The Itô SDEs and Fokker–Planck equations with Osgood and Sobolev coefficients

Dejun Luo

Institute of Applied Mathematics, AMSS, CAS

12^{th} Workshop on Markov Processes and Related Topics

JSNU, July 13, 2016

Outline

1. Background
   - ODE with Sobolev coefficient
   - SDE with Sobolev coefficients
   - SDE with Osgood coefficients
   - Osgood–Sobolev functions
2. Main results
3. Key ingredients of the proof

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Cauchy–Lipschitz theory

It is well known that if a vector field $V:\mathbb{R}^d\to\mathbb{R}^d$ is globally Lipschitz continuous, then the ODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t}=V(X_t),\quad X_0=x\in\mathbb{R}^d \tag{1}$$

determines a unique flow of homeomorphisms on $\mathbb{R}^d$, such that the Radon–Nikodym density

$$\rho_t(x):=\frac{\mathrm{d}(\mathcal{L}^d\circ X_t^{-1})}{\mathrm{d}\mathcal{L}^d}(x)=\exp\bigg[-\int_0^t \mathrm{div}(V)\big[X_s\big(X_t^{-1}(x)\big)\big]\,\mathrm{d}s\bigg].$$

Thus $\mathcal{L}^d\circ X_t^{-1}\approx\mathcal{L}^d$.

In this case, the flow X_t is called quasi-invariant.

Di Perna–Lions theory

In applications (e.g. in kinetic theory and fluid mechanics), the vector fields V often have only Sobolev or even BV regularity.

Many people have tried to generalize the Cauchy–Lipschitz theory to the cases where V is not regular.

A breakthrough was made by Di Perna and Lions (1989).

- Di Perna–Lions (Invent. Math., 1989): If

$$V \in W^{1,1}_{loc}, \ \frac{|V|}{1+|x|} \in L^1 + L^\infty \text{ and } \mathrm{div}(V) \in L^\infty,$$

then ODE (1) generates a unique flow of measurable maps $X_t : \mathbb{R}^d \to \mathbb{R}^d$, such that $\mathcal{L}^d \circ X_t^{-1} \approx \mathcal{L}^d$.

Di Perna–Lions' approach is indirect, and it is in fact an extension of the classical characteristics method.

- If $V \in C_b^1(\mathbb{R}^n, \mathbb{R}^n)$, then the ODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = V(X_t) \tag{1}$$

generates a unique flow of diffeomorphisms $X_t : \mathbb{R}^n \to \mathbb{R}^n$. For any $u_0 \in C^1(\mathbb{R}^n)$, $u_t(x) := u_0(X_t(x))$ solves the transport equation

$$\frac{\partial u}{\partial t} - V \cdot \nabla u = 0, \quad u|_{t=0} = u_0. \tag{2}$$

- Conversely, let u_t^i be the solution of (2) with the initial condition $u_0^i(x) = x^i$ $(1 \le i \le n)$, then

$$X_t := (u_t^1, u_t^2, \ldots, u_t^n)$$

is the flow generated by the ODE.

Methodology of the Di Perna–Lions theory

Theorem 1

Existence and uniqueness of solutions in $L^\infty([0,T], L^1 \cap L^\infty)$ to the transport equation (2)
$\Longrightarrow$ *Existence and uniqueness of flow of measurable maps generated by ODE (1).*

Thus it is enough to study the transport equation (2), which is a first order linear PDE.

- The existence of (2) follows easily from the a-priori estimate of solutions.
- For the uniqueness, Di Perna and Lions proposed the notion of renormalized solution: $\forall\, \phi \in C_b^1(\mathbb{R})$,

$$\frac{\partial u}{\partial t} - V \cdot \nabla u = 0 \quad \Longrightarrow \quad \frac{\partial \phi(u)}{\partial t} - V \cdot \nabla \phi(u) = 0.$$

Key ingredient for renormalization property

- Commutator estimate:

  Let χ_n be a standard convolution kernel, define

  $$C_n(u, V) = \chi_n * (V \cdot \nabla u) - V \cdot \nabla(\chi_n * u).$$

  Di Perna and Lions (1989): If $u \in L^p_{loc}$ and $V \in W^{1,q}_{loc}$, then

  $$C_n(u, V) \xrightarrow{L^r_{loc}} 0 \quad (n \to \infty),$$

  where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Extension to BV vector field:

- Ambrosio (Invent. Math., 2004) proved similar commutator estimate for vector fields V with BV regularity.

  This enables him to show the well-posedness of transport eq. (2) and hence the ODE (1).

Other developments of Di Perna–Lions theory

ODE on Wiener space (W, H, μ):

- Ambrosio–Figalli (JFA, 2008):

$$V \in \mathbb{D}_1^p(W, H), \quad \int_W e^{\lambda|\delta(V)|}\,\mathrm{d}\mu < \infty \quad \forall\, \lambda > 0;$$

- Fang–Luo (BSM, 2010): $\exists\, \lambda > 0$ s.t. $\int_W e^{\lambda|\delta(V)|}\,\mathrm{d}\mu < \infty$. Explicit construction of the flow on classical Wiener space.
- Trevisan (PTRF, 2015): $V : W \to H$ has BV regularity.

ODE on Riemannian manifolds:

- Dumas–Golse–Lochak (1994): compact case.
- Fang–Li–Luo (2011): non-compact case,

$$C_t(u, V) = V \cdot \nabla H_t u - H_t(V \cdot \nabla u).$$

Remark 2

Unfortunately, the above method does not work for SDE

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}W_t + V(X_t)\,\mathrm{d}t$$

with Sobolev coefficients. The reason is if u *solves the corresponding stochastic transport equation*

$$\mathrm{d}_t u = \frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^*\nabla^2 u\big)\,\mathrm{d}t - \langle V_\sigma, \nabla u\rangle\,\mathrm{d}t - \langle \sigma^*\nabla u, \mathrm{d}W_t\rangle,$$

where $V_\sigma = V - \sum_l \nabla_{\sigma^{\cdot,l}}\sigma^{\cdot,l}$*, then* $\phi(u)$ *no longer satisfies this equation due to the Itô formula.*

Crippa–de Lellis' direct method

Theorem 3 (Crippa–de Lellis, 2008)

- *$V, \tilde V \in W^{1,p}$ are two bounded vector fields;*
- *X and $\tilde X$ are the regular Lagrangian flows associated to V and $\tilde V$, with the constants L and $\tilde L$, respectively.*

Then $\forall\, R > 0$ and $\delta > 0$,

$$\int_{B_R} \log\left[1 + \frac{\sup_{t\le T} |X_t(x) - \tilde X_t(x)|}{\delta}\right] \mathrm{d}x$$

$$\le (L + \tilde L)\|\nabla V\|_{L^1(B_{\tilde R})} + \frac{\tilde L}{\delta}\|V - \tilde V\|_{L^1(B_{\tilde R})},$$

where $\tilde R = R + T\big(\|V\|_{L^\infty} \vee \|\tilde V\|_{L^\infty}\big)$.

$X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a Regular Lagrangian flow of V if

(i) for a.e. $x \in \mathbb{R}^d$, $t \to X_t(x)$ is an **integral curve** of V;

(ii) $\forall\, t \in [0,T]$, $\mathcal{L}^d \circ X_t^{-1} \ll L\mathcal{L}^d$.

Uniqueness of flow

Assume that X and $\tilde{X}$ are two regular Lagrangian flows generated by $V \in W^{1,p}$, then

$$\int_{B_R} \log\left[1 + \frac{\sup_{t\le T}|X_t(x) - \tilde{X}_t(x)|}{\delta}\right] \mathrm{d}x \le 2L\|\nabla V\|_{L^1(B_{\tilde{R}})}.$$

For any $\eta > 0$, define

$$\Sigma_\eta = \left\{x \in B_R : \sup_{t\le T}|X_t(x) - \tilde{X}_t(x)| \ge \eta\right\}.$$

Then

$$\begin{aligned}\mathcal{L}^d(\Sigma_\eta) &\le \frac{1}{\log(1+\frac{\eta}{\delta})}\int_{B_R} \log\left[1 + \frac{\sup_{t\le T}|X_t(x) - \tilde{X}_t(x)|}{\delta}\right] \mathrm{d}x \\ &\le \frac{2L}{\log(1+\frac{\eta}{\delta})}\|\nabla V\|_{L^1(B_{\tilde{R}})} \to 0 \quad \text{as } \delta \downarrow 0.\end{aligned}$$

Pointwise characterization of Sobolev functions

$f \in W^{1,p}(\mathbb{R}^d) \iff \exists\, g \in L^p(\mathbb{R}^n)$ such that

$$|f(x) - f(y)| \le (g(x) + g(y))|x - y| \quad \text{for a.e. } x, y \in \mathbb{R}^d. \qquad (*)$$

$g = C_d \mathcal{M}|\nabla f|$, where C_d depends only on d, and $\mathcal{M}$ is the maximal operator:

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mathcal{L}^d(B_r(x))} \int_{B_r(x)} |f(y)|\, \mathrm{d}y.$$

$\mathcal{M} : L^p \to L^p$ is bounded for $p > 1$.

If $p = 1$, then

$$\int_{\mathbb{R}^d} \mathcal{M}f(x)\, \mathrm{d}x \le C_d \int_{\mathbb{R}^d} |f| \log(1 + |f|)\, \mathrm{d}x.$$

- Hajłasz (1996) define Sobolev spaces on metric space by $(*)$.

Outline

Generalized stochastic flow of Itô's SDE

- $\sigma : \mathbb{R}^d \to \mathcal{M}_{d,m}$, $V : \mathbb{R}^d \to \mathbb{R}^d$ measurable functions;
- W_t a standard B.M. on $\mathbb{R}^m$;
- $\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}W_t + V(X_t)\,\mathrm{d}t, \quad X_0 = x$;
- μ: a (finite) reference measure on $\mathbb{R}^d$.

A measurable map $X : \Omega \times \mathbb{R}^d \to C([0,T],\mathbb{R}^d)$ is called a μ-a.e. stochastic flow generated by SDE if

(i) $\forall\, t \in [0,T]$ and μ-a.e. $x \in \mathbb{R}^d$, $X_t(\cdot,x) : \Omega \to \mathbb{R}^d$ is $\mathcal{F}_t = \sigma(B_s : s \le t)$ measurable;

(ii) $\exists\, K_t : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ s.t. $\mu \circ X_t^{-1}(\omega,\cdot) = K_t(\omega,\cdot)\mu$;

(iii) for μ-a.e. $x \in \mathbb{R}^d$, the following equality holds $\mathbb{P}$-a.s.:

$$X_t(\omega,x) = x + \int_0^t \sigma(X_s(\omega,x))\,\mathrm{d}W_s + \int_0^t V(X_s(\omega,x))\,\mathrm{d}s,\ \forall\, t \in [0,T];$$

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}W_t + V(X_t)\,\mathrm{d}t. \tag{3}$$

- X. Zhang (2010): $\mu = \mathcal{L}^d$.
  $\frac{|V|}{1+|x|}, \mathrm{div}(V) \in L^\infty$ and one of the conditions below holds:
  - $V \in BV_{loc}, \sigma$ is constant (essentially reduced to ODE);
  - $|\nabla V| \in (L^1 \log L^1)_{loc}$ and $|\nabla \sigma|, \big(\sup_{|z|\le 1} |\sigma(\cdot - z)|\big)|\nabla \mathrm{div}(\sigma)| \in L^\infty$.

  $\Longrightarrow$ SDE (3) generates a unique $\mathcal{L}^d$-a.e. stochastic flow X_t of measurable maps.
- Fang-Luo-Thalmaier (2010): $\mu = \gamma_d$ (Gaussian measure).
  - $\sigma \in \cap_{q>1} \mathbb{D}_1^q(\gamma_d)$ and $V \in \mathbb{D}_1^{q_0}(\gamma_d)$ for some $q_0 > 1$;
  - σ, V have linear growth;
  - $\exists\, \lambda_0 > 0$ such that $\int_{\mathbb{R}^d} \exp\big[\lambda_0\big(|\mathrm{div}_{\gamma_d}(V)| + |\nabla\sigma|^2 + |\mathrm{div}_{\gamma_d}(\sigma)|^2\big)\big]\,\mathrm{d}\gamma_d < +\infty$.

  $\Longrightarrow$ SDE (3) generates a unique γ_d-a.e. stochastic flow X_t and Radon–Nikodym density $K_t \in L^1 \log L^1(\mathbb{P} \times \gamma_d)$.

- X. Zhang (2013): $\mathrm{d}\mu = (1+|x|^2)^{-\alpha}\,\mathrm{d}x$, $\alpha > d/2$.
  - $\sigma \in W^{1,2q}_{loc}, V \in W^{1,q}_{loc}$ for some $q > 1$;
  - σ, V have linear growth;
  - $\forall\, p \geq 1$, it holds $\int_{\mathbb{R}^d} \exp\big[p\big([\mathrm{div}(V)]^- + |\nabla\sigma|^2\big)\big]\,\mathrm{d}\mu < +\infty$.

  Then similar results hold.
- Luo (2015): Fix $q > 1$ and let $\mathrm{d}\mu = (1+|x|^2)^{-\alpha}\,\mathrm{d}x$ for some $\alpha > q + d/2$.
  - $\sigma \in W^{1,2q}_{loc}, V \in W^{1,q}_{loc}$;
  - $\forall\, p > 0$, $\int_{\mathbb{R}^d} \exp\Big\{p\Big([\mathrm{div}(V)]^- + \frac{|V|}{1+|x|} + \big(\frac{|\sigma|}{1+|x|}\big)^2 + |\nabla\sigma|^2\Big)\Big\}\,\mathrm{d}\mu < +\infty$.

  $\Longrightarrow$ SDE (3) generates a unique stochastic flow X_t.

  σ and V do not necessarily have linear growth and they may be locally unbounded.

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SDE with Osgood coefficients

In the last two decades, lots of studies on SDEs with non-Lipschitz coefficients:

$$\|\sigma(x)-\sigma(y)\|^2 + 2\langle x-y, V(x)-V(y)\rangle \le C\rho(|x-y|^2).$$

E.g. $\rho(s) = s$, $s\log\frac{1}{s}$, $s(\log\frac{1}{s})(\log\log\frac{1}{s})$...

- Malliavin (1999): canonic B.M. on Diff(S^1) corresponds to the critical norm $H^{3/2}$;
- S. Fang (2002): more detailed construction;
- Airault–Ren (2002): explicit estimate on modulus of continuity of the canonic B.M. on Diff(S^1);
- S. Fang–T. Zhang (2005): pathwise uniqueness under general Osgood continuity, convergence of Euler scheme and large deviation principle under log-Lipschitz condition.
- X. Zhang (2005): homeomorphic flow property for SDEs with log-Lipschitz coefficients.

Two types of conditions

- Osgood condition: $\exists\, C > 0$ and $\rho \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\int_{0+} \frac{\mathrm{d}s}{\rho(s)} = \infty$, s.t.

$$|V(x) - V(y)| \leq C\rho(|x - y|) \quad \forall\, x, y \in \mathbb{R}^d;$$

- Sobolev condition: $V \in W^{1,p} \iff \exists\, g \in L^p$ s.t.

$$|V(x) - V(y)| \leq (g(x) + g(y))|x - y| \quad \text{for-a.e. } x, y \in \mathbb{R}^d.$$

Question: can we combine the two conditions together?

- Osgood–Sobolev: $\exists\, g \in L^p$ and $\rho \in C(\mathbb{R}_+, \mathbb{R}_+)$ s.t.

$$|V(x) - V(y)| \leq (g(x) + g(y))\rho(|x - y|) \quad \text{for-a.e. } x, y \in \mathbb{R}^d.$$

Example: $V = V_1 + V_2$, where $V_1 \in W^{1,p}(\mathbb{R}^d)$ and $V_2(x) = (\varphi(x_1), \cdots, \varphi(x_d))$ with $\varphi(t) = \sum_{k=1}^{\infty} \frac{|\sin kt|}{k^2}$.

Outline

Osgood–Sobolev functions

Fix $p \geq 1$ and $\rho \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\rho \nearrow$ and $\int_{0+} \frac{\mathrm{d}s}{\rho(s)} = \infty$.

Osgood–Sobolev condition:

$$|f(x) - f(y)| \leq (g(x) + g(y))\rho(|x - y|) \quad \text{for-a.e. } x, y \in \mathbb{R}^d. \quad (4)$$

Define

$$M_\rho^{1,p} = \big\{ f \in L^p(\mathbb{R}^d) : \exists\, g \in L^p(\mathbb{R}^d) \text{ s.t. } (4) \text{ holds} \big\}.$$

For $f \in M_\rho^{1,p}$, let

$$D(f) := \big\{ g \in L^p(\mathbb{R}^d) \text{ s.t. } (4) \text{ holds} \big\},$$
$$\|f\|_{M_\rho^{1,p}} := \|f\|_{L^p} + \inf_{g \in D(f)} \|g\|_{L^p}.$$

Properties of Osgood–Sobolev functions

Proposition 4

- $\|\cdot\|_{M_\rho^{1,p}}$ *is a norm on* $M_\rho^{1,p}$;
- $\exists$ *a unique* $g_f \in D(f)$ *s.t.* $\|g_f\|_{L^p} = \inf_{g\in D(f)} \|g\|_{L^p}$;
- $M_\rho^{1,p}$ *is a Banach space;*
- *If* ρ *is concave, then* $\forall\, f \in M_\rho^{1,p}$ *and* $\varepsilon > 0$, $\exists$ *an Osgood function* f_ε *s.t.*
  - $\mathcal{L}^d(\{f \neq f_\varepsilon\}) < \varepsilon$,
  - $\|f - f_\varepsilon\|_{M_\rho^{1,p}} < \varepsilon$;
- *(Ivanishko) Compactness of Sobolev embeddings;*
- ......

ODE with Osgood–Sobolev coefficients

- Local space: $f \in M^{1,p}_{\rho,loc}$ if $f \in L^p_{loc}$ and $\exists\, g \in L^p_{loc}$ s.t.

$$|f(x) - f(y)| \le (g(x) + g(y))\rho(|x-y|) \quad \text{for-a.e. } x, y \in \mathbb{R}^d.$$

H. Li–Luo (2015): Assume

- $V : \mathbb{R}^d \to \mathbb{R}^d$ is bounded;
- $V \in M^{1,1}_{\rho,loc}$;
- the distributional divergence $\operatorname{div}(V)$ of V exists and is bounded.

Then the ODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = V(X_t)$$

generates a unique flow X_t of maps s.t. $\mathcal{L}^d \circ X_t^{-1} \ll \mathcal{L}^d$.

Main idea

Recall that when $V \in W^{1,1}_{loc}$, the key is to estimate

$$\int_{B_R} \log\left[1 + \frac{\sup_{t\le T}|X_t(x) - \tilde X_t(x)|}{\delta}\right] \mathrm{d}x,$$

where $\delta > 0$. The function

$$\log\left(1 + \frac{\xi}{\delta}\right) = \int_0^\xi \frac{\mathrm{d}s}{s+\delta}, \quad \xi > 0.$$

By analogy, if $V \in M^{1,1}_{\rho,loc}$, we should consider

$$\psi_\delta(\xi) = \int_0^\xi \frac{\mathrm{d}s}{\rho(s)+\delta} \quad (\xi > 0)$$

and estimate

$$\int_{B_R} \psi_\delta\left(\sup_{t\le T}|X_t(x) - \tilde X_t(x)|\right) \mathrm{d}x.$$

Outline

Hypothesis

Assume $\rho \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfy

- $\rho(0) = 0$;
- $\rho'(s) \geq 0$;
- $\int_{0+} \frac{\mathrm{d}s}{\rho(s)} = +\infty$.

Let $q \geq 1$. V is a measurable vector field on $\mathbb{R}^d$.

$(\mathbf{H}_q)$ $\exists\, g \in L^q_{loc}(\mathbb{R}^d, \mathbb{R}_+)$ such that for a.e. $x, y \in \mathbb{R}^d$,

$$|\langle V(x) - V(y), x - y\rangle| \leq (g(x) + g(y))\rho(|x - y|^2).$$

First result

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}W_t + V(X_t)\,\mathrm{d}t. \tag{5}$$

Theorem 5

Let $q > 1$ *and* $\mathrm{d}\mu(x) = (1 + |x|^2)^{-q-d/2}\,\mathrm{d}x$*. Assume that*

- $\sigma \in W^{1,2}_{loc}$, V *verifies* $(\boldsymbol{H}_q)$ *and the distributional divergence* $\mathrm{div}(V)$ *exists;*
- $\forall\, p > 0$,

$$\int_{\mathbb{R}^d} \exp\left\{p\left[(\mathrm{div}(V))^- + \frac{|V|}{1+|x|} + \frac{|\sigma|^2}{1+|x|^2} + |\nabla\sigma|^2\right]\right\} \mathrm{d}\mu < \infty. \tag{6}$$

Then SDE (5) *generates a unique* μ*-a.e. stochastic flow* X_t*.*

Moreover, the Radon–Nikodym density $K_t := \frac{\mathrm{d}(\mu\circ X_t^{-1})}{\mathrm{d}\mu}$ *belongs to* $L^\infty([0,T], L^p(\mathbb{P}\times\mu))$ $(\forall\, p > 0)$*.*

Second result

Theorem 6

Assume that

- $\sigma \in C_b^2(\mathbb{R}^d, \mathbb{R}^{d\times m})$;
- V *verifies* ($\boldsymbol{H}_1$) *and the distributional divergence* $\mathrm{div}(V)$ *exists s.t.*

$$[\mathrm{div}(V)]^- \in L^\infty(\mathbb{R}^d), \quad \frac{|V|}{1+|x|} \in L^\infty(B_r^c)$$

  for some $r > 0$.

Then SDE (5) *generates a unique* $\mathcal{L}^d$*-a.e. stochastic flow* X_t.

The first condition can be replaced by

- $\sigma \in W^{2,2}_{loc}$ s.t. $|\nabla\sigma|, \left(\sup_{|z|\le 1} |\sigma(\cdot - z)|\right)|\nabla \mathrm{div}(\sigma)| \in L^\infty$.

Corollary 7

Under the conditions of Theorems 5 and 6, let

$$k_t(x) = \mathbb{E}K_t(\omega, x),$$

where $K_t = \frac{\mathrm{d}(\mu\circ X_t^{-1})}{\mathrm{d}\mu}$ *is the Radon–Nikodym density.*

Then k_t *solves the Fokker–Planck equation*

$$\partial_t u_t = L^* u_t, \quad u|_{t=0} \equiv 1, \tag{7}$$

where L^* *is the adjoint operator of*

$$L\varphi = \frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^*\,\nabla^2\varphi\big) + \langle V, \nabla\varphi\rangle.$$

Third result: uniqueness of FPE

Consider the Fokker–Planck equation:

$$\partial_t u_t = L^* u_t, \quad u|_{t=0} \equiv u_0. \tag{8}$$

L^*: the adjoint operator of $L\varphi = \frac{1}{2}\mathrm{Tr}\big(\sigma\sigma^* \nabla^2\varphi\big) + \langle V, \nabla\varphi\rangle$.

Theorem 8

Assume that

- σ *and* V *are essentially bounded;*
- $\exists\, g \in L^1_{loc}(\mathbb{R}^d, \mathbb{R}_+)$ *such that*
    - $\|\sigma(x) - \sigma(y)\|^2 \le (g(x) + g(y))\rho(|x-y|^2)$;
    - $|\langle V(x) - V(y), x - y\rangle| \le (g(x) + g(y))\rho(|x-y|^2)$.

Then for any $u_0 \in L^1 \cap L^\infty$, *the Fokker–Planck eq.* (8) *has at most one solution in* $L^\infty([0,T], L^1 \cap L^\infty)$.

Outline

Moment estimate of flows

Lemma 9

Assume that

- $q > 1$ *and* $\mathrm{d}\mu(x) = (1 + |x|^2)^{-q-d/2}\,\mathrm{d}x$*;*
- $\sigma, V \in L^{2q}(\mu)$*;*
- $\Lambda_{p,T} := \sup_{t \le T} \|K_t\|_{L^p(\mathbb{P}\times\mu)} < +\infty$*, where*
    - p *is the conjugate number of* q*;*
    - K_t *the Radon–Nikodym density with respect to* μ*.*

Then we have

$$\int_{\mathbb{R}^d} \mathbb{E} \sup_{0\le t\le T} |X_t(x)|^2 \,\mathrm{d}\mu(x) \le C_1 + C_T \Lambda_{p,T}\big(\|\sigma\|_{L^{2q}(\mu)}^2 + \|b\|_{L^{2q}(\mu)}^2\big),$$

where $C_1 = 3\int_{\mathbb{R}^d} |x|^2 \,\mathrm{d}\mu(x) < +\infty$*.*

Stability of flows

Given $\sigma, \tilde{\sigma} \in L^{2q}_{loc}(\mathbb{R}^d, \mathcal{M}_{d\times m})$ and $V, \tilde{V} \in L^q_{loc}(\mathbb{R}^d, \mathbb{R}^d)$. Consider

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}W_t + V(X_t)\,\mathrm{d}t,$$
$$\mathrm{d}\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t)\,\mathrm{d}W_t + \tilde{V}(\tilde{X}_t)\,\mathrm{d}t.$$

Radon–Nikodym densities:

$$K_t = \frac{\mathrm{d}(\mu \circ X_t^{-1})}{\mathrm{d}\mu}, \quad \tilde{K}_t = \frac{\mathrm{d}(\mu \circ \tilde{X}_t^{-1})}{\mathrm{d}\mu}.$$

Level set:

$$G_R(\omega) := \big\{x \in \mathbb{R}^d : \|X_\cdot(\omega, x)\|_{\infty,T} \vee \|\tilde{X}_\cdot(\omega, x)\|_{\infty,T} \le R\big\},$$

where $\|\cdot\|_{\infty,T}$ is the uniform norm on $C([0,T], \mathbb{R}^d)$.

Recall that $\rho \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ satisfy

- $\rho(0) = 0$;
- $\rho'(s) \geq 0$;
- $\int_{0+} \frac{\mathrm{d}s}{\rho(s)} = +\infty$.

Define the auxiliary function

$$\psi_\delta(\xi) = \int_0^\xi \frac{\mathrm{d}s}{\rho(s) + \delta} \quad (\xi > 0).$$

Then

$$\psi_\delta'(\xi) = \frac{1}{\rho(\xi) + \delta} > 0, \quad \psi_\delta''(\xi) = -\frac{\rho'(\xi)}{(\rho(\xi) + \delta)^2} \leq 0.$$

Lemma 10 (Stability estimate)

Assume that

- $q > 1$ *and* $\mathrm{d}\mu(x) = (1+|x|^2)^{-q-d/2}\,\mathrm{d}x$;
- $\sigma \in W^{1,2q}_{loc}$ *and* $V \in M^{1,q}_{\rho,loc}$;
- $\Lambda_{p,T} := \sup_{0\le t\le T} \big(\|K_t\|_{L^p(\mathbb{P}\times\mu)} \vee \|\tilde K_t\|_{L^p(\mathbb{P}\times\mu)}\big) < +\infty$.

Then for any $\delta > 0$,

$$\begin{aligned}
&\mathbb{E}\int_{G_R} \psi_\delta\big(\|X-\tilde X\|^2_{\infty,T}\big)\,\mathrm{d}\mu \\
&\le C_{d,R,T}\Lambda_{p,T}\bigg[\|\nabla\sigma\|^2_{L^{2q}(B_{3R})} + \|g\|_{L^q(B_R)} + \frac{1}{\delta}\|\sigma-\tilde\sigma\|^2_{L^{2q}(B_R)} \\
&\qquad + \frac{1}{\sqrt{\delta}}\Big(\|\sigma-\tilde\sigma\|_{L^{2q}(B_R)} + \|V-\tilde V\|_{L^q(B_R)}\Big)\bigg].
\end{aligned}$$

Uniqueness of flow X_t follows immediately from Lemma 10.

Existence of flow. Regularize the coefficients:

$$\sigma_n = (\sigma * \chi_n)\phi_n, \quad V_n = (V * \chi_n)\phi_n,$$

where $\phi_n(x) = \phi(x/n)$ is a truncating function. Consider

$$\mathrm{d}X_t^n = \sigma_n(X_t^n)\,\mathrm{d}W_t + V_n(X_t^n)\,\mathrm{d}t.$$

Flow of diffeomorphisms X_t^n. Let $K_t^n = \frac{\mathrm{d}(\mu\circ(X_t^n)^{-1})}{\mathrm{d}\mu}$.

Conditions on σ, V imply $\Lambda_{p,T} = \sup_{n\geq 1}\sup_{t\leq T} \|K_t^n\|_{L^p(\mathbb{P}\times\mu)} < \infty$.

$$\begin{aligned}
&\mathbb{E}\int_{G_R^m\cap G_R^n} \psi_\delta\big(\|X^m - X^n\|_{\infty,T}^2\big)\,\mathrm{d}\mu \\
&\leq C_{d,R,T}\Lambda_{p,T}\bigg[\|\nabla\sigma\|_{L^{2q}(B_{3R})}^2 + \|g\|_{L^q(B_R)} + \frac{1}{\delta}\|\sigma_m - \sigma_n\|_{L^{2q}(B_R)}^2 \\
&\qquad + \frac{1}{\sqrt{\delta}}\Big(\|\sigma_m - \sigma_n\|_{L^{2q}(B_R)} + \|V_m - V_n\|_{L^q(B_R)}\Big)\bigg],
\end{aligned}$$

Taking

$$\delta_{m,n} = \big(\|\sigma_m - \sigma_n\|_{L^{2q}(B_R)} + \|V_m - V_n\|_{L^q(B_R)}\big)^2 \to 0$$

as $m, n \to \infty$, then

$$\mathbb{E} \int_{G_R^m \cap G_R^n} \psi_{\delta_{m,n}}\big(\|X^m - X^n\|_{\infty,T}^2\big)\, \mathrm{d}\mu \le \bar{C}_{d,R,T} < +\infty.$$

$$\Longrightarrow \quad \lim_{m,n\to\infty} \mathbb{E} \int_{\mathbb{R}^d} \big(1 \wedge \|X^m - X^n\|_{\infty,T}^2\big)\, \mathrm{d}\mu = 0.$$

By the moment estimate, for any $\alpha \in [1, 2)$,

$$\lim_{m,n\to\infty} \mathbb{E} \int_{\mathbb{R}^d} \|X^m - X^n\|_{\infty,T}^\alpha \, \mathrm{d}\mu = 0.$$

Hence $X^n \to X \in L^\alpha\big(\Omega \times \mathbb{R}^d, C([0,T], \mathbb{R}^d)\big)$.

Thanks for your attention!