Dissipation in parabolic SPDEs

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joint work with D. Khoshnevisan, K.W. Kim and C. Mueller

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Parabolic Anderson Model [PAM] - from the point view of particle system

Consider the following PAM:

$$\frac{\partial u_t(x)}{\partial t} = \triangle u_t(x) + \xi(x) u_t(x) \ \ (t \in \mathbf{R}^+, x \in \mathbf{Z}^d),$$

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- Define $u_t(x) = E[n(t,x)]$, then $u_t(x)$ solves the PAM.

Stochastic Heat Equations [SHE]

Consider the following SHE:

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- u_0 is nonrandom, measurable and $u_0(x)$ is uniformly bounded away from zero and infinity.

Mild solution

Solution in mild form

$$u_t(x) = \int_D p_t(x,y)u_0(y)dy + \int_{(0,t]\times D} p_{t-s}(x,y)\sigma(u_s(y))W(dsdy).$$

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Markov Property

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where $W_s(r,y) = W(s+r,y)$.

Lyapunov exponent and Intermittency

- Upper pth-moment Lyapunov exponent $\bar{\gamma}(p)$ of u at x_0:

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- Full intermittency: if, regardless of the value of x_0,

$$p \to \frac{\bar{\gamma}(p)}{p} \text{ is strictly increasing for all } p \geq 2.$$

What does intermittency mean?

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Apply the Hoph-Cole transformation $u_t(x) := \exp(h_t(x))$. $\{h_t(x)\}$ satisfies the following KPZ equation:

$$\frac{\partial}{\partial t} h_t(x) = \frac{\partial^2}{\partial x^2} h_t(x) + (\frac{\partial}{\partial x} h_t(x))^2 + \frac{\partial^2}{\partial t \partial x} W(t, x).$$

Chaotic Behaviors

$$c < \limsup_{R\to\infty} \frac{\log \sup_{|x|\le R} u_t(x)}{(\log R)^d} < C,$$

- PAM, $d = 2/3$ [Conus-Joseph-Khoshnevisan, 2013].
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The chaotic estimates for SHE give

$$a_t < \limsup_{|x|\to\infty} \frac{h_t(x)}{(\log |x|)^{2/3}} < A_t.$$

Moments

Consider

$$\frac{\partial}{\partial t}u_t(x) = \triangle u_t(x) + \lambda\sigma(u_t(x))\frac{\partial^2}{\partial t \partial x}W(t,x),\ x \in [-1,1]$$

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$$ct \leq \liminf_{\lambda\to\infty} \frac{\log\sqrt{\int_{-1}^{1} E|u_t(x)|^2 dx}}{\lambda^4} \leq \limsup_{\lambda\to\infty} \frac{\log\sqrt{\int_{-1}^{1} E|u_t(x)|^2 dx}}{\lambda^4} \leq Ct$$

[K.W. Kim-Khoshnevisan, 2015].

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$$\liminf_{\lambda\to\infty} \frac{\log\log \inf_x E|u_t(x)|^2}{\log \lambda} = \limsup_{\lambda\to\infty} \frac{\log\log \sup_x E|u_t(x)|^2}{\log \lambda} = 4$$

[Foondun-Joseph, 2014].

Dissipation Theorem. When λ goes to ∞

Consider

$$\frac{\partial}{\partial t} u_t(x;\lambda) = \triangle u_t(x;\lambda) + \lambda\sigma(u_t(x;\lambda)) \frac{\partial^2}{\partial t \partial x} W(t,x),\ x \in [-1,1]$$

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Theorem (Khoshnevisan-K.W. Kim-Mueller-S, 2016+)

For fixed $t > 0$

$$\sup_{x\in[-1,1]} u_t(x;\lambda) \to 0 \text{ in probability as } \lambda \to \infty.$$

Dissipation Theorem. When $t \to \infty$

$E|u_t(x;\lambda)|^2 \geq C\exp(C\lambda^4 t)$ [Foondun-E.Nualart, 2015].

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Theorem (Khoshnevisan-K.W. Kim-Mueller-S, 2016+)

For all $\lambda > 0$,

$$\begin{aligned} -\infty &< \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in[-1,1]} u_t(x;\lambda) \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in[-1,1]} u_t(x;\lambda) < 0 \ a.s. \end{aligned}$$

Dissipation Theorem, Hohenberg-Swift

$$\begin{aligned}\frac{\partial}{\partial t}\psi_t(x;\lambda) &= \triangle\psi_t(x;\lambda)+\psi_t(x;\lambda)-\psi_t^3(x;\lambda)\\ &\quad+\lambda\sigma(\psi_t(x;\lambda))\frac{\partial^2}{\partial t\partial x}W(t,x), x\in[-1,1].\end{aligned}$$

with periodic boundary condition.

Dissipation Theorem, Hohenberg-Swift

$$\frac{\partial}{\partial t}\psi_t(x;\lambda) = \triangle\psi_t(x;\lambda) + \psi_t(x;\lambda) - \psi_t^3(x;\lambda) + \lambda\sigma(\psi_t(x;\lambda))\frac{\partial^2}{\partial t\partial x}W(t,x), x \in [-1,1].$$

with periodic boundary condition.

Theorem (Khoshnevisan-K.W. Kim-Mueller-S, 2016+)

There exists non-random constants $0 < \lambda_1 < \infty$ *such that: whenever* $\lambda > \lambda_1$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in[-1,1]}\psi_t(x,\lambda) < 0 \ a.s.$$

Thank you