Decay property of stopped Markovian bulk-arriving queues with $c$-servers

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• Definition of Decay parameter

Let $\mathbb{E}$ be a countable set.

$Q = (q_{ij}; i, j \in \mathbb{E})$ be a stable $Q$-matrix.

$(p_{ij}(t); i, j \in \mathbb{E})$ is the Feller minimal $Q$-process.

$C$ is a communicating class of $\mathbb{E}$ and

$$\lim_{t \to \infty} p_{ij}(t) = 0, \quad i, j \in C.$$ 

By Kingman (1963), there exists a number $\lambda_C \geq 0$ such that for all $i, j \in C$,

$$\frac{1}{t} \log p_{ij}(t) \to -\lambda_C \quad \text{as } t \to \infty$$

$\lambda_C$ is called the decay parameter for $C$. 
On the other hand, let

\[ \mu_{ij} = \inf \{ \lambda \geq 0 : \int_{0}^{\infty} e^{\lambda t} p_{ij}(t) dt = \infty \} \]

\[ = \sup \{ \lambda \geq 0 : \int_{0}^{\infty} e^{\lambda t} p_{ij}(t) dt < \infty \}. \]

It is easily seen that \( \mu_{ij} \) does not depend on \( i, j \in C \), the common value is denoted by \( \mu \). Moreover,

\[ \lambda_{C} = \mu. \]

(see, for example, Pollett (2006)).
Background

• Problems:
  ▶ $\lambda_C = ?$;
  ▶ The $\lambda_C$-recurrency of the process.

• Known progress:
  (i) Finite Markov chains.

(ii) BDP (Chen M.F.).
Specially, BDP: $q_{i-1} = a$, $q_{i+1} = b$, then $\lambda_C = (\sqrt{a} - \sqrt{b})^2$.

(iii) MBP: $q_{ij} = ib_{j-i+1}$, then $\lambda_C = -B'(q)$ where

$$B(s) = \sum_{j=0}^{\infty} b_j s^j$$

and $q$ is the smallest nonnegative root of $B(s) = 0$. 

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(iv) Stopped $M^X/M/1$ queue (Li and Chen, 2008):

\[
q_{ij} = \begin{cases} 
  b_{j-i+1}, & \text{if } i \geq 1, j \geq i - 1 \\
  0, & \text{otherwise}
\end{cases}
\]

\[
\lambda_C = \sup\{\lambda \geq 0 : B(s) + \lambda s = 0 \text{ has a root in } (0, +\infty)\}
\]

where $B(s) = \sum_{k=0}^{\infty} b_k s^k$. 
(iv) Stopped $M^X/M/1$ queue (Li and Chen, 2008):

\[ q_{ij} = \begin{cases} 
  b_{j-i+1}, & \text{if } i \geq 1, j \geq i - 1 \\
  0, & \text{otherwise}
\end{cases} \]

\[ \lambda_C = \sup \{ \lambda \geq 0 : B(s) + \lambda s = 0 \text{ has a root in } (0, +\infty) \} \]

where $B(s) = \sum_{k=0}^{\infty} b_k s^k$. 
(v) Controlled $M^X/M/1$ queue (Li and Chen, 2013):

$$q_{ij} = \begin{cases} 
  h_j, & \text{if } i = 0, j \geq 0 \\
  b_{j-i+1}, & \text{if } i \geq 1, j \geq i - 1 \\
  0, & \text{otherwise,}
\end{cases}$$

where

$$\begin{cases} 
  h_j \geq 0 \ (j > 0), \ 0 < \sum_{j=1}^{\infty} h_j = -h_0 < \infty \\
  b_j \geq 0 \ (j \neq 1), \ 0 < \sum_{j\neq 1} b_j = -b_1 < \infty.
\end{cases}$$

$$\lambda_Z = \min\{-\frac{B(s*)}{s*}, -\frac{B(s_h)}{s_h}\}$$
• Motivation

General controlled $M^X/M/1$ queue:

$$q_{ij} = \begin{cases} 
    h_j, & \text{if } i = 0, j \geq 0 \\
    h^{(i)}_{j-i+1}, & \text{if } 1 \leq i < c, j \geq i - 1 \\
    b_{j-i+1}, & \text{if } i \geq c, j \geq i - 1 \\
    0, & \text{otherwise},
\end{cases}$$

where

$$\begin{align*}
    h_j &\geq 0 \ (j > 0), \ 0 < \sum_{j=1}^{\infty} h_j = -h_0 < \infty \\
    h^{(i)}_j &\geq 0 \ (j \neq 1), \ 0 < \sum_{j\neq 1} h^{(i)}_j = -h^{(i)}_1 < \infty, \ 1 \leq i < c \\
    b_j &\geq 0 \ (j \neq 1), \ 0 < \sum_{j\neq 1} b_j = -b_1 < \infty.
\end{align*}$$
This talk is concentrated on the decay parameter of stopped $M^X / M / c$.

Let $Q = (q_{ij}; i, j \in \mathbb{Z}_+) \text{ be defined as follows:}$

\[
q_{ij} = \begin{cases} 
\min(i, c)b_0, & \text{if } i \geq 1, j = i - 1, \\
b_1 - \lfloor\min(i, c) - 1\rfloor b_0, & \text{if } i \geq 1, j = i, \\
b_{j-i+1}, & \text{if } i \geq 1, j \geq i + 1, \\
0, & \text{otherwise,}
\end{cases}
\]

(1.1)

where

\[
b_j \geq 0 \ (j \neq 1), \ 0 < \sum_{j\neq 1} b_j = -b_1 < \infty.
\]

(1.2)
Background

This talk is concentrated on the decay parameter of stopped \( M^X/M/c \).

Let \( Q = (q_{ij}; i, j \in \mathbb{Z}_+) \) be defined as follows:

\[
q_{ij} = \begin{cases} 
\min(i, c)b_0, & \text{if } i \geq 1, j = i - 1, \\
b_1 - [\min(i, c) - 1]b_0, & \text{if } i \geq 1, j = i, \\
b_{j-i+1}, & \text{if } i \geq 1, j \geq i + 1, \\
0, & \text{otherwise},
\end{cases}
\]

(1.3)

where

\[
b_j \geq 0 \ (j \neq 1), \ 0 < \sum_{j \neq 1} b_j = -b_1 < \infty.
\]

(1.4)
Definition 1. Let $Q = (q_{ij}; i, j \in \mathbb{Z}_+)$ be a $Q$-matrix defined in (1.3)–(1.4). The corresponding transition function $P(t) = (p_{ij}(t); i, j \in \mathbb{Z}_+)$ is called a stopped $M^X/M/c$ queueing process.

If $c = 1$ then it is a stopped $M^X/M/1$ queueing model. In the following, we assume that $b_0 > 0$ and $\sum_{j=2}^{\infty} b_j > 0$ (to avoid trivial cases).
Let
\[ K = \min\{ j \geq c; b_{j-c+2} > 0 \} (< \infty), \quad \mathbb{Z}_K^+ = \{1, 2, \cdots, K\} \]
and \( Q_K^+ = (q_{ij}; 1 \leq i, j \leq K) \) be the restriction of \( Q \) on \( \mathbb{Z}_K^+ \). Denote
\[ \lambda_K = \min\{ \lambda \geq 0; \det(\lambda I + Q_K^+) = 0 \}. \]
Actually, \( \lambda_K \) is the decay parameter for \( \mathbb{Z}_K^+ \).
Define

\[ B(s) = \sum_{k=0}^{\infty} b_k s^k \]

and

\[ B_i(s) = B(s) + (i - 1)b_0(1 - s), \quad i = 1, 2, \ldots, c. \]

\[ \rho = \limsup_{n \to \infty} \frac{1}{n} \sqrt[n]{b_n}. \]

Clearly, \( \rho \geq 1 \).

Now, let \( \rho_0 = \sup\{ s > 0 : B_c(s) \leq 0 \} \).
Lemma 2.1. $B_c(s)$ is convex on $[0, \rho)$ and has either one or two positive roots. More specifically,

(i) If either $\rho < +\infty$ and $B_c(\rho) \geq 0$ or $\rho = +\infty$ then $B_c(s) = 0$ has exactly two positive zeros $q_s$ and $q_L$ with $0 < q_s \leq q_L < \rho$.

(ii) If $\rho < +\infty$, $B_c(\rho) < 0$ then $B_c(s) = 0$ has exactly one positive root $q_s = 1$.

Define

$$\lambda^* = \sup\{\lambda \geq 0 : \lambda s + B_c(s) = 0 \text{ has a root in } [q_s, \rho_0]\}. \quad (2.1)$$
Lemma 2.2 Suppose that $B'_c(1) \leq 0$. Then for any $\lambda \in [0, \lambda_K]$, there exist positive $(m_j; j = 1, 2, \ldots, K)$ such that

$$\begin{cases}
\sum_{i=1}^{j+1} m_i q_{ij} = -\lambda m_j, & j = 1, 2, \ldots, K - 1, \\
\sum_{i=1}^{K} m_i q_{iK} \leq -\lambda m_K.
\end{cases} \tag{2.2}$$
For any $\lambda \in [0, \lambda_K]$, let $(m_i; 1 \leq i \leq K)$ be determined in Lemma 2.2 and define

$$F_\lambda(s) = m_1 b_0 + \sum_{i=1}^{c-1} m_1 s^{i-1} [B_c(s) - B_i(s)].$$

$F_\lambda(s)$ is called test function which will play an important role in determining $\lambda_C$.

Also let

$$\bar{\lambda} = \min(\lambda_K, \lambda^*)$$

and $\bar{s}$ be the smallest nonnegative root of $\bar{\lambda} s + B_c(s) = 0$. 
Conclusions

• Decay parameter

We now consider the decay parameter $\lambda_C$ of stopped $M^X/M/c$ process. For this purpose, we shall determine $\lambda_C$ in two cases that $B'_c(1) > 0$ and $B'_c(1) \leq 0$ separately.

**Theorem 3.1.** Let $Q$ be defined in (1.3)-(1.4). If $B'_c(1) > 0$, then

$$\lambda_C = \lambda^* = -\frac{B_c(s_*)}{s_*}.$$  

**Sketch of proof.** We only need to prove $\lambda_C \geq \lambda^*$. Indeed, let $x_j = s_j^* \ (j \geq 1)$, then $(x_j; j \geq 1)$ is a $\lambda^*$-subinvariant vector for $Q$ on $C$. 

Next we consider the case $B'_c(1) \leq 0$. We will find that the exact value of the decay parameter $\lambda_C$ in the case $F_{\lambda}(\bar{s}) \geq 0$ is quite different from the case $F_{\lambda}(\bar{s}) < 0$.

**Lemma 3.1.** Suppose that $B'_c(1) \leq 0$. Then for any $\lambda \in [0, \bar{\lambda}]$, if $F_{\lambda}(s_{\lambda}) \geq 0$, then $\frac{F_{\lambda}(s)}{\lambda s + B_c(s)}$ can be expanded as a Taylor series of $s \in [0, s_{\lambda})$ and with positive coefficients, where $s_{\lambda}$ is the smallest nonnegative root of $\lambda s + B_c(s) = 0$. 
Conclusions

**Sketch of proof.**

**Step 1.** Note that

\[ s^{i-1} - s^{i-1} - s^i + s^i = (s_\lambda - s)[s^{i-1} + (s_\lambda - 1)(s^{i-2} + s^{i-3}s + \cdots + s^{i-2})], \]

we have

\[
\frac{F_\lambda(s)}{\lambda s + B_c(s)} = F_\lambda(s_\lambda) \cdot \frac{1}{\lambda s + B_c(s)} + \frac{s_\lambda - s}{\lambda s + B_c(s)} \sum_{i=1}^{c-1} m_i(c - i) b_0 \\
\cdot [s^{i-1} + (s_\lambda - 1)(s^{i-2} + s^{i-3}s + \cdots + s^{i-2})].
\]

**Step 2.** If \( B'_c(1) \leq 0 \), then \( s_\lambda \geq 1 \), and hence

\[
\frac{F_\lambda(s)}{\lambda s + B_c(s)}
\]

can be expanded as a Taylor series of \( s \in [0, s_\lambda) \) and with positive coefficients.
Theorem 3.2. Suppose that $B'_c(1) \leq 0$. If $F_{\lambda}(\bar{s}) \geq 0$, then

$$\lambda_C = \bar{\lambda} = \min(\lambda_K, \lambda^*)$$

Sketch of proof.
(i) $\lambda_C \leq \bar{\lambda}$: Easy!
(ii) $\lambda_C \geq \bar{\lambda}$.

Step 1. By Lemma 2.2, there exists \( \{m_j > 0; \ j = 1, 2, \ldots, c - 1\} \) such that

$$m_{j+1} = \frac{-\bar{\lambda}m_j - \sum_{i=1}^{j} m_iq_{ij}}{q_{j+1j}}, \quad j = 1, 2, \ldots, c - 1.$$  \hspace{1cm} (3.1)
Step 2. By Lemma 3.1, \( \frac{F_\bar{\lambda}(s)}{\bar{\lambda}s + B_c(s)} \) can be expanded as a Taylor series for \( s \in [0, \bar{s}) \), i.e.,

\[
\frac{F_\bar{\lambda}(s)}{\bar{\lambda}s + B_c(s)} = \sum_{k=1}^{\infty} u_k s^{k-1}, \quad s \in [0, \bar{s})
\] (3.2)

where the coefficients \( u_k > 0 \ (k \geq 1) \).

Step 3. \( u_j = m_j \ (j = 1, 2, \ldots, c - 1) \) and hence \((u_j; j \geq 1)\) is a \( \bar{\lambda} \)-invariant measure for \( Q \) on \( C \).
Now we consider the case $F_{\bar{\lambda}}(\bar{s}) < 0$. Denote

$$G(s) := \frac{F_{-Bc}(s)}{s(s)}$$

which is a continuous function on $[q_s, \rho_0]$. Since $\bar{s} \in [q_s, \rho_0]$ and $G(\bar{s}) = F_{\bar{\lambda}}(\bar{s}) < 0$, we know that $\{s \in [q_s, \rho_0]; \ G(s) < 0\} \neq \emptyset$.

Define

$$\hat{s} = \inf\{s \in [q_s, \rho_0] | G(s) < 0\} \quad (3.3)$$

and

$$\hat{\lambda} = -\frac{B_c(\hat{s})}{\hat{s}}. \quad (3.4)$$
Theorem 3.3. Suppose that $B'_c(1) \leq 0$. If $F_{\hat{\lambda}}(\bar{s}) < 0$, then

$$\lambda_C = \hat{\lambda}.$$
Conclusions

Sketch of proof.

Step 1. $\lambda_C \geq \hat{\lambda}$. First prove $G(\hat{s}) = F\hat{\lambda}(\hat{s}) = 0$. Note that $G(\bar{s}) < 0$, we have $\bar{\lambda} > \hat{\lambda}$. Similarly as the proof of Theorem 3.2, we get $\lambda_C \geq \hat{\lambda}$.

Step 2. Note that $-\frac{B_c(s)}{s}$ is increasing and continuous, if $\lambda_C > \hat{\lambda}$, then $s_C > \hat{s}$, where $s_C$ is the smallest nonnegative root of $\lambda_C s + B_c(s) = 0$.

By (3.3), there exists $s_2 \in (\hat{s}, s_C)$ such that $G(s_2) < 0$. Denote $\lambda_2 = -\frac{B_c(s_2)}{s_2}$, we have $\lambda_C > \lambda_2 > \hat{\lambda}$. 
**Step 3.** It can be proved that there exists a $\lambda_2$-invariant measure $(u_k; \ k \in C)$ for $Q$ on $C$, satisfying

$$
\sum_{k=1}^{\infty} u_k s^k = \frac{F_{\lambda_2}(s)}{\lambda_2 s + B_c(s)}, \ s \in [0, s_2).
$$

Hence, \( \frac{F_{\lambda_2}(s)}{\lambda_2 s + B_c(s)} \geq 0 \) for \( s \in [0, s_2) \) and therefore,

$$
\lim_{s \uparrow s_2} \frac{F_{\lambda_2}(s)}{\lambda_2 s + B_c(s)} \geq 0. \quad (3.5)
$$

However, by the continuity of $G(s)$, we have $\lim_{s \uparrow s_2} G(s) = G(s_2) < 0$. Then

$$
\lim_{s \uparrow s_2} \frac{F_{\lambda_2}(s)}{\lambda_2 s + B_c(s)} < 0,
$$

which is a contradiction with (3.5).
Conclusions

• Summary

Case (i): $B'_c(1) > 0$. Then $\lambda_C = \lambda^*$. 

Case (ii): $B'_c \leq 0$ and $F_{\lambda}(\bar{s}) \geq 0$. Then $\lambda_C = \min(\lambda_K, \lambda^*)$, where $\lambda_K$ is the smallest eigenvalue of $-Q^*_K$. 

Case (iii): $B'_c \leq 0$ and $F_{\lambda}(\bar{s}) < 0$. Then $\lambda_C = \hat{\lambda} < \min(\lambda_K, \lambda^*)$. 

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Theorem 3.4. There exists a unique (up to constant multiples) $\lambda_C$-invariant measure $\{m_i; \ i \geq 1\}$ for $Q$ on $C$, whose generating function $\sum_{i=1}^{\infty} m_i s^{i-1}$ is given by

$$\sum_{i=1}^{\infty} m_i s^{i-1} = \frac{m_1 b_0 + \sum_{i=1}^{c-1} m_i s^{i-1}[B_c(s) - B_i(s)]}{\lambda_C s + B_c(s)}, \quad s \in [0, s_C)$$

where $\{m_i > 0; \ i = 1, 2, \ldots, c - 1\}$ is given in Lemma 2.2.
Conclusions

- An example

Consider \( b_0 = a, \ b_2 = 1, \ b_k = 0 \ (k \geq 3) \) and \( c = 3 \).

\[
B_3(s) = s^2 - (3a + 1)s + 3a, \quad K = \min\{j \geq 3; b_{j-1} > 0\} = 3.
\]

Then

\[
q_s = \min(1, 3a), \quad \lambda^* = (\sqrt{3a} - 1)^2, \quad s^*_* = \sqrt{3a}.
\]

\[
\lambda_K = \min\{\lambda \geq 0 : \det(\lambda I + Q^+_K) = 0\}
\]

where

\[
det(\lambda I + Q^+_K) = \lambda^3 - (6a + 3)\lambda^2 + (11a^2 + 7a + 3)\lambda - (6a^3 + 2a^2 + a + 1).
\]

Denote \( \bar{\lambda} = \min(\lambda_K, (\sqrt{3a} - 1)^2) \). The smallest nonnegative root of \( \bar{\lambda}s + B_3(s) = 0 \) is given by

\[
\bar{s} = \frac{3a + 1 - \bar{\lambda} - \sqrt{(3a + 1 - \bar{\lambda})^2 - 12a}}{2}.
\]
For $\lambda \in [0, \lambda_K]$, $F_\lambda(s) = a(3 - 2s) + \frac{1}{2}(a + 1 - \lambda)s(1 - s)$.

By Theorems 3.1-3.3, we have

(i) If $3a < 1$ then $\lambda_C = (\sqrt{3a} - 1)^2$.

(ii) If $3a \geq 1$ and $a(3 - 2\bar{s}) + \frac{1}{2}(a + 1 - \bar{\lambda})\bar{s}(1 - \bar{s}) \geq 0$, then $\lambda_C = \bar{\lambda}$.

(iii) If $3a \geq 1$ and $a(3 - 2\bar{s}) + \frac{1}{2}(a + 1 - \bar{\lambda})\bar{s}(1 - \bar{s}) < 0$, then $\lambda_C = \hat{\lambda} = -\frac{B_3(\hat{s})}{\hat{s}}$, where $\hat{s}$ is the smallest positive root of $G(s) = F_{-B(s)}/s(s) = 0$, i.e., $s^3 - (1 + 2a)s^2 + 9as - 9a = 0.$


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Thank you!