# Classical and Non-commutative Martingale Inequalities

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# Outline

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# Notations and definitions

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, and $\{\mathcal{F}_n\}_{n\geq 1}$ be a nondecreasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$.
An adapted sequence $f = (f_n)$ is called a martingale if for any $n \geq 1$, $f_n \in L^1(\Omega, \mathcal{F}_n, P)$ and

$$\mathbb{E}_n(f_{n+1}) = f_n.$$

Martingale difference: $d_n f = f_n - f_{n-1}$, $n \geq 1$ (with the convention that $f_0 = 0$)

# Notations and definitions

the Doob maximal function:

$$M_n(f) = \sup_{1\le m\le n} |f_m|, \qquad M(f) = \sup_{n\ge 1} |f_n|,$$

the square function:

$$S_n(f) = \Big( \sum_{1\le m\le n} \big|d_m f\big|^2 \Big)^{\frac{1}{2}}, \qquad S(f) = \Big( \sum_{n\ge 1} |d_n f|^2 \Big)^{\frac{1}{2}}$$

# Classical martingale inequalities

Theorem (Doob, [Stochastic Process, 1953 ])For $1 < p \leq \infty$,

$$\|M(f)\|_{L^p} \leq \frac{p}{p-1} \sup_n \|f\|_{L^p}.$$

Remark. This is not true for $p = 1$ and weak (1,1) type inequality holds.

# BGD inequalities

Theorem (Burkholder-Gundy, [Acta Math, 1970 ]) For $1 < p < \infty$,

$$\|M(f)\|_{L^p} \approx \|S(f)\|_{L^p} \approx \sup_n \|f_n\|_{L^p}.$$

Theorem (Davis, [Ann. Probab., 1971 ]) For $p = 1$,

$$\|M(f)\|_{L^1} \approx \|S(f)\|_{L^1}$$

These are the most important results in martingale theory. Since then, Doob, Merry, Burkholder, Bourgain, Garsir, Pisier..., Long, Liu....

# Vector-valued extensions: the martingale $f = (f_n)$ with value in Banach space X

Theorem (Pisier, [Isreal J. Math., 1983 ]) Let $2 \leq q < \infty$. Then a Banach space $X$ has an equivalent $q$-uniformly convex norm iff for every $1 < p < \infty$ (or equivalently, for some $1 < p < \infty$) there exists a positive constant $c$ such that

$$\Big\| \Big( \sum_{n \geq 1} \|f_n - f_{n-1}\|_X^q \Big)^{1/q} \Big\|_p \leq c \sup_{n \geq 1} \|f_n\|_{L_p(X)}$$

for all finite $L_p$-martingales $f$ with values in $X$. Again, the validity of the converse inequality amounts to saying that $X$ has an equivalent $q$-uniformly smooth norm $(1 < q \leq 2)$.

J.M.A.M. van Neerven and L. Weis. Stochastic integration of functions with values in a Banach space. Studia Math, 166, 2005.
J.M.A.M. van Neerven, M.C. V eraar, and L. Weis. Stochastic integration in UMD Banach spaces. Ann Probab., 35, 2007.
Lutz Weis, Stochastic integration in Banach spaces - a survey, arXiv:1304.7575, 2014

Natural question: What happens for the endpoint case $p = \infty$?

# Probability version of Carleson measure

Definition (Jiao, [Probab. Theore. Relat. Feild, 2009]) Let $\mu = dP \otimes dm$ be a nonnegative measure on $\Omega \times \mathbb{N}$, where $\mathbb{N}$ is equipped with the counting measure $dm$. $\mu$ is called a Carleson measure if

$$\|\mu\|_C =: \sup \frac{\mu(\widehat{\tau})}{P(\tau < \infty)} < \infty,$$

where the supremum runs over all stopping times $\tau$ and where $\widehat{\tau}$ denotes the "tent" over $\tau$:

$$\widehat{\tau} = \{(w, k) \in \Omega \times \mathbb{N} : \tau(w) \leq k, \tau(w) < \infty\}.$$

# Positive answer to the endpoint case $p = \infty$

Theorem (Jiao, [Probab. Theore. Relat. Feild, 2009]) Let $X$ be a Banach space and $2 \le q < \infty$. Then the following statements are equivalent:

(1) There exists a positive constant $c$ such that for any finite $X$-valued martingale

$$\sup_{\tau} \frac{1}{P(\tau < \infty)} \int_{\widehat{\tau}} \|df_k\|^q dP \otimes dm \le c^q \|f\|_{BMO}^q .$$

(2) $X$ has an equivalent norm which is $q-$uniformly convex.

Remark. The statement (1) means that $\|df_k\|^q dP \otimes dm$ is a Carleson measure on $\Omega \times \mathbb{N}$ for every $f \in BMO(X)$.

Remark. Pisier, Martingales in Banach spaces, Cambridge Studies in Advanced Mathematics, 2016 .

# Noncommutative Martingale inequalities

Let $(\mathcal{M}, \tau)$ be a noncommutative probability space, i.e. $\tau(1) = 1$.

Example 1. $\mathcal{M} = L_\infty(\Omega, P)$, $\tau = \int_\Omega$; $\quad \tau(1) = P(\Omega) = 1$

Example 2. $\mathcal{M} = \mathbb{M}_n(\mathbb{C})$, $\tau = \frac{1}{n} Tr$

Let $(\mathcal{M}_n)_{n\geq 1}$ be a nondecreasing sequence of von Neumann subalgebras of $\mathcal{M}$. A measurable sequence $x = (x_n)$ is called a noncommutative martingale if for any $n \geq 1$, $x_n \in L^1(\mathcal{M}_n, \tau)$ and

$$\mathcal{E}_n(x_{n+1}) = x_n.$$

Example. Matrix valued martingales

# Main difficulties

- How to define the Doob maximal operator: $\sup_n |f_n|$ ?
- How to define the square function?

$$\Big\|\Big(\sum_n |x_n|^2\Big)^{1/2}\Big\|_p \approx \Big\|\Big(\sum_n |x_n^*|^2\Big)^{1/2}\Big\|_p \quad ?$$

Answer: No!

Example. Let $(\mathcal{M}, \tau) = (M_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr})$. Set $x_k = e_{k,0}$, $0 \leq k < n$. It is immediate that

$$\Big\|\Big(\sum_{k=0}^{n-1} |x_k|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathcal{M})} = n^{1/2-1/p}, \quad \Big\|\Big(\sum_{k=0}^{n-1} |x_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{L_p(\mathcal{M})} = 1.$$

- $|x+y| \leq |x| + |y|$ ? No!
- The stopping time is not available...
- ...

# Noncommutative Burkholder-Gundy inequalities

Theorem (Pisier-Xu, 1997, Commun. Math. Phys.)
For $2 \le p < \infty$,

$$\|x\|_{L^p(\mathcal{M})} \approx \max\left\{\left\|S_c(x)\right\|_{L^p(\mathcal{M})}, \left\|S_r(x)\right\|_{L^p(\mathcal{M})}\right\}$$

For $1 < p < 2$,

$$\|x\|_{L^p(\mathcal{M})} \approx \inf_{x=y+z}\left\{\left\|S_c(y)\right\|_{L^p(\mathcal{M})} + \left\|S_r(z)\right\|_{L^p(\mathcal{M})}\right\},$$

where

$$S_c(x) = \Big(\sum_n |dx_n|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_n |dx_n^*|^2\Big)^{1/2}.$$

# Noncommutative extensions

Junge (Doob's maximal inequality ) [2002; J.Reine Angew. Math.]
Randrianantoanina (Square function for noncommutative martingale)[2007, Ann. Prob.]
Junge-Xu (Noncommutative Burkholder/Rosenthal inequalities)[2003, Ann. Prob.; 2008, Isreal J. Math.]
Parcet-Randrianantoanina (Gundy's decomposition) [2006, Proc. Lond.Math. Sco.]
Junge-Xu (Noncommutative maximal ergodic theorems )[2007, J. Amer. Math. Sco.]
Randrianantoanina (Noncommutative martingale transforms)[2009, J.Funct.Anal.]

Remark. p=1, the Davis inequality fails.

# Noncommutative extensions: the $\Phi$-moment case

Notation $\Phi : [0,\infty) \to [0,\infty)$, increasing, convex, continuous
$p_\Phi, q_\Phi$: Boyd index of $\Phi$
Example. $\Phi(t) = t^p, 1 \le p < \infty$ , $p_\Phi = q_\Phi = p$.

Theorem (Bekjan-Chen, 2012, Probab. Theore. Relat. Feild)

*For $2 < p_\Phi \le q_\Phi < \infty$ and any finite noncommutative martingale $x$,*

$$\tau(\Phi(|x|)) \approx_\Phi \max\Big\{\tau(\Phi[(\sum_{k\ge 0}|dx_k|^2)^{1/2}]), \tau(\Phi[(\sum_{k\ge 0}|dx_k^*|^2)^{1/2}])\Big\};$$

*For $1 < p_\Phi \le q_\Phi < 2$,*

$$\tau(\Phi(|x|)) \approx_\Phi \inf_{dx_k=y_k+z_k}\Big\{\tau(\Phi[\Big(\sum_{k\ge 0}|y_k|^2)^{1/2}])+\tau(\Phi[(\sum_{k\ge 0}|z_k^*|^2)^{1/2}])\Big\}.$$

# Noncommutative extensions: symmetric operator spaces

Symmetric operator space: $E(\mathcal{M})$...
Boyd index of $E$: $p_E, q_E$
Example: $E(\mathcal{M}) = L_p(\mathcal{M})$, $p_E = q_E = p$

Theorem (Dirkson, 2015, Transactions Amer. Math. Sco.)

For $2 < p_E \leq q_E < \infty$ and any finite noncommutative $E(\mathcal{M})$-martingale $x$,

$$\|x\|_{E(\mathcal{M})} \approx \max\left\{\left\|S_c(x)\right\|_{E(\mathcal{M})}, \left\|S_r(x)\right\|_{E(\mathcal{M})}\right\};$$

For $1 < p_E \leq q_E < 2$ and any finite noncommutative $E(\mathcal{M})$-martingale $x$,

$$\|x\|_{E(\mathcal{M})} \approx \inf_{x=y+z}\left\{\left\|S_c(y)\right\|_{E(\mathcal{M})} + \left\|S_r(z)\right\|_{E(\mathcal{M})}\right\};$$

# The sharp case

Notation: $p$-convex and $q$-concave

**Remark 2:** Let $\Phi$ be $p$-convex and $q$-concave, then

$$p \leq p_\Phi \leq q_\Phi \leq q.$$

Theorem (Jiao, Sukochev, Xie and Zanin, 2016, JFA)

*If $\Phi$ is 2-convex and $q$-concave for some $2 < q < \infty$, then*

$$\tau(\Phi(|x|)) \approx_\Phi \max\Big\{\tau(\Phi[(\sum_{k\geq 0}|dx_k|^2)^{1/2}]), \tau(\Phi[(\sum_{k\geq 0}|dx_k^*|^2)^{1/2}])\Big\};$$

*If $\Phi$ is $p$-convex for some $1 < p < 2$ and 2-concave, then,*

$$\tau(\Phi(|x|)) \approx_\Phi \inf_{dx_k = y_k + z_k}\Big\{\tau(\Phi[\Big(\sum_{k\geq 0}|y_k|^2)^{1/2}]) + \tau(\Phi[(\sum_{k\geq 0}|z_k^*|^2)^{1/2}])\Big\}.$$

# The sharp case

Notation: $E \in Int(L_p, L_q)$

**Remark 1:** Let $E \in Int(L_p, L_q)$, then

$$p \leq p_E \leq q_E \leq q.$$

Theorem (Jiao, Sukochev, Xie and Zanin, 2016, JFA)

*Let $x$ be an arbitrary finite noncommutative martingale.*

*(i) If $E \in Int(L_p(0,1), L_2(0,1))$ for some $1 < p < 2$, then*

$$\|\sum_{k\geq 0} x_k\|_{E(\mathcal{M})} \approx_E \inf_{x_k = y_k + z_k} (\|(\sum_{k\geq 0} |y_k|^2\Big)^{1/2}\|_{E(\mathcal{M})} + \|(\sum_{k\geq 0} |z_k^*|^2)^{1/2}\|_{E(\mathcal{M})}).$$

*(ii) If $E \in Int(L_2(0,1), L_q(0,1))$ for some $2 < q < \infty$, then*

$$\|\sum_{k\geq 0} x_k\|_{E(\mathcal{M})} \approx_E \max\Big\{\|(\sum_{k\geq 0} |x_k|^2)^{1/2}\|_{E(\mathcal{M})}, \|(\sum_{k\geq 0} |x_k^*|^2)^{1/2}\|_{E(\mathcal{M})}\Big\}.$$

# Examples

**Example** Let $\Phi(t) = t^p \log(1 + t^q)$ with $p > 1$ and $q > 0$. It is easy to check that $\Phi$ is an Orlicz function with $p_\Phi = p$ and $q_\Phi = p + q$.

(i) Suppose that $p = 2$. It is not hard to see that $\Phi$ is 2-convex and $(2 + q)$-concave, and hence the corresponding Burkholder-Gundy inequality holds due to the last theorem (ii).

(ii) Suppose that $p + q = 2$. Then $\Phi$ is 2-concave and $p$-convex with $p > 1$, and hence the corresponding Burkholder-Gundy inequality holds due to the last theorem (i).

# Question

- For general sequence of noncommutative random variables $(x_n)$

$$\Big\| \Big( \sum_n |x_n|^2 \Big)^{1/2} \Big\|_p \approx \Big\| \Big( \sum_n |x_n^*|^2 \Big)^{1/2} \|_p \quad ?$$

- What condition?

Theorem (Jiao-Sukochev-Zanin, 2016, J. London Math. Soc.)

*Let $(\mathcal{M}, \tau)$ be a noncommutative probability space and let $x_n$, $n \geq 0$, be mean zero and freely independent random variables. Then*

$$\Big\|\Big(\sum_n |x_n|^2\Big)^{1/2}\Big\|_p \approx \Big\|\Big(\sum_n |x_n^*|^2\Big)^{1/2}\Big\|_p, \quad 0 < p \leq \infty.$$

Thanks for your attention!