Williams decomposition for superprocesses

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The talk is based on a working paper with Renming Song and Rui Zhang.

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Outline

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Williams' decompositions

D. Williams (Proc. London Math. Soc., 1974) decomposed the Brownian excursion with respect to its maximum.

D. Aldous (Ann. Probab., 1991) recognized the genealogy of a quadratic (branching mechanism $\psi(z) = z^2$) continuous state branching process can be recognized in the Brownian excursion.

The genealogical structure of a general continuous branching process can be recognized in spectrally positive Lévy process (from Z. Li's talk).

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Later Williams' decomposition also refers to decompositions of branching processes with respect to their height.

Let X be a non-homogeneous superprocess. It models the evolution of a large population, where the location of the individuals is allowed to affect their reproduction law. We assume the extinction time H of this population is finite.

We are interested in the following conditioning on the genealogical structure of X:

The distribution $X^{(h_0)}$ of X conditioned on $H = h_0$: we derive it using a spinal decomposition involving the ancestral lineage of the last individual alive (**Williams' decomposition**).

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Previous results

For superprocesses with homogeneous branching mechanism, the spatial motion is independent of the genealogical structure. As a consequence, the law of the ancestral lineage of the last individual alive does not distinguish from the original motion. Therefore, in this setting, the description of $X^{(h_0)}$ may be deduced from Abraham and Delmas (2009) where no spatial motion is taken into account.

For nonhomogeneous branching mechanisms on the contrary, the law of the ancestral lineage of the last individual alive should depend on the distance to the extinction time h_0.

Using the Brownian snake, Delmas and Hénard (2013) provide a description of the genealogy for superprocesses with the following non-homogeneous branching mechanism

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We would like to find **conditions** such that the Williams' decomposition works for superprocesses with **general non-homogeneous branching** mechanisms. The conditions should be easy to check and satisfied by a lot of superpossess.

Outline

Superprocesses

The superprocess $X = \{X_t : t \geq 0\}$ we are going to work with is determined by two objects:

(i) a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E, which is Hunt process on E.

(ii) a branching mechanism Ψ of the form

$$\Psi(x,z) = -\alpha(x)z + b(x)z^2 + \int_{(0,+\infty)} (e^{-zy} - 1 + zy)n(x,dy), x \in E, z > 0, \tag{1}$$

where $\alpha \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0,\infty)$ satisfying

$$\sup_{x \in E} \int_{(0,+\infty)} (y \wedge y^2)n(x,dy) < \infty. \tag{2}$$

$\mathcal{M}_F(E)$: the space of finite measures on E. $\langle f, \mu \rangle := \int_E f(x)\mu(dx)$.

The superprocess X is a Markov process taking values in $\mathcal{M}_F(E)$. For any $\mu \in \mathcal{M}_F(E)$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$. Then for every $f \in \mathcal{B}_b^+(E)$ and $\mu \in \mathcal{M}_F(E)$,

$$-\log \mathbb{P}_\mu \left(e^{-\langle f, X_t \rangle} \right) = \langle u_f(t, \cdot), \mu \rangle,$$

where $u_f(t, x)$ is the unique positive solution to the equation

$$u_f(t, x) + \Pi_x \int_0^{t \wedge \zeta} \Psi(\xi_s, u_f(t-s, \xi_s))\beta(\xi_s) ds = \Pi_x f(\xi_t),$$

For any $f \in \mathcal{B}_b(E)$ and $(t, x) \in (0, \infty) \times E$, define

$$T_t f(x) := \Pi_x \left[e^{\int_0^t \alpha(\xi_s)\, ds} f(\xi_t) \right]. \qquad (3)$$

It is well-known that $T_t f(x) = \mathbb{P}_{\delta_x} \langle f, X_t \rangle$ for every $x \in E$.

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Assumptions

Define $\|\mu\| := \langle 1, \mu\rangle$; $v(t,x) := -\log \mathbb{P}_{\delta_x}(\|X_t\| = 0)$. Note that, since $\mathbb{P}_{\delta_x}\|X_t\| = T_t 1(x) > 0$, we have $\mathbb{P}_{\delta_x}(\|X_t\| = 0) < 1$.

Assumption 1 For $\forall t > 0$,

$$\sup_{x\in E} v(t,x) < \infty \quad (\Leftrightarrow \inf_{x\in E} \mathbb{P}_{\delta_x}(\|X_t\| = 0) > 0) \quad \text{and}$$

$$\lim_{t\to\infty} \sup_{x\in E} v(t,x) = 0 \quad (\Leftrightarrow \lim_{t\to\infty} \inf_{x\in E} \mathbb{P}_{\delta_x}(\|X_t\| = 0) = 1). \tag{4}$$

Remark 1 Now we give a sufficient condition for Assumption 1.

$$\Psi(x,z) \geq \widetilde{\Psi}(z) := az + bz^2 + \int_0^\infty (e^{-yz} - 1 + yz)n(dy), \tag{5}$$

where $a \geq 0$, $\int_0^\infty (y \wedge y^2)n(dy) < \infty$ and $\widetilde{\Psi}$ satisfies the Grey condition: $\int_0^\infty \frac{1}{\widetilde{\Psi}(z)}\,dz < \infty$.

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Remark 2 In 2014, Duquesne and Labbé proved that:

1) a Continuous State Branching Process (CSBP) with general branching mechanism such that the Grey condition holds has an **Eve**.

2) If the Grey condition does not hold CSBP may have (finitely and infinitely) **many settlers**. Moreover, under some conditions on the Lévy measure of the branching mechanism, there is even **dust**.

Assumptions

Recall that $v(t,x) := -\log \mathbb{P}_{\delta_x}(\|X_t\| = 0)$.

Assumption 2 We assume that, for any $x \in E$ and $t > 0$,

$$w(t,x) := -\frac{\partial v}{\partial t}(t,x)$$

exists, and for any $u > 0$ and $0 < r < t$,

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Excursion measures

We use $\mathbb{D}$ to denote the space of $\mathcal{M}_F(E)$-valued right continuous functions $t \mapsto \omega_t$ on $(0, \infty)$ having zero as a trap.

One can associate with $\{\mathbb{P}_{\delta_x} : x \in E\}$ a family of σ-finite measures $\{\mathbb{N}_x : x \in E\}$ defined on $(\mathbb{D}, \mathcal{A})$ such that $\mathbb{N}_x(\{0\}) = 0$,

$$\int_{\mathbb{D}} (1 - e^{-\langle f, \omega_t \rangle}) \mathbb{N}_x(d\omega) = -\log \mathbb{P}_{\delta_x}(e^{-\langle f, X_t \rangle}), \quad f \in \mathcal{B}_b^+(E), \ t > 0. \quad (6)$$

See El Karoui and Roelly (1991), Le Gall (1999), Zenghu Li (2002) and Dynkin and Kuznetsov (2004) for further details.

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Spine

Since $v(t+s,x) = -\log \mathbb{P}_\mu e^{-\langle v(s,\cdot), X_t\rangle}$, then we have, for $s,t>0$,

$$v(t+s,x) + \Pi_x \int_0^t \Psi(\xi_u, v(t+s-u,\xi_u))\,du = \Pi_x(v(s,\xi_t)). \tag{7}$$

By Assumption 2, both sides of the above equation is differentiable with respect to s and we get that

$$w(t+s,x)+\Pi_x \int_0^t \Psi'_z(\xi_u, v(t+s-u,\xi_u))w(t+s-u,\xi_u)\,du = \Pi_x(w(s,\xi_t)), \tag{8}$$

which implies that

$$w(t+s,x) = \Pi_x\left(\exp\left\{-\int_0^t \Psi'_z(\xi_u, v(t+s-u,\xi_u))\,du\right\} w(s,\xi_t)\right). \tag{9}$$

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Define, for $t \in [0, h)$,

$$Y_t^h := \frac{w(h-t, \xi_t)}{w(h, x)} e^{-\int_0^t \Psi_z'(\xi_u, v(h-u, \xi_u))\, du}.$$

Lemma For any $x \in E$ and $t < h$, $\Pi_x(Y_t^h) = 1$. Under Π_x, $\{Y_t^h, t < h\}$ is a nonnegative martingale.

Now we define a martingale change of measure by, for $t < h$,

$$\left.\frac{\Pi_x^h}{\Pi_x}\right|_{\mathcal{F}_t} := Y_t^h.$$

Then $\{\xi_t, 0 \le t < h; \Pi_x^h\}$ is a conservative Markov process(Spine). If ν is a probability measure on E, define $\Pi_\nu^h := \int_E \Pi_x^h \,\nu(dx)$.

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Main result

We put

$$H := \inf\{t \geq 0 : \|X_t\| = 0\},$$

$$H(\omega) := \inf\{t \geq 0 : \|\omega_t\| = 0\}, \quad \text{for } \omega \in \mathbb{D}.$$

We aim to reconstruct the process $\{X_t, t < h\}$ conditioned on $H = h$.

Theorem (Main Result)

Spine Let $\xi^h := \{\xi_t, 0 \leq t < h\}$ be a Markov process according to the measure Π_ν^h, where $\nu(dx) = \frac{w(h,x)}{\langle w(h,\cdot),\mu\rangle}\mu(dx)$. Given the trajectory of ξ^h, in the following, we will give three independent proceses:

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Continuous immigration Suppose that $\mathcal{N}^{1,h}(ds, d\omega)$ is a Poisson random measure on $[0, h) \times \mathbb{D}$ with density measure $2\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}\beta(\xi_s)b(\xi_s)\mathbb{N}_{\xi_s}(d\omega)ds$. Define, for $t \in [0, h)$,

$$X_t^{1,h,\mathbb{N}} := \int_0^t \int_{\mathbb{D}} \omega_{t-s}\mathcal{N}^{1,h}(ds, d\omega); \tag{10}$$

Jump immigration Suppose that $\mathcal{N}^{2,h}(ds, d\omega)$ is a Poisson random measure on $[0, h) \times \mathbb{D}$ with density measure $\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s} \int_0^\infty yn(\xi_s, dy)\mathbb{P}_{y\delta_{\xi_s}}(X \in d\omega)\, ds$. Define, for $t \in [0, h)$,

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Immigration at time 0 Let $X_t^{0,h}, 0 \leq t < h$, be a superprocess distributed according to the probability measure $\mathbb{P}_\mu(X \in \cdot | H < h)$.

Define

$$\Lambda_t^h := X_t^{0,h} + X_t^{1,h,\mathbb{N}} + X_t^{2,h,\mathbb{P}}. \quad (12)$$

Then $\{\Lambda_t^h, t < h\}$ has the same distribution as $\{X_t, t < h\}$ conditioned on $H = h$.

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Example 1 Let $\{P_t\}_{t\geq 0}$ be the semigroup of ξ. Suppose that P_t is conservative and preserves $C_b(E)$. Let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ be the infinitesimal generator of P_t in $C_b(E)$. Also assume that

(A) $\Psi(x,z) = -\alpha(x)z + b(x)z^2$, where $\sup_{x\in E} \alpha(x) \leq 0$ and $\inf_{x\in E} b(x) > 0$ and $1/b \in \mathcal{D}(\mathcal{A})$.
(This implies Assumption 1)

(B) $-\alpha(x) - b(x)\mathcal{A}(1/b)(x) \in \mathcal{D}(\mathcal{A}(1/b))$.
(This implies Assumption 2)

This example covers Delmas and Hénard (2013).

In the following examples, we always suppose the branching mechanism is given by

$$\Psi(x,z) = -\alpha(x)z + b(x)z^2 + \int_{(0,+\infty)} (e^{-zy} - 1 + zy)n(x,dy),$$

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Example 2 Assume ξ is a diffusion with infinitesimal generator

$$L = \sum a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum b_j(x) \frac{\partial}{\partial x_j}$$

satisfy the following conditions:

(A) (Uniform ellipticity) There exists a constant $\gamma > 0$ such that

$$\sum a_{i,j}(x) u_i u_j \geq \gamma \sum u_j^2.$$

(B) a_{ij} and b_j are bounded, continuous in x and satisfy Hölder's conditions.

Then the (ξ, Ψ)-superprocess X satisfies Assumption 1.

Suppose further that, for any $M > 0$, there exists c such that

$$|\Psi(x,z) - \Psi(y,z)| \leq c|x-y|, \quad x, y \in \mathbb{R}^d, z \in [0, M].$$

Then X also satisfies Assumption 2.

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$$L = \sum a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum b_j(x) \frac{\partial}{\partial x_j}$$

satisfy the following conditions:

- **(A)** (Uniform ellipticity) There exists a constant $\gamma > 0$ such that
$$\sum a_{i,j}(x) u_i u_j \geq \gamma \sum u_j^2.$$
- **(B)** a_{ij} and b_j are bounded, continuous in x and satisfy Hölder's conditions.

Then the (ξ, Ψ)-superprocess X satisfies Assumption 1.

Suppose further that, for any $M > 0$, there exists c such that

$$|\Psi(x, z) - \Psi(y, z)| \leq c|x - y|, \quad x, y \in \mathbb{R}^d, z \in [0, M].$$

Then X also satisfies Assumption 2.

Example 3 Suppose that $B = \{B_t\}$ is a Brownian motion in R^d and $S = \{S_t\}$ is an independent subordinator with Laplace exponent φ, that is

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\varphi(\lambda)}, \qquad t > 0, \lambda > 0.$$

The process $\xi_t = B_{S_t}$ is called a subordinate Brownian motion in $\mathbb{R}^d$. Then the (ξ, Ψ)-superprocess X satisfies Assumption 1.

1) Suppose further that φ satisfies the following conditions:

1. $\int_0^1 \frac{\varphi(r^2)}{r} dr < \infty$.
2. There exist constants $\delta \in (0, 2]$ and $a_1 \in (0, 1)$ such that

$$a_1 \lambda^{\delta/2} \varphi(r) \le \varphi(\lambda r), \qquad \lambda \ge 1, r \ge 1.$$

2) Suppose that for any $M > 0$, there exists c such that

$$|\Psi(x, z) - \Psi(y, z)| \le c|x - y|, \quad x, y \in \mathbb{R}^d, z \in [0, M].$$

Then X also satisfies Assumption 2.

Remark Actually, by the same arguments and the results from Kim-Song-Vondracek (Preprint, 2016), one check that in the example above, we could have replaced the subordinate Brownian motion by the non-symmetric jump process considered there, which contains the non-symmetric stable-like process discussed in Chen-Zhang (Probab. Theory Relat. Fields, 2016+).

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Outline

An application of the main result

Assumption 3 For any bounded open set $B \subset E$ and any $t > 0$, the function

$$x \to -\log \mathbb{P}_{\delta_x}\Big(\int_0^t X_s(B^c)\, ds = 0\Big)$$

is finite for $x \in B$ and locally bounded.

Remark Suppose ξ is a diffusion with generator L satisfying (A) and (B) in Example 2, and suppose X is a (ξ, Ψ)-superdiffusion. If the branching mechanism $\Psi(x, z)$ satisfies that, for some $\alpha \in (1, 2]$, $\Psi(x, z) \geq z^\alpha$ for all $x \in \mathbb{R}^d$ then Assumption 3 is satisfied.

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An application of the main result

Corollary Assume that Assumption 3 holds and that for any $\mu \in \mathcal{M}_F(E)$,

$$\lim_{t\uparrow h} \xi_t =: \xi_{h-} \text{ exists}, \quad \Pi_\nu^h - a.s., \tag{13}$$

where $\nu(dx) = \frac{w(h,x)}{\langle w(h,\cdot),\mu\rangle}\mu(dx)$. Then there exists a E-valued random variable Z such that

$$\lim_{t\uparrow H} \frac{\Lambda_t}{\|\Lambda_t\|} = \delta_Z \quad \text{(weak)}, \quad \mathbb{P}_\mu - a.s.$$

Conditioned on $\{H = h\}$, Z has the same law as $\{\xi_{h-}, \Pi_\nu^h\}$.

In 1992, Tribe proved that if the spatial motion is Feller process and the branching mechanism is binary ($\Psi(z) = z^2$). Compared with Tribe (1992), we assume that the spatial motion ξ is a diffusion (special), while our branching mechanisms is more general.

In 2014, Duquesne and Labbé proved that a Continuous State Branching Process (CSBP) with general branching mechanism such that the Grey condition holds has an **Eve**.

In some sense, our result gives a special dependent version of the result of Duquesne and Labbé (2014) under the Grey condition.

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Thank you!