Singular perturbation analysis for countable Markov chains

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The 12th Workshop on Markov Processes and Related Topics, Xuzhou, July 13-17, 2016, BNU and JSNU

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Outline

Perturbed DTMC

Let $\widetilde{X}_n$ be a time-homogeneous discrete-time Markov chain (DTMC) on a countable state space $\mathbb{E}$ with an irreducible stochastic transition matrix $\widetilde{P}$. Suppose that $\widetilde{P}$ is nearly decomposable, i.e.

$$\widetilde{P} = P + \Delta,$$

- the (small) perturbation matrix Δ satisfies that $\Delta \boldsymbol{e} = 0$.
- the unperturbed transition matrix P is decomposable (reducible), the state space $\mathbb{E}$ is decomposed into denumerable irreducible and ergodic classes $\mathbb{E}_n$ for $n \in \hat{\mathbb{E}}$, where the set $\hat{\mathbb{E}} := \{0, 1, \cdots, \ell\}$, $0 \leq \ell \leq \infty$. Thus $\mathbb{E} = \bigcup_{n \in \hat{\mathbb{E}}} \hat{\mathbb{E}} \times \mathbb{E}_n$.

The unperturbed transition matrix P can be written as

$$P = \begin{bmatrix} P_0 & 0 & 0 & \cdots \\ 0 & P_1 & 0 & \cdots \\ 0 & 0 & P_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

When $\ell = 0$, the perturbation is called regular perturbation; when $\ell \geq 1$, the perturbation is called singular perturbation.

Perturbed CTMC

Let $\widetilde{X}_t$ be a time-homogeneous CTMC on a countable space $\mathbb{E}$ with an irreducible, conservative and possibly unbounded generator $\widetilde{Q}$. Suppose that $\widetilde{Q}$ is nearly decomposable, i.e.

$$\widetilde{Q} = Q + \Delta,$$

where Δ is the small and Q is decomposable:

$$Q = \begin{pmatrix} Q_0 & 0 & 0 & \cdots \\ 0 & Q_1 & 0 & \cdots \\ 0 & 0 & Q_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Similarly, regular perturbation when $\ell = 0$, and singular perturbation when $\ell \geq 1$

For singular perturbation, the elements of P (Q) and Δ reflect the strong and weak interactions, respectively.

A wide range of large-scale systems are distinguished by this feature and hence are nicely modeled with the help of decomposable Markov chains, such as

▷ Markov decision processes, e.g., Bielecki and Stetlner (1998), Yin and Zhang (2000);

▷ control problems, e.g., Delebecque and Quadrat (1981), Yin and Zhang (2003);

▷ queueing networks, e.g., Latouche and Schweitzer (1995), Yin and Zhang (2008)

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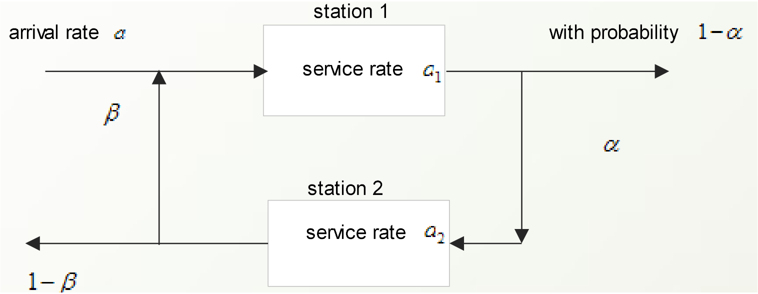
Example 1: a 2d discrete-time queue

Consider a DT queueing system with two stations:



which is modified from Yin and Zhang (2008) by letting that Station 1 has an unlimited room and the arrival customers follow a geometric distribution with a constant parameter.

Let $\widetilde{X}_j(n)$ be the number of customers at Station j at time n. Then $(\widetilde{X}_2(n), \widetilde{X}_1(n))$ is a two-dimensional DTMC on the state space $\mathbb{E} = \bigcup_{n\in\hat{\mathbb{E}}} \hat{\mathbb{E}} \times \mathbb{E}_n$, with $\hat{\mathbb{E}} = \mathbb{E}_n = \{0, 1, 2, \cdots\}$.

Its transition matrix $\widetilde{P}$ is given by $\widetilde{P} = P + \Delta$, where P is decomposable and $P_n, n \geq 1$ and Δ are determined by

$$P_0 = \begin{bmatrix} a & 1-a & 0 & \cdots \\ \mu_1 & \theta_0 & \lambda_1 & \cdots \\ 0 & \mu_1 & \theta_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad P_n = \begin{bmatrix} c & 1-c & 0 & \cdots \\ \mu_2 & \theta_1 & \lambda_2 & \cdots \\ 0 & \mu_2 & \theta_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

$$\Delta = \begin{bmatrix} \overline{\Delta}_0 & \overline{\Delta}_1 & 0 & \cdots \\ \Delta_{-1} & \Delta_0 & \Delta_1 & \cdots \\ 0 & \Delta_{-1} & \Delta_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \overline{\Delta}_0 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & -\widehat{\gamma}_{12} & 0 & \cdots \\ 0 & 0 & -\widehat{\gamma}_{12} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\overline{\Delta}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ a\widehat{\gamma}_{12} & (1-a)\widehat{\gamma}_{12} & 0 & 0 & \cdots \\ 0 & a\widehat{\gamma}_{12} & (1-a)\widehat{\gamma}_{12} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\Delta_{-1} = \begin{bmatrix} a\gamma_2 & \widehat{\delta}_0 & \overline{\delta}_0 & 0 & 0 & \cdots \\ a\gamma_1\gamma_2 & \delta_1 & \widehat{\delta}_1 & \overline{\delta}_1 & 0 & \cdots \\ 0 & a\gamma_1\gamma_2 & \delta_1 & \widehat{\delta}_1 & \overline{\delta}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\Delta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ aa_2\widehat{\gamma}_{12} & (1-a)a_2\widehat{\gamma}_{12} & 0 & 0 & \cdots \\ 0 & aa_2\widehat{\gamma}_{12} & (1-a)a_2\widehat{\gamma}_{12} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and

$$\Delta_0 = \begin{bmatrix} -\overline{\gamma}_2 & 0 & 0 & 0 & \cdots \\ a\widehat{\gamma}_{12}\gamma_2 & \widehat{\theta}_1 & (1-a)\widehat{\gamma}_{12}\widehat{\gamma}_{21} & 0 & \cdots \\ 0 & a\widehat{\gamma}_{12}\gamma_2 & \widehat{\theta}_1 & (1-a)\widehat{\gamma}_{12}\widehat{\gamma}_{21} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Assume that parameters α and $(1 - a_2)$ are small$\Rightarrow$ Δ is small.

$\Rightarrow$ the changes of the queue length process and departure process corresponding to Station 2 are relatively slow compared with those corresponding to Station 1, which results in that the transition probability between any two levels is very small.

Example 2: a M M Birth-death process

Consider a Markov modulated state dependent birth-death process.

The environment process is a slowly varying M/M/1 queue, in which customers arrive at this queue according to a Poisson process with rate εq and are served at rate εp. Let $X(t)$ be the number of customers in the queue at time t.

Define a birth-death process $Y(t)$ as follows: if at time t, $X(t) = n$ and $Y(t) = i$, then $Y(t)$ jumps up to $i + 1$ at the birth rate $r_i b_n$ and jumps down to $i - 1$ at the death rate $s_i a_n$ for non-negative functions r_i and and s_i on $\{0, 1, 2, \cdots\}$.

The process $(X(t), Y(t))$ is a two-dimensional level dependent QBD process with infinitely many levels and phases.

Its generator is $\tilde{Q} = Q + \Delta$, where Q is decomposable with $Q_n, n \geq 0$ given by

$$Q_n = \begin{pmatrix} -r_0 b_n & r_0 b_n & 0 & \cdots \\ s_1 a_n & -r_1 b_n - s_1 a_n & r_1 b_n & \cdots \\ 0 & s_2 a_n & -r_2 b_n - s_2 a_n & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and Δ has the same form as that of Example 1 with

$$\Delta_1 = \begin{pmatrix} \varepsilon q & 0 & 0 & \cdots \\ 0 & \varepsilon q & 0 & \cdots \\ 0 & 0 & \varepsilon q & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \Delta_{-1} = \begin{pmatrix} \varepsilon p & 0 & 0 & \cdots \\ 0 & \varepsilon p & 0 & \cdots \\ 0 & 0 & \varepsilon p & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\overline{\Delta}_1 = \Delta_1$, $\overline{\Delta}_0 = -\overline{\Delta}_1$, $\Delta_0 = -(\Delta_{-1} + \Delta_1)$.

Our goal

Understand how perturbation affect the long-run behavior:

(i) Find conditions on Δ and P_i to ensure that $\widetilde{P}$ is positive recurrent with stationary distribution $\widetilde{\boldsymbol{\pi}}$, and

$$\widetilde{\boldsymbol{\pi}} = \pi^{(0)} \sum_{n=0}^{\infty} U^n,$$

for some matrix U and probability vector $\pi^{(0)}$.

(ii) When (i) is addressed well, we consider the bound on the difference $\widetilde{\boldsymbol{\pi}} - \pi^{(0)} = \pi^{(0)} \sum_{n=1}^{\infty} U^n$.

Literature

The singular perturbation problems are investigated by posing the Dobelin condition, see, e.g.

$\diamond$: Korolyuk and Turbin (1993), Bielecki and Stetlner (1998) for singular perturbation of Markov chains on general measurable state,

$\diamond$: Yin and Zhang (2002, 2008) for two-time scales singular perturbation.

The Dobelin condition is quite restrictive for DTMCs on an infinitely countable state space, see e.g. Hou and L (2004).

Altman et al. (2004): first adopt geometric ergodicity condition.

$\diamond$: the bounds are not well investigated,

$\diamond$: $\widetilde{P}$ and P_n are aperiodic,

$\diamond$: the perturbation Δ is linear (i.e. $\Delta = \varepsilon G$),

$\diamond$: the generator $\widetilde{Q}$ and Q are bounded.

We will investigate these issues by extending the ideas for regularly perturbed MCs, see e.g.

$\diamond$: Kartashov (1986), Mouhoubi and Aissani (2010), L (2012) for DTMCs,
$\diamond$: Mitrophanov (2006), Heidergott etal. (2010), L (2015) for CTMCs.

Outline

1. Background
2. Singular perturbation for DTMCs
3. Singular perturbation for CTMCs

Notations

Recall that each class P_n is positive recurrent, whose stationary probability vector is π_n and stationary probability matrix is Π_n.

Define the fundamental matrix by $R_n = (I - P_n + \Pi_n)^{-1}$.

Let

$$\mathbb{\Pi} = \begin{bmatrix} \Pi_0 & 0 & 0 & \cdots \\ 0 & \Pi_1 & 0 & \cdots \\ 0 & 0 & \Pi_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \mathbb{R} = \begin{bmatrix} R_0 & 0 & 0 & \cdots \\ 0 & R_1 & 0 & \cdots \\ 0 & 0 & R_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

To introduce the aggregated CTMC (e.g. Delebecque 1983), define the matrices M and W by

$$M = \begin{bmatrix} \pi_0 & 0 & 0 & \cdots \\ 0 & \pi_1 & 0 & \cdots \\ 0 & 0 & \pi_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{\hat{\mathbb{E}}\times\mathbb{E}}, \quad W = \begin{bmatrix} \mathbf{e}_0 & 0 & 0 & \cdots \\ 0 & \mathbf{e}_1 & 0 & \cdots \\ 0 & 0 & \mathbf{e}_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{\mathbb{E}\times\hat{\mathbb{E}}}.$$

The generator $\hat{Q}$ of the aggregated chain is defined by

$$\hat{Q} = M\Delta W, \quad (\hat{q}_{ij} = \pi_i \Delta_{ij} \mathbf{e}_j, \; i,j \in \hat{\mathbb{E}}),$$

whose stationary distribution and deviation matrix are denoted respectively by $\hat{\pi}$ and $D_{\hat{Q}}$ if they exist.

a key matrix

Introduce a key matrix U as follows

$$U = \Delta(\mathbb{R} - \mathbb{\Pi})(I + \Delta W D_{\hat{Q}} M).$$

Remark:

(i) When $\ell = 0$, $U = \Delta(R - \Pi)$, which plays a key role for regular perturbation analysis.

(ii) Compared with Altman et al. (2004), this definition permits that $\widetilde{P}$ and each class P_n are periodic and the perturbation Δ is general.

v-norm and Assumption 1

Let $\boldsymbol{v} \geq 1$. For a measure $\boldsymbol{\mu}$, its $\boldsymbol{v}$-norm is defined by $\|\boldsymbol{\mu}\|_{\boldsymbol{v}} = \sum_{i\in\mathbb{E}} |\mu_i| v_i$. For a matrix A, its $\boldsymbol{v}$-norm is defined by

$$\|A\|_{\boldsymbol{v}} = \sup_{i\in\mathbb{E}} v_i^{-1} \sum_{j\in\mathbb{E}} |A_{ij}| v_j.$$

Assumption 1: Let $\boldsymbol{v} \geq 1$. Assume that

(i) all the matrices P, Δ, Π, $\hat{\boldsymbol{\pi}} M$, $\mathbb{R}$ and $WD_{\hat{Q}}M$ have finite $\boldsymbol{v}$-norm; and

(ii) there exists a finite positive integer N and a constant $\delta_N \in (0,1)$ such that $\delta_N = \|U^N\|_{\boldsymbol{v}} < 1$.

Main results

Theorem 1: Suppose that Assumption 1 holds. Then
(i) $\widetilde{P}$ has a unique stationary distribution, given by

$$\tilde{\pi} = \hat{\pi} M \sum_{k=0}^{\infty} U^k.$$

(ii) For $N = 1$ (the case $N > 1$ can be obtained)

$$\left\| \widetilde{\pi} - \hat{\pi} M \sum_{k=0}^{m} U^k \right\|_{\boldsymbol{v}} \leq \|\hat{\pi} M\|_{\boldsymbol{v}} \frac{\|U\|_{\boldsymbol{v}}^{m+1}}{1 - \|U\|_{\boldsymbol{v}}}.$$

Remark:

▷ Note that for regular perturbation $\hat{\boldsymbol{\pi}} M = \pi$ $(\pi P = \pi)$, while for singular perturbation $\hat{\boldsymbol{\pi}} M = (\hat{\pi}_0 \boldsymbol{\pi}_1, \hat{\pi}_1 \boldsymbol{\pi}_1, \cdots)$.

▷ We call $\sum_{k=0}^{m} \hat{\boldsymbol{\pi}} M U^k$ the $(m{+}1)$-th order approximation of the stationary distribution $\widetilde{\boldsymbol{\pi}}$.

▷ For regular perturbation, the first approximation $\hat{\boldsymbol{\pi}} M$ (i.e., $m = 0$) becomes the stationary distribution of P, which has caused much concern in the literature.

▷ To prove this theorem, we need to make use of associativity for the multiplications of (possibly negative) infinite matrices. (i) of Assumption 1 can guarantee the associativity.

We now state Lyapunov drift conditions (see Chen (2003)), which are equivalent to geometric and exponential ergodicity, respectively.

D1($\boldsymbol{V}, \lambda, b, \{i\}$): For a transition matrix P, suppose that there exist a finite vector $\boldsymbol{V}$, $\boldsymbol{V} \geq \boldsymbol{e}$, some state i and positive numbers $b < \infty, \lambda < 1$ such that

$$P\boldsymbol{V} \leq \lambda \boldsymbol{V} + b\mathbb{1}_{\{i\}}.$$

D2($\boldsymbol{V}, \lambda, b, \{i\}$): For a q-matrix Q, suppose that there exist a finite function $\boldsymbol{V}$, $\boldsymbol{V} \geq \boldsymbol{e}$, some state i and positive constants $\lambda > 0, b < \infty$ such that

$$Q\boldsymbol{V} \leq -\lambda \boldsymbol{V} + b\mathbb{1}_{\{i\}}.$$

Assumption 2: Assume that
(i) **D1(**$V, \lambda, b, \{i_0(n)\}$**)** holds uniformly for each P_n; and
(ii) **D2(**$\hat{V}, \hat{\lambda}, \hat{b}, \{\hat{i}_0\}$**)** holds for the generator $\hat{Q}$.

Theorem 2: Suppose that Assumption 2 holds and $\|\Delta\|_{\boldsymbol{v}} < \infty$, where $v_{ni} := \hat{V}_n V_i$, $n \in \hat{\mathbb{E}}, i \in \mathbb{E}_n$. Then Assumption 1 holds and

$$\|U\|_{\boldsymbol{v}} \leq x(1 + y\|\Delta\|_{\boldsymbol{v}} + z),$$

where x, y, z are given by

$$x = \frac{c_1}{1-\lambda}, \quad y = \frac{c_2(1+c_3)^2}{\hat{\lambda}}, \quad z = c_2 c_3$$

with $c_1 := \|\Delta(I - \Pi)\|_{\boldsymbol{v}}$, $c_2 := \sup_{n \in \hat{\mathbb{E}}} \pi_n(V)$ and $c_3 := \hat{\pi}(\hat{V})$.
Note that $c_1 \leq (1 + c_2)\|\Delta\|_{\boldsymbol{v}}$ $c_2 \leq b/(1-\lambda)$, and $c_3 \leq \hat{b}/\hat{\lambda}$.

Back to Example 1

Assumption-1: Suppose that (i) $a_1 + a\alpha < a + aa_1\alpha$; and (ii) $a_2 + \alpha < 1 + aa_2\alpha$.

▷ Note that (i) of Assumption-1 is equivalent to $\rho := \frac{\lambda_1}{\mu_1} < 1$.

▷ Under (i) of Assumption-1, $\mathbf{D1}(\boldsymbol{V}, \lambda, b, \{i_0(n)\})$ holds uniformly in n for $i_0(n) = 0$, $\boldsymbol{V}_j = \left(\sqrt{\frac{1}{\rho}}\right)^j, j \geq 0$, $\lambda = 1 - \left(\sqrt{\lambda_1} - \sqrt{\mu_1}\right)^2$ and $b = a + (1-a)\sqrt{\frac{1}{\rho}} - \lambda_0$.

$$\pi_{n,0} = \frac{\frac{a_1}{\rho}}{\frac{a_1}{\rho} + \frac{1}{1-\rho}}, \quad \pi_{n,i} = \frac{\rho^{i-1}}{\frac{a_1}{\rho} + \frac{1}{1-\rho}}, \quad i \geq 1.$$

The generator of the aggregated CTMC $\hat{Q}$ is

$$\hat{Q} = \begin{bmatrix} d_0 & d_1 & 0 & \cdots \\ c_0 & c_1 & c_2 & \cdots \\ 0 & c_0 & c_1 & \cdots \\ \vdots & \vdots & \ddots & \cdots \end{bmatrix},$$

where $d_i = \boldsymbol{\pi}_0 \Delta_{0i} \mathbf{e}$, and $c_i = \boldsymbol{\pi}_0 \Delta_i \mathbf{e}$.

▷ Note that (ii) of Assumption-1 is equivalent to $\hat{\rho} := \frac{c_2}{c_0} < 1$.

▷ When $\hat{\rho} < 1$, $\mathbf{D2}(\hat{\boldsymbol{V}}, \hat{\lambda}, \hat{b}, \{\hat{i}_0\})$ holds for $\hat{i}_0 = 0$, $\hat{V}_j = \left(\sqrt{\frac{1}{\hat{\rho}}}\right)^j, j \geq 0, \hat{\lambda} = \left(\sqrt{c_2} - \sqrt{c_0}\right)^2, \hat{b} = d_0 + d_1\sqrt{\frac{1}{\hat{\rho}}} + \overline{\lambda}_d$.

Define the drift vector $\boldsymbol{v}$ for the whole space by

$$\boldsymbol{v}_{ni} = \hat{V}_n V_i = \left(\sqrt{\frac{1}{\hat{\rho}}}\right)^n \left(\sqrt{\frac{1}{\rho}}\right)^i.$$

Assumption-2: Both parameters α and $(1 - a_2)$ are small enough such that $x\,(1 + y\|\Delta\|_v + z) < 1$.

Under the above two assumptions, by Theorem 2, we have

$$\left\| \widetilde{\pi} - \hat{\pi} M \sum_{k=0}^{m} U^k \right\|_{\boldsymbol{v}} \leq \frac{\left[x\,(1 + y\|\Delta\|_{\boldsymbol{v}} + z)\right]^{m+1}}{1 - x\,(1 + y\|\Delta\|_{\boldsymbol{v}} + z)} \|\hat{\pi} M\|_{\boldsymbol{v}},$$

Note that $\|\Delta\|_{\boldsymbol{v}}$, $\|\hat{\pi} M\|_{\boldsymbol{v}}$, x, y, and z can be determined by the model parameters.

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Singular perturbation for CTMCs

Let π_n and D_n respectively be the stationary distribution and the deviation matrix corresponding to Q_n whenever they exist.

Let

$$D_Q = \begin{bmatrix} D_0 & 0 & 0 & \cdots \\ 0 & D_1 & 0 & \cdots \\ 0 & 0 & D_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The generator $\hat{Q}$ of the aggregated CTMC is defined by

$$\hat{Q} = M\Delta W.$$

Let $\hat{\pi}$ and $D_{\hat{Q}}$ respectively be the stationary distribution and the deviation matrix of $\hat{Q}$ whenever they exist.

To establish the counterpart of the discrete-time perturbation results, define

$$U = \Delta D_Q (I + \Delta W D_{\hat{Q}} M).$$

Assumption 1': Let $\boldsymbol{v} \geq 1$ be a vector on $\mathbb{E}$.
(i) All the matrices Q, Δ, Π, $\hat{\pi} M$, D_Q and $W D_{\hat{Q}} M$ have finite $\boldsymbol{v}$-norm;
(ii) there exists a finite number N and a constant $\delta_N \in (0, 1)$ such that $\delta_N = \| U^N \|_{\boldsymbol{v}} < 1$.

Theorem 1' Suppose that Assumption 1' holds. Then the same results as that in Theorem 1 hold.

Assumption 2': Assume that
(i) **D2**($\boldsymbol{V}, \lambda, b, \{i_0(n)\}$) holds uniformly in n for the same $\boldsymbol{V}, \lambda$ and b; and
(ii) **D2**($\hat{\boldsymbol{V}}, \hat{\lambda}, \hat{b}, \{\hat{i}_0\}$) holds for $\hat{Q}$.

Theorem 2' Suppose that Assumption 2' and $\|\Delta\|_{\boldsymbol{v}} < \infty$, where $v_{ni} := \hat{V}_n V_i$ for $n \in \hat{\mathbb{E}}, i \in \mathbb{E}_n$. Then (i) of Assumption 1' holds and

$$\|U\|_{\boldsymbol{v}} \le x \left[1 + \frac{(1 + \hat{\pi}(\hat{\boldsymbol{V}}))^2}{\hat{\lambda}} \sup_{n \in \hat{\mathbb{E}}} \boldsymbol{\pi}_n(\boldsymbol{V}) \|\Delta\|_{\boldsymbol{v}}\right],$$

where the value of x is given by

$$x = \|\Delta D_Q\|_{\boldsymbol{v}} \le \frac{(1 + \sup_{n \in \hat{\mathbb{E}}} \boldsymbol{\pi}_n(\boldsymbol{V}))^2}{\lambda} \|\Delta\|_{\boldsymbol{v}}.$$

To prove the above result, one key point is to use the following result

Theorem [L 2015] For a CTMC with a (possibly unbound) Q-matrix. Suppose that $\mathbf{D2}(\boldsymbol{V}, \lambda, b, \{i\})$ holds. Then we have

$$\|D\|_{\boldsymbol{v}} \le \frac{(1+\boldsymbol{\pi}(\boldsymbol{V}))^2}{\lambda} \le \frac{(\lambda+b)^2}{\lambda^3}.$$

Back to Example 2

Assumption: Assume that (i) $\inf_{n\geq 0}(a_n - b_n) > 0$, $\sup_{n\geq 0} b_n < \infty$; and (ii) $p < q$.

(i) of this assumption implies that $\mathbf{D2}(V, \lambda, b, \{i_0(n)\})$ holds uniformly in n for $i_0(n) = 0$, $V_j = j+1, j \geq 0$, $\lambda = \frac{\inf_{n\geq 0}(a_n - b_n)}{2}$, and $b = \sup_{n\geq 0} b_n$.

The stationary distribution $\{\boldsymbol{\pi}_n, n \in \hat{\mathbb{E}}\}$, dependent on n, is given by

$$\pi_{n,0} = \frac{1}{1 - \log(1 - b_n/a_n)}, \quad \pi_{n,k} = \frac{1}{k}\left(\frac{b_n}{a_n}\right)^k \pi_{n,0}, \quad k \geq 1.$$

The generator $\hat{Q}$ of the aggregated CTMC is given by

$$\hat{Q} = \begin{pmatrix} -\varepsilon q & \varepsilon q & 0 & \cdots \\ \varepsilon p & -\varepsilon(p+q) & \varepsilon q & \cdots \\ 0 & \varepsilon p & -\varepsilon(p+q) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(ii) of the assumption implies that $\hat{\rho} := \frac{q}{p} < 1$, and **D2**$(\hat{\boldsymbol{V}}, \hat{\lambda}, \hat{b}, \{\hat{i}_0\})$ holds for $\hat{i}_0 = 0$, $\hat{V}_j = \left(\sqrt{\frac{1}{\hat{\rho}}}\right)^j, j \geq 0$, $\hat{\lambda} = \varepsilon\left(\sqrt{p} - \sqrt{q}\right)^2$, $\hat{b} = \varepsilon(p - \sqrt{pq})$. Furthermore, we have

$$\hat{\pi}_0 = 1 - \hat{\rho}, \quad \hat{\pi}_n = (1-\hat{\rho})\hat{\rho}^{n-1}, \quad n \geq 1, \quad \hat{\boldsymbol{\pi}}(\hat{\boldsymbol{V}}) = 1 + \sqrt{\hat{\rho}}.$$

Define the drift vector $\boldsymbol{v}$ for the whole space by

$$\boldsymbol{v}_{ni} = \hat{V}_n V_i = \left(\sqrt{\frac{1}{\hat{\rho}}}\right)^n (i+1), \quad n \geq 0, i \geq 0.$$

Let $c = \sup_{n\geq 0} \boldsymbol{\pi}_n(\boldsymbol{V})$ and

$$u = \frac{(1+c)^2(\sqrt{p}+\sqrt{q})^2}{\lambda}\left[1 + c(2+\sqrt{q/p})^2\frac{(\sqrt{p}+\sqrt{q})^4}{(p-q)^2}\right].$$

Under the above assumption, if $0 < \varepsilon < 1/u$, then we have

$$\left\|\widetilde{\boldsymbol{\pi}} - \hat{\boldsymbol{\pi}} M \sum_{k=0}^{m} U^k\right\|_{\boldsymbol{v}} \leq c(1+\sqrt{\hat{\rho}})\frac{(\varepsilon u)^{m+1}}{1-\varepsilon u}.$$

Thank you for your attention!