Stabilization of regime-switching processes by feedback control based on discrete time observations

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Given an unstable regime-switching diffusion of the form (1) with $b = 0$, it is required to find a feedback control $b(X(t), \Lambda(t))$ so that the controlled system

$$dX(t) = \big(a(X(t), \Lambda(t)) - b(X(t), \Lambda(t))\big)dt + \sigma(X(t), \Lambda(t))dW(t) \tag{1}$$

becomes stable. Here $(\Lambda(t))$ is a Markov chain on a state space $\mathcal{S} = \{1, 2, \ldots, N\}$, $2 \le N < \infty$, and independent of Wiener process $(W(t))$.

- Aim: Construct feedback control using discrete-time observations of $(X(t))$ and $(\Lambda(t))$.

- Xuerong, Mao, (Automatica 49, 2013), first studied the stabilization based on the discrete-time observation of $(X(t))$. There, he designed a discrete-time feedback control so that the controlled system

$$
\begin{aligned}
\mathrm{d}X(t) = \big(a(X(t), \Lambda(t)) - b(X([t/\tau]\tau), \Lambda(t))\big)\mathrm{d}t \\
+ \sigma(X(t), \Lambda(t))\mathrm{d}W(t)
\end{aligned}
$$

  becomes stable.

- Showed that when τ is less than some upper bound τ^*, which depends on the Lipschitz constants and bounds of coefficients a, b, σ, the above controlled system is mean-square stable.

The following works developed this theory:

- X. Mao, W. Liu, L. Hu, Q. Luo, J. Lu, Systems Control Lett. 73, 2014.
- S. You, W. Liu, J. Lu, X. Mao, Q. Qiu, SIAM J. Control Optim. 53 (2), 2015.
- Q. Qiu, W. Liu, L. Hu, X. Mao, S. You, Statistics and Prob. Letters, 2016.

For instance, X. Mao et al. (2014) studied the following system

$$\begin{aligned} \mathrm{d}X(t) = [A(\Lambda(t))X(t) - D(\Lambda(t))X(\delta(t))]\mathrm{d}t \\ + \sum_{k=1}^{d} B_k(\Lambda(t))X(t)\mathrm{d}w_k(t), \end{aligned}$$

where A, $B_k : \mathcal{S} \to \mathbb{R}^{d\times d}$, $W(t) = (w_1(t), w_2(t), \ldots, w_d(t))$ is a d-dimensional Brownian motion.

Theorem (X.Mao et al. 2014)

Assume $\exists$ *symmetric positive-definite matrices* Γ_i, $i \in \mathcal{S}$, *such that*

$$\bar{\Gamma}_i := \Gamma_i(A_i + D_i) + (A_i + D_i)^T \Gamma_i + \sum_{k=1}^{d} B_{ki}^T \Gamma_i B_{ki} + \sum_{j=1}^{N} q_{ij} \Gamma_j$$

are all negative-definite matrices. Set

$$-\lambda = \max_{i \in \mathcal{S}} \lambda_{\max}(\bar{\Gamma}_i), \quad M_{od} = \max_i \|Q_i D_i\|^2$$

If τ *is sufficiently small so that* $\lambda_\tau < \lambda/2$ *and* $K(\tau) < 1/2$, *where*

$$\lambda_\tau = \sqrt{\frac{2M_{od}K(\tau)}{1 - 2K(\tau)}}, \; K(\tau) = \tau C_1 e^{\tau C_2}, \; C_1, \, C_2 \text{ constants},$$

then $(X(t))$ *is mean-square stable.*

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Purpose

- In this talk, I design simply a linear feedback control based on not only the discrete-time observations of $(X(t))$ but also on those of $(\Lambda(t))$. Precisely, investigate the stability of the system

$$\mathrm{d}X(t) = [a(X(t), \Lambda(t)) - b(\Lambda([t/\tau]\tau))X([t/\tau]\tau)]\mathrm{d}t + \sigma(X(t), \Lambda(t))\mathrm{d}W(t), \tag{2}$$

  where $\tau > 0$ constant.
- The key point is the term $\Lambda([t/\tau]\tau)$.

Conditions on Coefficients

The coefficients $a : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}^d$, $b : \mathcal{S} \to [0, \infty)$, $\sigma : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}^{d\times d}$ satisfy the conditions:

(H1) $\exists$ nonnegative functions $C(\cdot)$ and $c(\cdot)$ on $\mathcal{S}$ such that

$$c(i)|x|^2 \le 2\langle a(x,i), x\rangle + \|\sigma(x,i)\|_{\rm HS}^2 \le C(i)|x|^2, \ \ (x,i) \in \mathbb{R}^d \times \mathcal{S},$$

where $\|\sigma(x,i)\|_{\rm HS}^2 = \mathrm{trace}(\sigma\sigma^*)(x,i)$ with σ^* denoting the transpose of the matrix σ.

(H2) $\exists$ constant $\bar{K}$ such that

$$|a(x,i) - a(y,i)| + \|\sigma(x,i) - \sigma(y,i)\|_{\rm HS} \le \bar{K}|x-y|, \ x, y \in \mathbb{R}^d, \ i \in \mathcal{S}.$$

(H3) $\exists$ constant M_a such that $|a(x,i)| \le M_a|x|$ for all $(x,i) \in \mathbb{R}^d \times \mathcal{S}$.

Conditions **(H1)** and **(H2)** ensure the existence and uniqueness of strong solution of the functional SDE.

Theorem 1(Shao, 2016) Assume **(H1)-(H3)** hold. Denote by (π_i) the stationary distribution of $(\Lambda(t))$. Set $\bar{C} = \max_{i\in\mathcal{S}} C(i)$, $\bar{b} = \max_{i\in\mathcal{S}} b(i)$, and

$$K(\tau) = 2(2\bar{C} + 3M_a + \bar{b}^2)\tau e^{(2\bar{C}+M_a+1)\tau}.$$

Assume τ is sufficiently small so that $K(\tau) < 1$.

1. If
$$\sum_{i\in\mathcal{S}} \pi_i \Big(C(i) - 2\Big(1 - \Big(\frac{K(\tau)}{1-K(\tau)}\Big)^{\frac{1}{2}}\Big) b(i)\Big) < 0, \qquad (3)$$
then $\lim_{t\to\infty} \mathbb{E}|X(t)|^2 = 0$.

2. If
$$\sum_{i\in\mathcal{S}} \pi_i \Big(c(i) - 2\Big(1 + \Big(\frac{K(\tau)}{1-K(\tau)}\Big)^{\frac{1}{2}}\Big) b(i)\Big) > 0, \qquad (4)$$
then $\lim_{t\to\infty} \mathbb{E}|X(t)|^2 = \infty$.

Application to 1-dim linear systems

To see the sharpness of the criteria in Theorem 1, we consider the following linear system on the line.

$$\begin{aligned}\mathrm{d}X(t) = \big(a(\Lambda(t))X(t) - b(\Lambda([t/\tau]\tau))X([t/\tau]\tau)\big)\mathrm{d}t \\ + \sigma(\Lambda(t))X(t)\mathrm{d}W(t), \quad X(0) = x \in \mathbb{R},\end{aligned} \tag{5}$$

Corollary 2 Set $\bar{C} = \max_{i\in\mathcal{S}}(2a(i) + \sigma(i)^2)$, $\bar{b} = \max_{i\in\mathcal{S}} b(i)$, $M_a = \max_{i\in\mathcal{S}} |a(i)|$.

Set $K(\tau) = 2(2\bar{C} + 2M_a^2 + \bar{b}^2 + 1)\tau e^{(2\bar{C}+2M_a^2+1)\tau}$.

If $\sum_{i\in\mathcal{S}} \pi_i\Big(2a(i) + \sigma(i)^2 - 2\Big(1-\Big(\frac{K(\tau)}{1-K(\tau)}\Big)^{\frac{1}{2}}\Big)b(i)\Big) < 0$,

then $\lim_{t\to\infty} \mathbb{E}X(t)^2 = 0$.

If $\sum_{i\in\mathcal{S}} \pi_i\Big(2a(i) + \sigma(i)^2 - 2\Big(1+\Big(\frac{K(\tau)}{1-K(\tau)}\Big)^{\frac{1}{2}}\Big)b(i)\Big) > 0$,

then $\lim_{t\to\infty} \mathbb{E}X(t)^2 = \infty$.

Key steps in the proof

On the product Probability Space

$$(\Omega, \mathscr{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathscr{B}(\Omega_1) \times \mathscr{B}(\Omega_2), \mathbb{P}_1 \times \mathbb{P}_2),$$

$\omega = (\omega_1, \omega_2) \in \Omega$, $\omega_1(\cdot)$ is a Brownian motion, and $\omega_2(\cdot)$ is a Poisson random measure.

- **Lemma 3**.

$$\mathbb{E}^{\Lambda}|X(t) - X([t/\tau]\tau)|^2(\omega_2) \leq \frac{K(\tau)}{1 - K(\tau)} \mathbb{E}^{\Lambda}|X(t)|^2(\omega_2), \ \ \mathbb{P}_2\text{-a.s.}$$

Key steps in the proof

- **Lemma 4**. Under the same conditions as Theorem 1, the following estimates hold

$$\mathbb{E}|X(t)|^2 \le |x(0)|^2 \mathbb{E}\Big[\mathrm{e}^{\int_0^t C(\Lambda(r)) - \Big(2 - 2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\Big) b(\Lambda([r/\tau]\tau)) \mathrm{d}r} \Big],$$

and

$$\mathbb{E}|X(t)|^2 \ge |x(0)|^2 \mathbb{E}\Big[\mathrm{e}^{\int_0^t c(\Lambda(r)) - \Big(2 + 2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\Big) b(\Lambda([r/\tau]\tau)) \mathrm{d}r} \Big].$$

In view of the estimates provided by Lemma 4, the mean-square stability of $(X(t))$ is determined by the long time behaviour of two terms in the form separately

$$\int_0^t f(\Lambda(s))\mathrm{d}s, \qquad \int_0^t g(\Lambda([s/\tau]\tau))\mathrm{d}s, \quad f, g \text{ bounded on } \mathcal{S}.$$

The long time behavior of $\int_0^t f(\Lambda(s))\mathrm{d}s$ can be determined by the strong ergodic theorem for $(\Lambda(t))$. However, to deal with long time behavior of $\int_0^t g(\Lambda([s/\tau]\tau))\mathrm{d}s$, we need to study further a skeleton process $Y_n = \Lambda(n\tau)$, $n \in \mathbb{N}$, of the Markov chain $(\Lambda(t))$. $(Y_n)_n$ has the same stationary distribution as $(\Lambda(t))$.

Set $\Theta(t,\omega_2)=\int_0^t C(\Lambda(r)(\omega_2))-\Big(2-2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\Big)b(\Lambda(\delta(r))(\omega_2))\mathrm{d}r$.

$$\begin{aligned}
&\lim_{t\to\infty}\frac{\Theta(t,\omega_2)}{t}\\
&=\lim_{t\to\infty}\frac{1}{t}\int_0^t C(\Lambda(r)(\omega_2))\mathrm{d}r-\Big(2-2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\Big)\frac{1}{t}\int_0^t b(\Lambda(\delta(r))(\omega_2))\mathrm{d}r\\
&=\sum_{i\in\mathcal{S}}\pi_i C(i)-\Big(2-2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\Big)\lim_{t\to\infty}\frac{[t/\tau]\tau}{t}\cdot\frac{1}{[t/\tau]\tau}\sum_{n=0}^{[t/\tau]}b(\Lambda(n\tau)(\omega_2))\tau\\
&=\sum_{i\in\mathcal{S}}\pi_i\Big[C(i)-\Big(2-2\sqrt{\frac{K(\tau)}{1-K(\tau)}}\Big)b(i)\Big]=:\alpha,\quad \mathbb{P}_2\text{-a.s.}
\end{aligned}$$

Then, as $\alpha<0$,

$$\limsup_{t\to\infty}\mathbb{E}|X(t)|^2\le\limsup_{t\to\infty}|x(0)|^2\mathbb{E}\mathrm{e}^{\Theta(t)}\le|x(0)|^2\mathbb{E}\limsup_{t\to\infty}\mathrm{e}^{\alpha t}=0,$$

which implies that $\lim_{t\to\infty}\mathbb{E}|X(t)|^2=0$.

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Let us go back to consider how to stabilize the following unstable stochastic system with feedback control

$$\mathrm{d}X(t) = b(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t). \tag{6}$$

Traditionally, design a feedback control $u(t, X(t))$ in order for the controlled system

$$\mathrm{d}X_t = (b(t, X(t)) + u(t, X(t)))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t)$$

to be stable. To be costless and more realistic, one can design a feedback control $u(t, X([t/\tau]\tau))$ based on the discrete-time observations so that

$$\mathrm{d}X(t) = (b(t, X(t)) + u(t, X([t/\tau]\tau)))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t)$$

to be stable. At present stage, one only needs the observations of $X(t)$ at $t = \tau, 2\tau, 3\tau, \ldots$

Based on our study on regime-switching diffusion process, one can design a new approach to stabilize the given system (6) to save the number of observation times.

$$\mathrm{d}X_t = (b(t, X(t)) + u(t, X([t/\tau]\tau), \Lambda([t/\tau]\tau))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t),$$

where $(\Lambda(t))$ is a Markov chain on $\mathcal{S} = \{1, 2\}$, and $u(t, x, 1) \equiv 0$, which means that we do nothing when $\Lambda(t)$ is at the stage 1. Let $\alpha(t)$ denote the time spent by (Λ_t) at the state "1" during the time interval $[0, t]$. It is known that

$$\lim_{t\to\infty} \frac{\mathbb{E}[\alpha(t)]}{t} = \frac{q_2}{q_1 + q_2}. \tag{7}$$

For example, when $q_1 = q_2$, above equality implies that the process $(\Lambda(t))$ spends half of time to stay at the state "1" in average for large t. Since we do not need to observe X_t when the Markov chain (Λ_t) is at state "1", this method can greatly reduce the number of observation times of $(X(t))$ in average.

Theorem 5 (Li and Shao 2016) Assume that there exist a function $\rho \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R}^+)$, a state $i_0 \in \mathcal{S}$, constants $c_{i_0} > 0$, $p > 0$ and continuous nonnegative functions ψ_i for $i \in \mathcal{S}\backslash\{i_0\}$ such that

(i) $\rho(t,x) \geq |x|^p \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$;

(ii) $L^{(i_0)}\rho(t,x) \leq -c_{i_0}\rho(t,x) \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$;

(iii) $L_t^{(i)}\rho(t,x) \leq -\psi_i(t)\rho(t,x) \quad \forall\, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$, $i \in \mathcal{S}\backslash\{i_0\}$.

Then there exists a constant $\gamma > 0$ such that

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|X_t|^p}{t} \leq -\gamma < 0.$$

Moreover, the process (X_t) is also almost surely asymptotically stable.

Further works on this topic

J.H. Bao, Shao, C.G. Yuan
The stability of Functional SDE with regime-switching diffusions. Prepare

Shao, F.B. Xi
The stabilization of regime-switching jump-diffusions based on discrete-time observations. Prepare

J. Li, Shao,
Algebraic stability of non-homogeneous regime-switching diffusion processes, Submitted, 2016.

Shao,
Stabilization of regime-switching processes by feedback control based on discrete time observations, Submitted, 2016.

Thank You !