ and κ is a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$0 < \kappa_0 \leq \kappa(x,z) \leq \kappa_1 \,, \quad \kappa(x,z) = \kappa(x,-z) \,, \tag{1}$$

and for some $\beta \in (0,1)$,

$$|\kappa(x,z) - \kappa(y,z)| \leq \kappa_2 |x-y|^\beta \,. \tag{2}$$

Define

$$\mathcal{L}_\alpha^\kappa f(x) = \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} (f(x+z) - f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \, dz \,. \tag{3}$$

This is a non-symmetric and non-local stable-like operator.

These operator can be regarded as the non-local counterpart of elliptic operators in non-divergence form. In this context the Hölder continuity of $\kappa(\cdot, z)$ in (2) is a natural assumption.

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In [PTRF16], Z.-Q. Chen and X. Zhang proved the existence and uniqueness of a non-negative jointly continuous function $p^\kappa_\alpha(t, x, y)$ in $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ solving the equation

$$\partial_t p^\kappa_\alpha(t, x, y) = \mathcal{L}^\kappa_\alpha p^\kappa_\alpha(t, \cdot, y)(x)\,, \quad x \neq y\,,$$

and satisfying four properties - an upper bound, Hölder's estimate, fractional derivative estimate and continuity.

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and satisfying four properties - an upper bound, Hölder's estimate, fractional derivative estimate and continuity.

Their main result is as follows:

Theorem (Chen-Zhang)

There exists a unique non-negative jointly continuous function $p^\kappa_\alpha(t,x,y)$, $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$, solving

$$\partial_t p^\kappa_\alpha(t,x,y)=\mathcal{L}^\kappa_\alpha p^\kappa_\alpha(t,\cdot,y)(x)\,,\quad x\neq y\,, \tag{4}$$

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and satisfying the following properties:

(i) There exists $c_1 > 0$ such that for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$p^\kappa_\alpha(t,x,y) \leq c_1 t(t^{\frac{1}{\alpha}} + |x-y|)^{-d-\alpha}.$$

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(ii) For every $\gamma\in(0,\alpha\wedge 1)$, there exists $c_2>0$ such that for all $t\in(0,1]$ and $x,x',y\in\mathbb{R}^d$,

$$|p^\kappa_\alpha(t,x,y)-p^\kappa_\alpha(t,x',y)|\le c_2|x-x'|^\gamma t^{1-\frac{\gamma}{\alpha}}(t^{\frac{1}{\alpha}}+|x-y|\wedge|x'-y|)^{-d-\alpha}.$$

Theorem (Chen-Zhang) (cont)

(iii) For all $x \neq y$ in $\mathbb{R}^d$, the map $t \mapsto \mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t, \cdot, y)(x)$ is continuous in $(0, 1]$ and

$$|\mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t, \cdot, y)(x)| \leq c_3(t^{\frac{1}{\alpha}} + |x - y|)^{-d-\alpha}.$$

Theorem (Chen-Zhang) (cont)

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$$|\mathcal{L}_\alpha^\kappa p_\alpha^\kappa(t,\cdot,y)(x)| \leq c_3(t^{\frac{1}{\alpha}} + |x-y|)^{-d-\alpha}.$$

(iv) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y) f(y)\, dy - f(x) \right| = 0\,.$$

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Moreover, the following conclusions are valid:

Theorem (Chen-Zhang) (cont)

(1) For all $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y)dy = 1.$$

Theorem (Chen-Zhang) (cont)

(1) For all $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa_\alpha(t, x, y)dy = 1.$$

(2) For all $s, t \in (0, 1]$ with $s + t \in (0, 1]$, and all $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa_\alpha(s, x, z)p^\kappa_\alpha(t, x, y)dz = p^\kappa_\alpha(s + t, x, y).$$

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$$\int_{\mathbb{R}^d} p_\alpha^\kappa(s, x, z)p_\alpha^\kappa(t, x, y)dz = p_\alpha^\kappa(s + t, x, y).$$

(3) There exists $c_4 > 0$ such that for all $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,

$$p_\alpha^\kappa(t, x, y) \geq c_4 t(t^{\frac{1}{\alpha}} + |x - y|)^{-d-\alpha}.$$

Theorem (Chen-Zhang) (cont)

(4) For any $f \in C_b^2(\mathbb{R}^d)$,

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y)(f(y) - f(x))dy = \mathcal{L}_\alpha^\kappa f(x)$$

and the convergence is uniform.

Theorem (Chen-Zhang) (cont)

(4) For any $f \in C_b^2(\mathbb{R}^d)$,

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and the convergence is uniform.

(5) The C_0-semigroup $(P_t^\kappa : t \geq 0)$ defined by

$$P_t^\kappa f(x) = \int_{\mathbb{R}^d} p_\alpha^\kappa(t,x,y)f(y)dy$$

is analytic in $L^p(\mathbb{R}^d)$ for all $p \in [1,\infty)$.

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Our goal is to extend the results of the Chen-Zhang paper to more general operators than the ones defined in (3). These operators will be non-symmetric and not necessarily stable-like. We will replace the kernel $\kappa(x,z)|z|^{-d-\alpha}$ with a kernel $\kappa(x,z)J(z)$ where κ still satisfies (1) and (2), but $J(z)$ is the Lévy density of a rather general symmetric Lévy process.

Setting and assumptions

Suppose that $S = (S_t : t \geq 0)$ is a subordinator, that is, a non-negative Lévy process with $S_0 = 0$. Let ϕ be the Laplace exponent of S, that is,

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad t > 0, \lambda > 0.$$

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A function $\phi : (0, \infty) \mapsto (0, \infty)$ is the Laplace exponent of a subordinator iff it is a Bernstein function satisfying $\lim_{\lambda \to 0} \phi(\lambda) = 0$. Recall that a nonnegative function ϕ on $(0, \infty)$ is a Bernstein function if it is C^∞ and $(-1)^{n-1}\phi^{(n)} \geq 0$.

Setting and assumptions II

Suppose that S has no drift. Then ϕ admits the following expression

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\mu(dt), \quad \lambda > 0,$$

where μ is a measure on $(0, \infty)$ such that $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$. μ is called the Lévy measure of S.

Setting and assumptions II

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Suppose that $B = (B_t : t \geq 0)$ is a Brownian motion in R^d with generator Δ. Suppose that B and S are independent. Then the process $(X_t : t \geq 0)$ defined by $X_t := B_{S_t}$ is a Lévy process and it is called a subordinate Brownian motion. The generator X is $-\phi(-\Delta)$.

Setting and assumptions III

When $\phi(\lambda) = \lambda^{\alpha/2}$, the resulting subordinate Brownian motion is a symmetric α-stable process. Without loss of generality, we will assume $\phi(1) = 1$.

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The Lévy exponent of X is $\Phi(\xi)=\phi(|\xi|^2)$. The Lévy measure of X has a density $J(x)=J(|x|)$ with

$$J(r)=\int_{(0,\infty)}(4\pi t)^{-d/2}e^{-\frac{r^2}{4t}}\mu(dt)\,,\quad r>0.$$

Setting and assumptions IV

Our main assumption is the following *weak lower scaling condition at infinity*: There exist $\delta_1 \in (0, 2]$ and $a_1 \in (0, 1)$ such that

$$a_1 \lambda^{\delta_1} \Phi(r) \le \Phi(\lambda r), \quad \lambda \ge 1, r \ge 1. \tag{5}$$

This condition implies that $\lim_{\lambda \to \infty} \Phi(\lambda) = \infty$ and hence $\int_{\mathbb{R}^d \setminus \{0\}} j(|y|)dy = \infty$.

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The weak lower scaling condition governs the short-time small-space behavior of the subordinate Brownian motion. We also need a weak condition on the behavior of Φ near zero. We assume that

$$\int_0^1 \frac{\Phi(r)}{r} \, dr = C_* < \infty. \tag{6}$$

Setting and assumptions V

Assume that κ is a function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1) and (2). We define

$$\mathcal{L}^{\kappa} f(x) := \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\kappa,\varepsilon} f(x)$$

where

$$\mathcal{L}^{\kappa,\varepsilon} f(x) := \int_{|z|>\varepsilon} (f(x+z) - f(x))\kappa(x,z)J(z)\, dz, \quad \varepsilon > 0.$$

Setting and assumptions V

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$$\mathcal{L}^{\kappa,\varepsilon} f(x) := \int_{|z|>\varepsilon} (f(x+z) - f(x))\kappa(x,z)J(z)\,dz, \quad \varepsilon > 0.$$

For $t > 0$ and $\in \mathbb{R}^d$ we define

$$\rho(t,x) = \rho^{(d)}(t,x) := \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right)\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d}.$$

In case when $\Phi(r) = r^{\alpha}$ we see that $\rho(t,x) = (t^{1/\alpha} + |x|)^{-d-\alpha}$.

Theorem 1

There exists a unique non-negative jointly continuous function $p^\kappa(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ solving

$$\partial_t p^\kappa(t,x,y)=\mathcal{L}^\kappa p^\kappa(t,\cdot,y)(x)\,,\quad x\neq y\,,$$

and satisfying the following properties:

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and satisfying the following properties:

(i) For every $T\geq 1$, there is a constant $c_1>0$ so that for all $t\in(0,T]$ and $x,y\in\mathbb{R}^d$,

$$p^\kappa(t,x,y)\leq c_1 t\rho(t,x-y)\,.$$

Theorem 1 (cont)

(ii) For any $x, y \in \mathbb{R}^d$, $x \neq y$, the mapping $t \mapsto \mathcal{L}^\kappa p^\kappa(t, \cdot, y)(x)$ is continuous in $(0, \infty)$, and, for each $T > 0$ there is a constant $c_2 > 0$ so that for all $t \in (0, T]$, $\varepsilon \in [0, 1]$ and $x, y \in \mathbb{R}^d$,

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t, \cdot, y)(x)| \leq c_2 \rho(t, x - y) .$$

Theorem 1 (cont)

(ii) For any $x, y \in \mathbb{R}^d$, $x \neq y$, the mapping $t \mapsto \mathcal{L}^\kappa p^\kappa(t, \cdot, y)(x)$ is continuous in $(0, \infty)$, and, for each $T > 0$ there is a constant $c_2 > 0$ so that for all $t \in (0, T]$, $\varepsilon \in [0, 1]$ and $x, y \in \mathbb{R}^d$,

$$|\mathcal{L}^{\kappa,\varepsilon} p^\kappa(t, \cdot, y)(x)| \leq c_2 \rho(t, x - y).$$

(iii) For any bounded and uniformly continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p^\kappa(t, x, y) f(y)\, dy - f(x) \right| = 0.$$

Theorem 2

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(1) For all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, $p^\kappa(t, x, y) \geq 0$ and

$$\int_{\mathbb{R}^d} p^\kappa(t, x, y)\, dy = 1 .$$

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(1) For all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, $p^\kappa(t, x, y) \geq 0$ and

$$\int_{\mathbb{R}^d} p^\kappa(t, x, y)\, dy = 1\,.$$

(2) For all $s, t > 0$ and all $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p^\kappa(t, x, z) p^\kappa(s, z, y)\, dz = p^\kappa(t + s, x, y)\,.$$

Theorem 2

(3) For every $T \geq 1$, there is a constant $c_4 > 0$ such that for all $0 < s \leq t \leq T$ and $x, x', y \in \mathbb{R}^d$ with either $x \neq y$ or $x' \neq y$,

$$
\begin{aligned}
&|p^\kappa(s, x, y) - p^\kappa(t, x', y)| \\
&\leq c_4 \left(|t - s| + |x - x'| t \Phi^{-1}(t^{-1})\right) \left(\rho(s, x - y) \vee \rho(s, x' - y)\right).
\end{aligned}
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$$|p^\kappa(s,x,y) - p^\kappa(t,x',y)| \\ \leq c_4 \left(|t-s| + |x-x'|t\Phi^{-1}(t^{-1})\right)\left(\rho(s,x-y) \vee \rho(s,x'-y)\right).$$

(4) For every $T \geq 1$, there exists $c_5 => 0$ so that for all $x, y \in \mathbb{R}^d$, $x \neq y$, and $t \in (0, T]$,

$$|\nabla_x p^\kappa(t,x,y)| \leq c_5 \Phi^{-1}(t^{-1}) t \rho(t, x-y).$$

Theorem 3

(a) Let $\varepsilon > 0$. For any $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, we have

$$\lim_{t\downarrow 0} \frac{1}{t}\left(P_t^\kappa f(x) - f(x)\right) = \mathcal{L}^\kappa f(x),$$

and the convergence is uniform.

Theorem 3

(a) Let $\varepsilon > 0$. For any $f \in C_b^{2,\varepsilon}(\mathbb{R}^d)$, we have

$$\lim_{t\downarrow 0} \frac{1}{t}\left(P_t^\kappa f(x) - f(x)\right) = \mathcal{L}^\kappa f(x),$$

and the convergence is uniform.

(b) The semigroup $(P_t^\kappa)_{t\geq 0}$ of $\mathcal{L}^\kappa$ is analytic in $L^p(\mathbb{R}^d)$ for every $p \in [1, \infty)$.

For a lower bound, we will assume the following weak upper scaling condition: there exist $\delta_2 \in (0,2)$ and $a_2 > 0$ such that

$$\Phi(\lambda r) \le a_2 \lambda^{\delta_2} \Phi(r)\,, \quad \lambda \ge 1, r \ge 1\,. \tag{7}$$

Theorem 4

For a lower bound, we will assume the following weak upper scaling condition: there exist $\delta_2 \in (0,2)$ and $a_2 > 0$ such that

$$\Phi(\lambda r) \le a_2 \lambda^{\delta_2} \Phi(r)\,, \quad \lambda \ge 1, r \ge 1\,. \tag{7}$$

Theorem 4

For every $T \ge 1$, there exists $c_6 > 0$ such that for all $t \in (0, T]$

$$p^\kappa(t,x,y) \ge c_6 \begin{cases} \Phi^{-1}(t^{-1})^d & \text{if } |x-y| \le 3\Phi^{-1}(t^{-1})^{-1}, \\ tj\left(|x-y|\right) & \text{if } |x-y| > 3\Phi^{-1}(t^{-1})^{-1}. \end{cases}$$

In particular, for every $T, M \ge 1$, there exists $c_7 > 0$ for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$ with $|x-y| \le M$,

$$p^\kappa(t,x,y) \ge c_7 t \rho(t, x-y)\,.$$

When the global lower and upper scaling conditions are both satisfied, the lower and upper bound differ by a multiplicative constant.

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Here are some examples:

(i) $\phi(\lambda) = \lambda^{\alpha_1} + \lambda^{\alpha_2}$, $0 < \alpha_1 < \alpha_2 < 1$;
(ii) $\phi(\lambda) = (\lambda + \lambda^{\alpha_1})^{\alpha_2}$, $\alpha_1, \alpha_2 \in (0, 1)$;
(iii) $\phi(\lambda) = (\lambda + m^{1/\alpha})^{\alpha} - m$, $\alpha \in (0, 1)$, $m > 0$;
(iv) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1 + \lambda))^{\alpha_2}$, $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, 1 - \alpha_1]$;
(v) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1 + \lambda))^{-\alpha_2}$, $\alpha_1 \in (0, 1)$, $\alpha_2 \in (0, \alpha_1)$;
(vi) $\phi(\lambda) = \lambda/\log(1 + \lambda^{\alpha})$, $\alpha \in (0, 1)$.

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(iii) $\phi(\lambda) = (\lambda + m^{1/\alpha})^{\alpha} - m$, $\alpha \in (0,1)$, $m > 0$;
(iv) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1+\lambda))^{\alpha_2}$, $\alpha_1 \in (0,1)$, $\alpha_2 \in (0, 1-\alpha_1]$;
(v) $\phi(\lambda) = \lambda^{\alpha_1}(\log(1+\lambda))^{-\alpha_2}$, $\alpha_1 \in (0,1)$, $\alpha_2 \in (0, \alpha_1)$;
(vi) $\phi(\lambda) = \lambda / \log(1+\lambda^{\alpha})$, $\alpha \in (0,1)$.

The functions in (i)–(v) satisfy (5), (6) and (7); while the function in (vi) satisfies (5) and (6), but does not satisfy (7). The function $\phi(\lambda) = \lambda/\log(1+\lambda)$ satisfies (5), but does not satisfy the other two conditions.

1. Background
2. Setting, Assumptions and Main Results
3. Outline of Proof

We follow the ideas and the road-map from the Chen-Zhang paper, and use the freezing coefficient method. At many stages we encounter substantial technical difficulties due to the fact that in the stable-like case one deals with power functions while in the present situation the power functions are replaced with a quite general Φ and its variants.

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Here is a lemma that will give a flavor of the things we have to deal with. For $\gamma, \beta \in \mathbb{R}$, let

$$\rho_\gamma^\beta(t,x) := \Phi^{-1}(t^{-1})^{-\gamma}(|x|^\beta \wedge 1)\rho(t,x), \quad t > 0, x \in \mathbb{R}^d .$$

Note that $\rho_0^0(t,x) = \rho(t,x)$.

Lemma

(a) For every $T \geq 1$, there exists $c > 0$ such that for $0 < t \leq 1$, all $\beta \in [0, \delta_1/2]$ and $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \rho_\gamma^\beta(t,x)\,dx \leq ct^{-1}\Phi^{-1}(t^{-1})^{-\gamma-\beta}.$$

Lemma

(a) For every $T \geq 1$, there exists $c > 0$ such that for $0 < t \leq 1$, all $\beta \in [0, \delta_1/2]$ and $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \rho_\gamma^\beta(t,x)\,dx \leq ct^{-1}\Phi^{-1}(t^{-1})^{-\gamma-\beta}.$$

(b) For every $T \geq 1$, there exists $C_0 > 0$ such that for all $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 \leq \delta_1/2$, $\gamma_1, \gamma_2 \in \mathbb{R}$ and $0 < s < t \leq 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_{\gamma_1}^{\beta_1}(t-s,x-z)\rho_{\gamma_2}^{\beta_2}(s,z)\,dz \\
&\leq C_0\Big((t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\gamma_1-\beta_1-\beta_2}\Phi^{-1}(s^{-1})^{-\gamma_2} \\
&\quad + \Phi^{-1}((t-s)^{-1})^{-\gamma_1}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma_2-\beta_1-\beta_2}\Big)\rho(t,x) \\
&+ C_0(t-s)^{-1}\Phi^{-1}((t-s)^{-1})^{-\gamma_1-\beta_1}\Phi^{-1}(s^{-1})^{-\gamma_2}\rho_0^{\beta_2}(t,x) \\
&+ C_0\Phi^{-1}((t-s)^{-1})^{-\gamma_1}s^{-1}\Phi^{-1}(s^{-1})^{-\gamma_2-\beta_2}\rho_0^{\beta_1}(t,x).
\end{aligned}$$

Lemma (cont)

(c) For all $\beta_1, \beta_2 \geq 0$ with $\beta_1 + \beta_2 \leq \delta_1/2$, $\theta, \eta \in [0,1]$, $\gamma_1 + \beta_1 + 2 - 2\theta > 0$, $\gamma_2 + \beta_2 + 2 - 2\eta > 0$, $0 < t \leq T$ and $x \in \mathbb{R}^d$, we have

$$\int_0^t \int_{\mathbb{R}^d} (t-s)^{1-\theta} \rho_{\gamma_1}^{\beta_1}(t-s, x-z) s^{1-\eta} \rho_{\gamma_2}^{\beta_2}(s,z)\, dz\, ds$$
$$\leq c_2 t^{2-\theta-\eta} \left(\rho^0_{\gamma_1+\gamma_2+\beta_1+\beta_2} + \rho^{\beta_1}_{\gamma_1+\gamma_2+\beta_2} + \rho^{\beta_2}_{\gamma_1+\gamma_2+\beta_1} \right)(t,x).$$

where

$$c_2 = 4C_0(T) B\left((\gamma_1+\beta_1)/2 + 1 - \theta, \gamma_2 + \beta_2/2 + 1 - \eta\right).$$

Let $\mathfrak{K} : \mathbb{R}^d \to (0, \infty)$ be a symmetric function, that is, $\mathfrak{K}(z) = \mathfrak{K}(-z)$. Assume that there are $0 < \kappa_0 \le \kappa_1 < \infty$ such that

$$\kappa_0 \le \mathfrak{K}(z) \le \kappa_1 \,, \qquad \text{for all } z \in \mathbb{R}^d \,. \tag{8}$$

Let $j^{\mathfrak{K}}(z) := \mathfrak{K}(z)J(z)$, $z \in \mathbb{R}^d$. Let $Z^{\mathfrak{K}}$ be the purely discontinuous Lévy process with Lévy measure $j^{\mathfrak{K}}(z)$. The infinitesimal generator of $Z^{\mathfrak{K}}$ is given by

$$\mathcal{L}^{\mathfrak{K}} f(x) = \mathrm{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x))\mathfrak{K}(z)J(z)\, dz.$$

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We first study the heat kernel estimates of this process.

For a fixed $y \in \mathbb{R}^d$, let $\mathfrak{K}_y(z) = \kappa(y, z)$ and let $\mathcal{L}^{\mathfrak{K}_y}$ be the freezing operator

$$\mathcal{L}^{\mathfrak{K}_y} f(x) = \mathcal{L}^{\mathfrak{K}_y,0} f(x) = \lim_{\varepsilon \downarrow 0} \mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x),$$

where

$$\mathcal{L}^{\mathfrak{K}_y,\varepsilon} f(x) = \int_{|z|>\varepsilon} \delta_f(x; z)\kappa(y, z)J(z)dz$$

$$\delta_f(x; z) = f(x + z) + f(x - z) - 2f(x).$$

Let $p_y(t, x) = p^{\mathfrak{K}_y}(t, x)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_y}$.

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Let $p_y(t,x) = p^{\mathfrak{K}_y}(t,x)$ be the heat kernel of the operator $\mathcal{L}^{\mathfrak{K}_y}$.

Define

$$q_0(t,x,y) := \left(\mathcal{L}^{\mathfrak{K}_x} - \mathcal{L}^{\mathfrak{K}_y}\right) p_y(t,\cdot)(x-y)\,.$$

Then we solve the integral equation

$$q(t,x,y) = q_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q(s,z,y)\,dz\,ds$$

by iteration:

$$q_n(t,x,y) := \int_0^t \int_{\mathbb{R}^d} q_0(t-s,x,z)q_{n-1}(s,z,y)\,dz\,ds$$

and

$$q(t,x,y) = \sum_{n=0}^{\infty} q_n(t,x,y).$$

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Finally, we define

$$p^{\kappa}(t,x,y) := p_y(t,x-y) + \int_0^t \int_{\mathbb{R}^d} p_z(t-s,x-z) q(s,z,y)\, dz\, ds\,.$$