Level statistics of eigenvalues for 1D random Schrödinger operators

S. Kotani
KwanseiGakuin University and Osaka University
(joint work with F. Nakano of Gakushuin University)

12th Workshop on Markov Processes and Related Topics
July 17, 2016
Jiangsu Normal University, Xuzhou, China

Contents

- **Object**: Schrödinger operators with random potentials

$$H_\omega = -\Delta + V_\omega(x), \quad x \in \mathbb{R}^d \text{ or } \mathbb{Z}^d$$

Contents

- **Object**: Schrödinger operators with random potentials

$$H_{\omega} = -\Delta + V_{\omega}(x), \quad x \in \mathbb{R}^d \text{ or } \mathbb{Z}^d$$

- **Points of interest**: Localization and delocalization

Contents

- **Object**: Schrödinger operators with random potentials

$$H_\omega = -\Delta + V_\omega(x), \quad x \in \mathbb{R}^d \text{ or } \mathbb{Z}^d$$

- **Points of interest**: Localization and delocalization
- **Two mathematical aspects to study**:

Contents

- **Object**: Schrödinger operators with random potentials

$$H_\omega = -\Delta + V_\omega(x), \quad x \in \mathbb{R}^d \text{ or } \mathbb{Z}^d$$

- **Points of interest**: Localization and delocalization
- **Two mathematical aspects to study**:

1. **Spectral properties**: Point spectrum or Continuous spectrum

Contents

- **Object**: Schrödinger operators with random potentials

$$H_\omega = -\Delta + V_\omega(x), \quad x \in \mathbb{R}^d \text{ or } \mathbb{Z}^d$$

- **Points of interest**: Localization and delocalization
- **Two mathematical aspects to study**:

1. **Spectral properties**: Point spectrum or Continuous spectrum
2. **Local structures of spectrum**: Poisson distribution or other dependent distribution

Model

- Schrödinger operators with random decaying potentials

$$H_\omega = -\frac{d^2}{dx^2} + a(x)F\left(X_x\left(\omega\right)\right),$$

where $\{X_x\}_{x\geq 0}$ is a B.M. on $\mathbb{T}^d$ and F is a smooth function on $\mathbb{T}^d$ satisfying

$$\int_{\mathbb{T}^d} F\left(x\right)dx = 0, \quad F \neq 0.$$

Model

- Schrödinger operators with random decaying potentials

$$H_\omega = -\frac{d^2}{dx^2} + a(x)F\left(X_x\left(\omega\right)\right),$$

where $\{X_x\}_{x\geq 0}$ is a B.M. on $\mathbb{T}^d$ and F is a smooth function on $\mathbb{T}^d$ satisfying

$$\int_{\mathbb{T}^d} F\left(x\right)dx = 0, \quad F \neq 0.$$

- $a(x)$ decays with order

$$a(x) \sim x^{-\alpha} \ \text{ as } \ x \to \infty \ \ \exists\alpha \geq 0.$$

Random matrix v.s. Random Schrödinger operator

- Discrete random Schrödinger operators $\Delta_d + V_n$

$$\begin{pmatrix} V_1 & 1 & 0 & 0 & \cdots & \cdots \\ 1 & V_2 & 1 & 0 & \cdots & \cdots \\ 0 & 1 & V_3 & 1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 1 & V_n & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

Random matrix v.s. Random Schrödinger operator

- Discrete random Schrödinger operators $\Delta_d + V_n$

$$\begin{pmatrix} V_1 & 1 & 0 & 0 & \cdots & \cdots \\ 1 & V_2 & 1 & 0 & \cdots & \cdots \\ 0 & 1 & V_3 & 1 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 1 & V_n & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

- Random matrix

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & \cdots \\ X_{12} & X_{22} & X_{23} & X_{24} & \cdots \\ X_{13} & X_{23} & X_{33} & X_{34} & \cdots \\ X_{14} & X_{24} & X_{34} & X_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Spectral properties

- The spectral properties for H_ω (K-Ushiroya 1988)
  On $[0,\infty)$ we have

$\alpha > 1/2$	a.c. spec.
$\alpha = 1/2$	$\exists E_c > 0$ s.t. $\begin{cases} \text{point spec. on } (0, E_c) \\ \text{s.c. spec. on } (E_c, \infty) \end{cases}$
$0 < \alpha < 1/2$	point spec.

Spectral properties

- The spectral properties for H_ω (K-Ushiroya 1988)
  On $[0, \infty)$ we have

$\alpha > 1/2$	a.c. spec.
$\alpha = 1/2$	$\exists E_c > 0$ s.t. $\begin{cases} \text{point spec. on } (0, E_c) \\ \text{s.c. spec. on } (E_c, \infty) \end{cases}$
$0 < \alpha < 1/2$	point spec.

- If $\alpha = 0$, Molchanov-Goldsheid-Pastur (1977): point spec.

Eigenvalues distribution

- Let $\{E_j(L)\}_{j\geq 1}$ be positive eigenvalues for the operator H_ω restricted to $[0, L]$ with Dirichlet boundary condition.

Eigenvalues distribution

- Let $\{E_j(L)\}_{j\geq 1}$ be positive eigenvalues for the operator H_ω restricted to $[0, L]$ with Dirichlet boundary condition.
- Fix a reference energy $E_0 > 0$ and define

$$\xi_L = \sum_{j\geq 1} \delta_{L\left(\sqrt{E_j(L)} - \sqrt{E_0}\right)}.$$

Eigenvalues distribution

- Let $\{E_j(L)\}_{j\geq 1}$ be positive eigenvalues for the operator H_ω restricted to $[0, L]$ with Dirichlet boundary condition.
- Fix a reference energy $E_0 > 0$ and define

$$\xi_L = \sum_{j\geq 1} \delta_{L\left(\sqrt{E_j(L)}-\sqrt{E_0}\right)}.$$

- If $H_\omega = -\frac{d^2}{dx^2}$, namely $a(x) = 0$, then

$$\sqrt{E_j(L)} = \frac{\pi j}{L} \Longrightarrow \xi_L = \sum_{j\geq 1} \delta_{\left(\pi j - L\sqrt{E_0}\right)}$$

Classical results on limit of eigenvalues distribution

- **Molchanov 1981** In $\mathbb{R}^1$ if $\alpha = 0$

$$H_\omega = -\frac{d^2}{dx^2} + F\left(X_x\left(\omega\right)\right)$$

then

$$\xi_L \underset{L \to \infty}{\to} \text{Poisson}\left(n\left(E_0\right) d\lambda\right) \quad (n(E) \text{ is the density of states})$$

Classical results on limit of eigenvalues distribution

- **Molchanov 1981** In $\mathbb{R}^1$ if $\alpha = 0$

$$H_\omega = -\frac{d^2}{dx^2} + F\left(X_x\left(\omega\right)\right)$$

then

$$\xi_L \underset{L\to\infty}{\to} \text{Poisson}\left(n\left(E_0\right)d\lambda\right) \quad (n(E) \text{ is the density of states})$$

- **Minami 1996** In $\mathbb{Z}^d$

$$H_\omega = \Delta_d + V_\omega\left(x\right),$$

$$\left\{V_\omega\left(x\right)\right\}_{x\in\mathbb{Z}^d} \quad \text{i.i.d. with smooth prob. density}$$

If E_0 is in the point spectral region, then

$$\xi_{\Lambda_L} \underset{L\to\infty}{\to} \text{Poisson}\left(n\left(E_0\right)d\lambda\right).$$

Main results

- If $\alpha > \dfrac{1}{2}$ (super critical) (K-Nakano 2014)

$$\xi_L \underset{L\to\infty}{\longrightarrow} \text{clock process}$$

Main results

- If $\alpha > \frac{1}{2}$ (super critical) (K-Nakano 2014)

$$\xi_L \underset{L\to\infty}{\longrightarrow} \text{clock process}$$

- If $\alpha = \frac{1}{2}$ (critical) (K-Nakano 2014)

$$\xi_L \underset{L\to\infty}{\longrightarrow} \text{Sine}_\beta-\text{process}$$

Main results

- If $\alpha > \frac{1}{2}$ (super critical) (K-Nakano 2014)

$$\xi_L \underset{L\to\infty}{\longrightarrow} \text{clock process}$$

- If $\alpha = \frac{1}{2}$ (critical) (K-Nakano 2014)

$$\xi_L \underset{L\to\infty}{\longrightarrow} \text{Sine}_\beta - \text{process}$$

- If $0 < \alpha < \frac{1}{2}$ (subcritical) (K-Nakano 2016)

$$\xi_L \underset{L\to\infty}{\longrightarrow} \text{Poisson}\left(\frac{d\lambda}{\pi}\right)$$

Supercritical case

- Clock process (Kotani-Nakano 2014)
  Assume the subsequence $\{L_j\}_{j\geq 1}$ satisfies

$$\sqrt{E_0}L_j = m_j\pi + \gamma + o(1) \quad \text{as} \quad j \to \infty$$

for some $m_j \in \mathbb{N}$ satisfying $m_j \to \infty$ and $\gamma \in [0, \pi)$. Then we have

$$\xi_\infty \underset{d}{=} \lim_{j\to\infty} \xi_{L_j}.$$

ξ_∞ is

$$\xi_\infty = \sum_{n\in\mathbb{Z}} \delta_{\theta_\gamma + n\pi},$$

where θ_γ is a random variable taking value in $[0, \pi]$.

Critical case

- For a $\mathbb{C}-$B.M $\{Z_t\}_{t\geq 0}$ let $\alpha_t^{\beta}(\lambda)$ be the solution to

$$d\alpha_t^{\beta}(\lambda) = \lambda e^{-t}dt + \frac{2}{\sqrt{\beta}}\mathrm{Re}\left\{\left(e^{i\alpha_t^{\beta}(\lambda)} - 1\right)dZ_t\right\}, \ \alpha_0^{\beta}(\lambda) = 0.$$

Critical case

- For a $\mathbb{C}-$B.M $\{Z_t\}_{t\geq 0}$ let $\alpha_t^\beta(\lambda)$ be the solution to

$$d\alpha_t^\beta(\lambda) = \lambda e^{-t}dt + \frac{2}{\sqrt{\beta}}\mathrm{Re}\left\{\left(e^{i\alpha_t^\beta(\lambda)} - 1\right)dZ_t\right\},\ \alpha_0^\beta(\lambda) = 0.$$

- $\alpha_t^\beta(\lambda)$ is non-decreasing in λ, and the limit

$$\exists\alpha_\infty^\beta(\lambda) = \lim_{t\to\infty}\alpha_t^\beta(\lambda) \in 2\pi\mathbb{Z}$$

exists a.s. and *Sine*$_\beta$-process ζ_β is defined by

$$\zeta_\beta([\lambda_1,\lambda_2]) = \frac{\alpha_\infty^\beta(\lambda_2) - \alpha_\infty^\beta(\lambda_1)}{2\pi} \quad \text{(Varko-Virag 2009).}$$

Critical case

- For a $\mathbb{C}-$B.M $\{Z_t\}_{t\geq 0}$ let $\alpha_t^{\beta}(\lambda)$ be the solution to
$$d\alpha_t^{\beta}(\lambda) = \lambda e^{-t}dt + \frac{2}{\sqrt{\beta}}\operatorname{Re}\left\{\left(e^{i\alpha_t^{\beta}(\lambda)} - 1\right)dZ_t\right\},\ \alpha_0^{\beta}(\lambda) = 0.$$
- $\alpha_t^{\beta}(\lambda)$ is non-decreasing in λ, and the limit
$$\exists\alpha_{\infty}^{\beta}(\lambda) = \lim_{t\to\infty}\alpha_t^{\beta}(\lambda) \in 2\pi\mathbb{Z}$$
exists a.s. and *Sine*$_{\beta}$-process ζ_{β} is defined by
$$\zeta_{\beta}([\lambda_1,\lambda_2]) = \frac{\alpha_{\infty}^{\beta}(\lambda_2) - \alpha_{\infty}^{\beta}(\lambda_1)}{2\pi} \quad \text{(Varko-Virag 2009).}$$
- The limit process $\xi_{\infty} \underset{d}{=} \lim_{L\to\infty}\xi_L$ exists and $\xi_{\infty} = \zeta_{\beta}$ with
$$\beta = \beta(E_0) = -4E_0\left(\operatorname{Re}\left(\left(\frac{1}{2}\Delta + 2i\sqrt{E_0}\right)^{-1}F, F\right)\right)^{-1} > 0.$$

Ideas of the proof

- For a solution φ to the eigen-equation $H\varphi = \kappa^2\varphi$ set

$$\theta = \arg\left(\frac{\varphi'}{\kappa} + i\varphi\right) \quad \text{(The Prüfer variable).}$$

Ideas of the proof

- For a solution φ to the eigen-equation $H\varphi = \kappa^2\varphi$ set

$$\theta = \arg\left(\frac{\varphi'}{\kappa} + i\varphi\right) \quad \text{(The Prüfer variable).}$$

- Then θ satisfies

$$\theta_t'(\kappa) = \kappa + \frac{1}{2\kappa}a(t)F(X_t)\,\mathrm{Re}\left(e^{2i\theta_t(\kappa)} - 1\right), \quad \theta_0(\kappa) = 0.$$

(replaced x by t.) $\theta_t(\kappa)$ is increasing in κ for fixed $t > 0$.

Ideas of the proof

- For a solution φ to the eigen-equation $H\varphi = \kappa^2 \varphi$ set

$$\theta = \arg\left(\frac{\varphi'}{\kappa} + i\varphi\right) \quad \text{(The Prüfer variable).}$$

- Then θ satisfies

$$\theta_t'(\kappa) = \kappa + \frac{1}{2\kappa} a(t) F(X_t) \operatorname{Re}\left(e^{2i\theta_t(\kappa)} - 1\right), \quad \theta_0(\kappa) = 0.$$

(replaced x by t.) $\theta_t(\kappa)$ is increasing in κ for fixed $t > 0$.

- Define

$$\Theta_t(\lambda) = \theta_t\left(\sqrt{E_0} + \frac{\lambda}{t}\right) - \theta_t\left(\sqrt{E_0}\right), \quad \phi_t = \pi\left\{\frac{\theta_t\left(\sqrt{E_0}\right)}{\pi}\right\},$$

where $\{x\} \in [0,1)$ denotes the fractional part of x. Then

$$\xi_L(f) = \sum_{k\in\mathbb{Z}} f\left(\Theta_L^{-1}(k\pi - \phi_L)\right).$$

Ideas of the proof

- The key equation

$$\theta_t'(\kappa) = \kappa + \frac{1}{2\kappa} a(t) F(X_t) \operatorname{Re}\left(e^{2i\theta_t(\kappa)} - 1\right), \quad \theta_0(\kappa) = 0$$

is equivalent to

$$\theta_t(\kappa) = \kappa t + \frac{1}{2\kappa}\int_0^t a(s) F(X_s) \operatorname{Re}\left(e^{2i\theta_s(\kappa)} - 1\right) ds.$$

Ideas of the proof

- The key equation

$$\theta_t'(\kappa) = \kappa + \frac{1}{2\kappa} a(t) F(X_t) \operatorname{Re}\left(e^{2i\theta_t(\kappa)} - 1\right), \quad \theta_0(\kappa) = 0$$

  is equivalent to

$$\theta_t(\kappa) = \kappa t + \frac{1}{2\kappa} \int_0^t a(s) F(X_s) \operatorname{Re}\left(e^{2i\theta_s(\kappa)} - 1\right) ds.$$

- A prototype for this integral is

$$\int_0^t \frac{\sin s}{s^\alpha} ds,$$

  and **Integration by parts** gives

$$\int_0^t \frac{\sin s}{s^\alpha} ds = \frac{1 - \cos t}{t^\alpha} - \alpha \int_0^t \frac{1 - \cos s}{s^{\alpha+1}} ds.$$

Subcritical case

- We can show for $\alpha > 0$ $\left(a\left(x\right) = x^{-\alpha} + o\left(x^{-\alpha}\right)\right)$

$$d\Theta_{nt}\left(\lambda\right) \doteqdot \lambda dt + n^{\frac{1}{2}-\alpha}\,\mathrm{Re}\left[\left(e^{2i\Theta_{nt}\left(\lambda\right)} - 1\right) t^{-\alpha} dZ_t\right]$$

Subcritical case

- We can show for $\alpha > 0$ $\left(a(x) = x^{-\alpha} + o\left(x^{-\alpha}\right)\right)$

$$d\Theta_{nt}(\lambda) \doteqdot \lambda dt + n^{\frac{1}{2}-\alpha} \operatorname{Re}\left[\left(e^{2i\Theta_{nt}(\lambda)} - 1\right) t^{-\alpha} dZ_t\right]$$

- If $0 < \alpha < 1/2$, Time change $t = s^{\gamma}$ $\left(\gamma = \frac{1}{1-2\alpha} > 1\right)$

$$d\Theta_{ns^{\gamma}}(\lambda) \doteqdot \lambda \gamma s^{\gamma-1} ds + n^{\frac{1}{2}-\alpha} \operatorname{Re}\left[\left(e^{2i\Theta_{ns^{\gamma}}(\lambda)} - 1\right) d\widetilde{Z}_s\right]$$

Subcritical case

- We can show for $\alpha > 0$ $\left(a(x) = x^{-\alpha} + o\left(x^{-\alpha}\right)\right)$

$$d\Theta_{nt}(\lambda) \doteqdot \lambda dt + n^{\frac{1}{2}-\alpha} \operatorname{Re}\left[\left(e^{2i\Theta_{nt}(\lambda)} - 1\right) t^{-\alpha} dZ_t\right]$$

- If $0 < \alpha < 1/2$, Time change $t = s^{\gamma}$ $\left(\gamma = \dfrac{1}{1-2\alpha} > 1\right)$

$$d\Theta_{ns^{\gamma}}(\lambda) \doteqdot \lambda \gamma s^{\gamma-1} ds + n^{\frac{1}{2}-\alpha} \operatorname{Re}\left[\left(e^{2i\Theta_{ns^{\gamma}}(\lambda)} - 1\right) d\widetilde{Z}_s\right]$$

- Allez-Dumaz (2014) showed

$$\lim_{\beta\to 0} \textit{Sine}_{\beta}\text{-process } \alpha_{\infty}^{\beta}(\lambda) = \text{Poisson}\left(\frac{d\lambda}{2\pi}\right).$$

Recall $\alpha_{\infty}^{\beta}(\lambda) = \lim_{t\to\infty} \alpha_t^{\beta}(\lambda)$ and $\alpha_t^{\beta}(\lambda)$ is defined by

$$d\alpha_t^{\beta}(\lambda) = \lambda e^{-t} dt + \frac{2}{\sqrt{\beta}} \operatorname{Re}\left\{\left(e^{i\alpha_t^{\beta}(\lambda)} - 1\right) dZ_t\right\}.$$

Observation

- For simplicity fix λ. Then

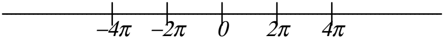
$$
\begin{aligned}
d\alpha_t^{\beta}(\lambda) &= \lambda e^{-t}dt + \frac{2}{\sqrt{\beta}}\operatorname{Re}\left\{\left(e^{i\alpha_t^{\beta}(\lambda)} - 1\right)dZ_t\right\} \\
&\Updownarrow \\
dX_t &= \lambda e^{-t}dt + \frac{2\sqrt{2}}{\sqrt{\beta}}\sin\frac{X_t}{2}dB_t.
\end{aligned}
$$

Observation

- For simplicity fix λ. Then

$$
\begin{aligned}
d\alpha_t^{\beta}(\lambda) &= \lambda e^{-t}dt + \frac{2}{\sqrt{\beta}}\mathrm{Re}\left\{\left(e^{i\alpha_t^{\beta}(\lambda)}-1\right)dZ_t\right\} \\
&\Updownarrow \\
dX_t &= \lambda e^{-t}dt + \frac{2\sqrt{2}}{\sqrt{\beta}}\sin\frac{X_t}{2}dB_t.
\end{aligned}
$$

- $\{X_t\}_{t\geq 0}$ moves on each interval $(2n\pi, 2(n+1)\pi)$ below randomly and it never reaches the left edge $2n\pi$ and once reaches the right edge $2(n+1)\pi$, then it never returns to $(2n\pi, 2(n+1)\pi)$.



Subcritical case

One can apply the argument by Allez-Dumaz to our $\Theta_{ns^\gamma}(\lambda)$ and obtain

Theorem

(Kotani-Nakano) $\{\Theta_{nt}(\lambda), \phi_{nt}\}$ *converges to* $\left\{\widehat{\Theta}_t(\lambda), \widehat{\phi}_t\right\}$, *the two processes are independent and*

(1) $\widehat{\phi}_t$ *has the uniform distribution on* $[0, \pi)$ *for each* $t > 0$.
(2)

$$\widehat{\Theta}_t(\lambda) = \pi \int_{[0,t]\times[0,\lambda]} \Pi(dsd\xi)$$

where $\Pi(dsd\xi)$ *is the Poisson random measure on* $\mathbb{R}^2$ *whose intensity measure is* $dsd\xi$.

Related results

(1) **CMV matrices** Killip-Stoiciu 2009

$$\Xi_k = \begin{pmatrix} \overline{\alpha}_k & \sqrt{1-|\alpha_k|^2} \\ \sqrt{1-|\alpha_k|^2} & -\alpha_k \end{pmatrix} \quad \text{with} \quad |\alpha_k| < 1$$

$$\mathcal{L} = diag\left(\Xi_0, \Xi_2, \Xi_4, \cdots\right), \ \mathcal{M} = diag\left(1, \Xi_1, \Xi_3, \cdots\right)$$

$$\text{CMV matrix } \mathcal{C} = \mathcal{C}\left(\alpha_0, \alpha_1, \alpha_2, \cdots\right) := \mathcal{L}\mathcal{M}$$

CMV matrix is unitary. Assume $\mathbb{E}\left(|\alpha_k|^2\right) = ck^{-\alpha} + o\left(k^{-\alpha}\right)$. Then

$\alpha > \frac{1}{2}$	Clock process
$\alpha = \frac{1}{2}$	Sine$_\beta$-process
$0 < \alpha < \frac{1}{2}$	Poisson point process

Random matrices

- (2) **Discrete Schrödinger** Krichevski-Valko-Virag 2012

$$\text{If } \alpha = \frac{1}{2}, \text{ then the limit is } \mathrm{Sine}_\beta\text{-process}$$

Random matrices

- (2) **Discrete Schrödinger** Krichevski-Valko-Virag 2012

$$\text{If } \alpha = \frac{1}{2}, \text{ then the limit is Sine}_\beta\text{-process}$$

- (3) **Random matrices** Valko-Virag 2009

$$\beta\text{-ensembles } \Lambda_n^\beta: \quad Z^{-1} e^{-\beta \sum_{k=1}^n \lambda_k^2/4} \prod_{j<k} \left|\lambda_j - \lambda_k\right|^\beta$$

Let $\{\mu_n\}_{n\geq 1}$ be a sequence s.t. $n^{1/6}\left(2\sqrt{n} - |\mu_n|\right) \to +\infty$.
Then

$$\sqrt{4n - \mu_n^2}\left(\Lambda_n^\beta - \mu_n\right) \to \text{Sine}_\beta\text{-process}$$

Random matrices

- (2) **Discrete Schrödinger** Krichevski-Valko-Virag 2012

$$\text{If } \alpha = \frac{1}{2}, \text{ then the limit is } \text{Sine}_\beta\text{-process}$$

- (3) **Random matrices** Valko-Virag 2009

$$\beta\text{-ensembles } \Lambda_n^\beta: \quad Z^{-1} e^{-\beta \sum_{k=1}^n \lambda_k^2/4} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

Let $\{\mu_n\}_{n\geq 1}$ be a sequence s.t. $n^{1/6}\left(2\sqrt{n} - |\mu_n|\right) \to +\infty$. Then

$$\sqrt{4n - \mu_n^2}\left(\Lambda_n^\beta - \mu_n\right) \to \text{Sine}_\beta\text{-process}$$

- Remark: In case $\alpha = \frac{1}{2}$ and $E_0 = E_c$ we have $\beta = 2$, and $Sine_\beta - process$ arises from Gaussian unitary ensemble.

Thank you for your attention