Decay property of stopped Markovian bulk-arriving queues with c-servers

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Background

- Definition of Decay parameter

Let $\mathbf{E}$ be a countable set.
$Q = (q_{ij}; i, j \in \mathbf{E})$ be a stable Q-matrix.
$(p_{ij}(t); i, j \in \mathbf{E})$ is the Feller minimal Q-process.
C is a communicating class of $\mathbf{E}$ and

$$\lim_{t\to\infty} p_{ij}(t) = 0, \quad i, j \in C.$$

By Kingman (1963), there exists a number $\lambda_C \geq 0$ such that for all $i, j \in C$,

$$\frac{1}{t}\log p_{ij}(t) \to -\lambda_C \ \text{ as } t \to \infty$$

λ_C is called the decay parameter for C.

Background

On the other hand, let

$$\begin{aligned}\mu_{ij} &= \inf\{\lambda \geq 0 : \int_0^\infty e^{\lambda t} p_{ij}(t)dt = \infty\} \\ &= \sup\{\lambda \geq 0 : \int_0^\infty e^{\lambda t} p_{ij}(t)dt < \infty\}.\end{aligned}$$

It is easily seen that μ_{ij} does not depend on $i, j \in C$, the common value is denoted by μ. Moreover,

$$\lambda_C = \mu.$$

(see, for example, Pollett (2006)).

Background

- Problems:
  - $\lambda_C = ?$;
  - The λ_C-recurrency of the process.

- Known progress:

(i) Finite Markov chains.

(ii) BDP(Chen M.F.).
Specially, BDP: $q_{i\ i-1} = a$, $q_{i\ i+1} = b$, then $\lambda_C = (\sqrt{a} - \sqrt{b})^2$.

(iii) MBP: $q_{ij} = ib_{j-i+1}$, then $\lambda_C = -B'(q)$ where

$$B(s) = \sum_{j=0}^{\infty} b_j s^j$$

and q is the smallest nonnegative root of $B(s) = 0$.

Background

(iv) Stopped $M^X/M/1$ queue (Li and Chen, 2008):

$$q_{ij} = \begin{cases} b_{j-i+1}, & \text{if } i \geq 1, j \geq i-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\lambda_C = \sup\{\lambda \geq 0 : B(s) + \lambda s = 0 \text{ has a root in } (0, +\infty)\}$$

where $B(s) = \sum_{k=0}^{\infty} b_k s^k$.

Background

(iv) Stopped $M^X/M/1$ queue (Li and Chen, 2008):

$$q_{ij} = \begin{cases} b_{j-i+1}, & \text{if } i \geq 1, j \geq i-1 \\ 0, & \text{otherwise} \end{cases}$$

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where $B(s) = \sum_{k=0}^{\infty} b_k s^k$.

Background

(v) Controlled $M^X/M/1$ queue (Li and Chen, 2013):

$$q_{ij} = \begin{cases} h_j, & \text{if } i = 0, j \geq 0 \\ b_{j-i+1}, & \text{if } i \geq 1, j \geq i-1 \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\begin{cases} h_j \geq 0 \ (j > 0), \ 0 < \sum_{j=1}^{\infty} h_j = -h_0 < \infty \\ b_j \geq 0 \ (j \neq 1), \ 0 < \sum_{j \neq 1} b_j = -b_1 < \infty. \end{cases}$$

$$\lambda_Z = \min\{-\frac{B(s^*)}{s^*}, -\frac{B(s_h)}{s_h}\}$$

Background

- Motivation

General controlled $M^X/M/1$ queue:

$$q_{ij} = \begin{cases} h_j, & \text{if } i = 0, j \geq 0 \\ h^{(i)}_{j-i+1}, & \text{if } 1 \leq i < c, j \geq i-1 \\ b_{j-i+1}, & \text{if } i \geq c, j \geq i-1 \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\begin{cases} h_j \geq 0 \ (j > 0),\ 0 < \sum_{j=1}^{\infty} h_j = -h_0 < \infty \\ h_j^{(i)} \geq 0 \ (j \neq 1),\ 0 < \sum_{j \neq 1} h_j^{(i)} = -h_1^{(i)} < \infty,\ 1 \leq i < c \\ b_j \geq 0 \ (j \neq 1),\ 0 < \sum_{j \neq 1} b_j = -b_1 < \infty. \end{cases}$$

Background

This talk is concentrated on the decay parameter of stopped $M^X/M/c$.

Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be defined as follows:

$$q_{ij} = \begin{cases} \min(i,c)b_0, & \text{if } i \geq 1,\ j = i-1, \\ b_1 - [\min(i,c)-1]b_0, & \text{if } i \geq 1,\ j = i, \\ b_{j-i+1}, & \text{if } i \geq 1,\ j \geq i+1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

where

$$b_j \geq 0\ (j \neq 1),\ 0 < \sum_{j\neq 1} b_j = -b_1 < \infty. \tag{1.2}$$

Background

This talk is concentrated on the decay parameter of stopped $M^X/M/c$.

Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be defined as follows:

$$q_{ij} = \begin{cases} \min(i,c)b_0, & \text{if } i \geq 1,\ j = i-1, \\ b_1 - [\min(i,c)-1]b_0, & \text{if } i \geq 1,\ j = i, \\ b_{j-i+1}, & \text{if } i \geq 1,\ j \geq i+1, \\ 0, & \text{otherwise,} \end{cases} \tag{1.3}$$

where

$$b_j \geq 0\ (j \neq 1),\ 0 < \sum_{j \neq 1} b_j = -b_1 < \infty. \tag{1.4}$$

Background

Definition 1. Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a Q-matrix defined in (1.3)–(1.4). The corresponding transition function $P(t) = (p_{ij}(t); i, j \in \mathbf{Z}_+)$ is called a stopped $M^X/M/c$ queueing process.

If $c = 1$ then it is a stopped $M^X/M/1$ queueing model. In the following, we assume that $b_0 > 0$ and $\sum_{j=2}^{\infty} b_j > 0$ (to avoid trivial cases).

Background

Let

$$K = \min\{j \geq c; b_{j-c+2>0}\}(< \infty), \qquad \mathbf{Z}_K^+ = \{1, 2, \cdots, K\}$$

and $Q_K^+ = (q_{ij}; 1 \leq i, j \leq K)$ be the restriction of Q on $\mathbf{Z}_K^+$. Denote

$$\lambda_K = \min\{\lambda \geq 0; det(\lambda I + Q_K^+) = 0\}.$$

Actually, λ_K is the decay parameter for $\mathbf{Z}_K^+$.

Preliminary

Define

$$B(s) = \sum_{k=0}^{\infty} b_k s^k$$

and

$$B_i(s) = B(s) + (i-1)b_0(1-s), \quad i = 1, 2, \cdots, c.$$

$$\rho = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{b_n}}.$$

Clearly, $\rho \geq 1$.
Now, let $\rho_0 = \sup\{s > 0 : \ B_c(s) \leq 0\}$.

Preliminary

Lemma 2.1. $B_c(s)$ is convex on $[0, \rho)$ and has either one or two positive roots. More specifically,
(i) If either $\rho < +\infty$ and $B_c(\rho) \geq 0$ or $\rho = +\infty$ then $B_c(s) = 0$ has exactly two positive zeros q_s and q_L with $0 < q_s \leq q_L < \rho$.
(ii) If $\rho < +\infty, \ B_c(\rho) < 0$ then $B_c(s) = 0$ has exactly one positive root $q_s = 1$.

Define

$$\lambda^* = \sup\{\lambda \geq 0 : \lambda s + B_c(s) = 0 \text{ has a root in } [q_s, \rho_0]\}. \quad (2.1)$$

Preliminary

Lemma 2.2 Suppose that $B_c'(1) \leq 0$. Then for any $\lambda \in [0, \lambda_K]$, there exist positive $(m_j;\ j = 1, 2, \ldots, K)$ such that

$$\begin{cases} \sum_{i=1}^{j+1} m_i q_{ij} = -\lambda m_j, \quad j = 1, 2, \ldots, K-1, \\ \sum_{i=1}^{K} m_i q_{iK} \leq -\lambda m_K. \end{cases} \tag{2.2}$$

Preliminary

For any $\lambda \in [0, \lambda_K]$, let $(m_i; 1 \leq i \leq K)$ be determined in Lemma 2.2 and define

$$F_\lambda(s) = m_1 b_0 + \sum_{i=1}^{c-1} m_1 s^{i-1} [B_c(s) - B_i(s)].$$

$F_\lambda(s)$ is called test function which will play an important role in determining λ_C.

Also let

$$\bar{\lambda} = \min(\lambda_K, \lambda^*)$$

and $\bar{s}$ be the smallest nonnegative root of $\bar{\lambda} s + B_c(s) = 0$.

Conclusions

- Decay parameter

We now consider the decay parameter λ_C of stopped $M^X/M/c$ process. For this purpose, we shall determine λ_C in two cases that $B_c'(1) > 0$ and $B_c'(1) \leq 0$ separately.

Theorem 3.1. Let Q be defined in (1.3)-(1.4). If $B_c'(1) > 0$, then

$$\lambda_C = \lambda^* = -\frac{B_c(s_*)}{s_*}.$$

Sketch of proof. We only need to prove $\lambda_C \geq \lambda^*$. Indeed, let $x_j = s_*^j \ (j \geq 1)$, then $(x_j; j \geq 1)$ is a λ^*-subinvariant vector for Q on C.

Conclusions

Next we consider the case $B_c'(1) \leq 0$. We will find that the exact value of the decay parameter λ_C in the case $F_{\bar{\lambda}}(\bar{s}) \geq 0$ is quite different from the case $F_{\bar{\lambda}}(\bar{s}) < 0$.

Lemma 3.1. Suppose that $B_c'(1) \leq 0$. Then for any $\lambda \in [0, \bar{\lambda}]$, if $F_\lambda(s_\lambda) \geq 0$, then $\frac{F_\lambda(s)}{\lambda s + B_c(s)}$ can be expanded as a Taylor series of $s \in [0, s_\lambda)$ and with positive coefficients, where s_λ is the smallest nonnegative root of $\lambda s + B_c(s) = 0$.

Conclusions

Sketch of proof.

Step 1. Note that

$$s^{i-1}-s_\lambda^{i-1}-s^i+s_\lambda^i=(s_\lambda-s)\left[s^{i-1}+(s_\lambda-1)(s_\lambda^{i-2}+s_\lambda^{i-3}s+\cdots+s^{i-2})\right],$$

we have

$$\frac{F_\lambda(s)}{\lambda s+B_c(s)} = F_\lambda(s_\lambda)\cdot\tfrac{1}{\lambda s+B_c(s)}+\tfrac{s_\lambda-s}{\lambda s+B_c(s)}\textstyle\sum_{i=1}^{c-1}m_i(c-i)b_0 \cdot\left[s^{i-1}+(s_\lambda-1)(s_\lambda^{i-2}+s_\lambda^{i-3}s+\cdots+s^{i-2})\right].$$

Step 2. If $B_c'(1)\le 0$, then $s_\lambda\ge 1$, and hence

$$\frac{F_\lambda(s)}{\lambda s+B_c(s)}$$

can be expanded as a Taylor series of $s\in[0,s_\lambda)$ and with positive coefficients.

Conclusions

Theorem 3.2. Suppose that $B_c'(1) \leq 0$. If $F_{\bar{\lambda}}(\bar{s}) \geq 0$, then

$$\lambda_C = \bar{\lambda} = \min(\lambda_K, \lambda^*).$$

Sketch of proof.
(i) $\lambda_C \leq \bar{\lambda}$: Easy!
(ii) $\lambda_C \geq \bar{\lambda}$.
Step 1. By Lemma 2.2, there exists $\{m_j > 0;\ j = 1, 2, \ldots, c-1\}$ such that

$$m_{j+1} = \frac{-\bar{\lambda} m_j - \sum_{i=1}^{j} m_i q_{ij}}{q_{j+1j}}, \quad j = 1, 2, \ldots, c-1. \tag{3.1}$$

Conclusions

Step 2. By Lemma 3.1, $\frac{F_{\bar{\lambda}}(s)}{\bar{\lambda}s+B_c(s)}$ can be expanded as a Taylor series for $s \in [0, \bar{s})$, i.e.,

$$\frac{F_{\bar{\lambda}}(s)}{\bar{\lambda}s + B_c(s)} = \sum_{k=1}^{\infty} u_k s^{k-1}, \quad s \in [0, \bar{s}) \tag{3.2}$$

where the coefficients $u_k > 0$ $(k \geq 1)$.
Step 3. $u_j = m_j$ $(j = 1, 2, \dots, c-1)$ and hence $(u_j; j \geq 1)$ is a $\bar{\lambda}$-invariant measure for Q on C.

Now we consider the case $F_{\bar{\lambda}}(\bar{s}) < 0$. Denote

$$G(s) := F_{-B_c(s)/s}(s),$$

which is a continuous function on $[q_s, \rho_0]$. Since $\bar{s} \in [q_s, \rho_0]$ and $G(\bar{s}) = F_{\bar{\lambda}}(\bar{s}) < 0$, we know that $\{s \in [q_s, \rho_0];\ G(s) < 0\} \neq \emptyset$. Define

$$\hat{s} = \inf\{s \in [q_s, \rho_0] | G(s) < 0\} \tag{3.3}$$

and

$$\hat{\lambda} = -\frac{B_c(\hat{s})}{\hat{s}}. \tag{3.4}$$

Conclusions

Theorem 3.3. Suppose that $B_c'(1) \leq 0$. If $F_{\bar{\lambda}}(\bar{s}) < 0$, then

$$\lambda_C = \hat{\lambda}.$$

Conclusions

Sketch of proof.

Step 1. $\lambda_C \geq \hat{\lambda}$. First prove $G(\hat{s}) = F_{\hat{\lambda}}(\hat{s}) = 0$. Note that $G(\bar{s}) < 0$, we have $\bar{\lambda} > \hat{\lambda}$. Similarly as the proof of Theorem 3.2, we get $\lambda_C \geq \hat{\lambda}$.

Step 2. Note that $-\frac{B_c(s)}{s}$ is increasing and continuous, if $\lambda_C > \hat{\lambda}$, then $s_C > \hat{s}$, where s_C is the smallest nonnegative root of $\lambda_C s + B_c(s) = 0$.
By (3.3), there exists $s_2 \in (\hat{s}, s_C)$ such that $G(s_2) < 0$. Denote $\lambda_2 = -\frac{B_c(s_2)}{s_2}$, we have $\lambda_C > \lambda_2 > \hat{\lambda}$.

Conclusions

Step 3. It can be proved that there exists a λ_2-invariant measure $(u_k;\ k \in C)$ for Q on C, satisfying

$$\sum_{k=1}^{\infty} u_k s^k = \frac{F_{\lambda_2}(s)}{\lambda_2 s + B_c(s)},\ \ s \in [0, s_2).$$

Hence, $\frac{F_{\lambda_2}(s)}{\lambda_2 s+B_c(s)} \geq 0$ for $s \in [0, s_2)$ and therefore,

$$\lim_{s\uparrow s_2} \frac{F_{\lambda_2}(s)}{\lambda_2 s + B_c(s)} \geq 0. \tag{3.5}$$

However, by the continuity of $G(s)$, we have $\lim_{s\uparrow s_2} G(s) = G(s_2) < 0$. Then

$$\lim_{s\uparrow s_2} \frac{F_{\lambda_2}(s)}{\lambda_2 s + B_c(s)} < 0,$$

which is a contradiction with (3.5).

Conclusions

- Summary

Case (i): $B_c'(1) > 0$. Then $\lambda_C = \lambda^*$.

Case (ii): $B_c' \leq 0$ and $F_{\bar{\lambda}}(\bar{s}) \geq 0$. Then $\lambda_C = \min(\lambda_K, \lambda^*)$, where λ_K is the smallest eigenvalue of $-Q_K^*$.

Case (iii): $B_c' \leq 0$ and $F_{\bar{\lambda}}(\bar{s}) < 0$. Then $\lambda_C = \hat{\lambda} < \min(\lambda_K, \lambda^*)$.

Conclusions

- λ_C-invariant measure

Theorem 3.4. There exists a unique (up to constant multiples) λ_C-invariant measure $\{m_i;\ i \geq 1\}$ for Q on C, whose generating function $\sum_{i=1}^{\infty} m_i s^{i-1}$ is given by

$$\sum_{i=1}^{\infty} m_i s^{i-1} = \frac{m_1 b_0 + \sum_{i=1}^{c-1} m_i s^{i-1}[B_c(s) - B_i(s)]}{\lambda_C s + B_c(s)}, \quad s \in [0, s_C)$$

where $\{m_i > 0;\ i = 1, 2, \ldots, c-1\}$ is given in Lemma 2.2.

Conclusions

- An example

Consider $b_0 = a,\ b_2 = 1,\ b_k = 0\ (k \geq 3)$ and $c = 3$.

$$B_3(s) = s^2 - (3a+1)s + 3a, \quad K = \min\{j \geq 3; b_{j-1} > 0\} = 3.$$

Then

$$q_s = \min(1, 3a), \quad \lambda^* = (\sqrt{3a} - 1)^2, \ s_* = \sqrt{3a}.$$

$$\lambda_K = \min\{\lambda \geq 0 : det(\lambda I + Q_K^+) = 0\}$$

where

$$det(\lambda I + Q_K^+) = \lambda^3 - (6a+3)\lambda^2 + (11a^2 + 7a + 3)\lambda - (6a^3 + 2a^2 + a + 1).$$

Denote $\bar{\lambda} = \min(\lambda_K, (\sqrt{3a} - 1)^2)$. The smallest nonnegative root of $\bar{\lambda}s + B_3(s) = 0$ is given by

$$\bar{s} = \frac{3a + 1 - \bar{\lambda} - \sqrt{(3a + 1 - \bar{\lambda})^2 - 12a}}{2}.$$

Conclusions

For $\lambda \in [0, \lambda_K]$, $F_\lambda(s) = a(3-2s) + \frac{1}{2}(a+1-\lambda)s(1-s)$.

By Theorems 3.1-3.3, we have

(i) If $3a < 1$ then $\lambda_C = (\sqrt{3a}-1)^2$.

(ii) If $3a \geq 1$ and $a(3-2\bar{s}) + \frac{1}{2}(a+1-\bar{\lambda})\bar{s}(1-\bar{s}) \geq 0$, then $\lambda_C = \bar{\lambda}$.

(iii) If $3a \geq 1$ and $a(3-2\bar{s}) + \frac{1}{2}(a+1-\bar{\lambda})\bar{s}(1-\bar{s}) < 0$, then $\lambda_C = \hat{\lambda} = -\frac{B_3(\hat{s})}{\hat{s}}$, where $\hat{s}$ is the smallest positive root of $G(s) = F_{-B(s)/s}(s) = 0$, i.e., $s^3 - (1+2a)s^2 + 9as - 9a = 0$.

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