Hunt's Hypothesis (H) and Getoor's Conjecture

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based on joint works with Wei Sun and Jing Zhang

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Outline

1. **Background**
2. Hunt's hypothesis (H)
3. Getoor's conjecture
4. Our results
   - Hu-Sun (2012, SPA)
   - Hu-Sun-Zhang (2015, PA)
   - Hu-Sun (2016, SCM)
   - Hu-Sun ((H) for the sum of two independent LPs)
   - Hu ((H) for Multi-dimensional LPs)
5. Some questions

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Background

• Hunt's hypothesis (H) belongs to Potential theory.

• When studying dual processes, in order to obtain satisfied results, people often need Hunt's Hypothesis (H).

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Meaning of Hunt's Hypothesis (H)

Let X be a standard Markov process on E.

Hunt's hypothesis (H): every semipolar set A of X is polar.

Intuitively speaking, (H) means that if A cannot be immediately hit by X with arbitrary starting point, then A will not be hit by X forever.

Example and counterexample

• (H) is satisfied by Brownian motion in $\mathbf{R}$: All semipolar and polar sets must be the empty set, since in this case, every point x is the regular point of the single set $\{x\}$.

In fact, all finite dimensional Brownian motions satisfy (H).

• (H) is not satisfied by uniform motion to the right in $\mathbf{R}$: semipolar set is countable set but polar set is the empty set.

Several potential principles (I)

Assume X is in duality with $\hat{X}$ on E w.r.t. σ-finite excessive measure m.

• Bounded positivity principle (P_α^*): If μ is a finite signed measure and $U^\alpha\mu$ is bounded, then $\mu U^\alpha \mu \geq 0$.

• Bounded energy principle (E_α^*): If μ is a finite measure with compact support and $U^\alpha\mu$ is bounded, then μ does not charge semipolar sets.

• Bounded maximum principle (M_α^*): If μ is a finite measure with compact support K and $U^\alpha\mu$ is bounded, then
$\sup\{U^\alpha\mu(x) : x \in E\} = \sup\{U^\alpha\mu(x) : x \in K\}$.

Several potential principles (II)

- Bounded regularity principle (R_α^*): If μ is a finite measure with compact support such that $U^\alpha\mu$ is bounded, then $U^\alpha\mu$ is regular.

Assume that all α-excessive functions are lower semicontinuous. Then

$$(P_\alpha^*) \Leftrightarrow (E_\alpha^*) \Leftrightarrow (M_\alpha^*) \Leftrightarrow (R_\alpha^*) \Leftrightarrow (\mathrm{H}).$$

Blumenthal, Getoor, Rao, Fitzsimmons.

Other equivalent characterizations

• (H) holds if and only if every natural additive functional of X is a continuous additive functional (Blumenthal and Getoor).

• (H) holds if and only if fine topology and cofine topology differ by polar sets, which means that the fine closure of any set and its cofine closure differ by polar sets (Blumenthal and Getoor, Glover).

• (H) is equivalent to the dichotomy of capacity (Fitzsimmons and Kanda (1992, Ann. Probab.)).

Some sufficient conditions for (H)

• If X is symmetric, i.e. $u(x,y) = u(y,x)$, then (H) holds.

• Glover and Rao (86, Ann. Prob.): α-subordinators of general Hunt processes satisfy (H).

• The sector condition implies that every semipolar set is essentially polar: Silverstein (1978,PTRF), Fitzsimmons (2001,PA), Han-Ma-Sun (2011, Acta. Math. Sinica).

Outline

Getoor's conjecture

About 50 years ago, R. K. Getoor conjectured that:

essentially all Lévy processes satisfy (H), except in certain extremely non-symmetric cases like uniform motions.

Lévy processes

$(\Omega, \mathcal{F}, P)$: probability space

$X = (X_t)_{t\geq 0}$: $\mathbf{R}^n$-valued Lévy process on $(\Omega, \mathcal{F}, P)$ with Lévy-Khintchine exponent ψ, i.e.

$$E[\exp\{i\langle z, X_t\rangle\}] = \exp\{-t\psi(z)\},\ z \in \mathbf{R}^n, t \geq 0.$$

Lévy-Khintchine formula:

$$\psi(z) = i\langle a, z\rangle + \frac{1}{2}\langle z, Az\rangle + \int_{\mathbf{R}^n}\left(1 - e^{i\langle z,x\rangle} + i\langle z, x\rangle 1_{\{|x|<1\}}\right)\mu(dx).$$

We also use (a, A, μ) to stand for Lévy-Khintchine exponent ψ.

Progress on Getoor's conjecture (I)

Kesten (69, Memoirs of AMS): When $n = 1$, if X is not a compound Poisson process, then every $\{x\}$ is non-polar if and only if

$$\int_0^\infty \mathrm{Re}([1+\psi(z)]^{-1})dz < \infty.$$

Bretagnolle (1971, LNM) gave a very simple proof of Kesten's result.

Port and Stone (69, Ann. Math. Statist.): For the asymmetric Cauchy process on the line every x is regular for $\{x\}$. Hence only the empty set is a semipolar set and therefore (H) holds.

Blumenthal and Getoor (70): All stable processes with index $\alpha \in (0, 2)$ on the line satisfy (H).

Progress on Getoor's conjecture (II)

Kanda (76, PTRF), Forst (75, Math. Ann.): (H) holds if X has bounded continuous transition densities and $|\text{Im}(\psi)| \leq M(1 + \text{Re}(\psi))$.

Rao (77, PTRF) gave a short proof of the Kanda-Forst theorem under the weaker condition that X has resolvent densities.

In particular, for $n > 1$ all stable processes of index $\alpha \neq 1$ satisfy (H). Kanda (78, PTRF) settled this problem for $\alpha = 1$ assuming the linear term vanishes.

The restrict "the linear term vanishes" can be canceled by the following result due to Rao, and so all stable processes satisfy (H).

Progress on Getoor's conjecture (III)

Rao (88, Proc. AMS): If all 1-excessive functions of X are lower semicontinuous and $|\mathrm{Im}(\psi)| \leq (1+\mathrm{Re}(\psi))f(1+\mathrm{Re}(\psi))$, where f is increasing on $[1,\infty)$ such that

$$\int_N^\infty \frac{1}{zf(z)}dz = \infty$$

for any $N \geq 1$, then X satisfies (H).

Examples for f: $f(x) \equiv M(\text{constant}),\ \ f(x) = \ln x,\ \ f(x) = \ln\ln x \cdots$.

Outline

Our starting point: Lévy-Itô decomposition

Let X be a Lévy process on $\mathbf{R}^n$ with exponent (a, A, μ). Then there exist a Brownian motion B_A on $\mathbf{R}^n$ with covariance matrix A and an independent Poisson random measure N on $\mathbf{R}^+ \times (\mathbf{R}^n \backslash \{0\})$ such that, for each $t \geq 0$,

$$X_t = bt + B_A(t) + \int_{|x|\geq 1} xN(t, dx) + \int_{|x|<1} x\tilde{N}(t, dx),$$

where $b = -a, \tilde{N}(t, A) = N(t, A) - t\mu(A)$.

$$X_t^{(I)} := bt + \sqrt{A}B_t,\ X_t^{(II)} := \int_{|x|\geq 1} xN(t, dx),\ X_t^{(III)} := \int_{|x|<1} x\tilde{N}(t, dx).$$

Theorem 1.1: Suppose A is non-degenerate. Then:

(i) X satisfies (H);

(ii) The Kanda-Forst condition $|\mathrm{Im}(\psi)| \leq M(1 + \mathrm{Re}(\psi))$ holds for some positive constant M;

(iii) X and $\tilde{X}$ have the same polar sets, where $\tilde{X}$ is symmetrization of X ($\tilde{X} := X - \bar{X}$).

Preparation for the second result

Denote $b := -a$ and $\mu_1 := \mu|_{\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n}$.

If $\int_{|x|<1} |x|\mu_1(dx) < \infty$, we set $b' := b - \int_{|x|<1} x\mu_1(dx)$.

Define the following solution condition:

(S) The equation $\sqrt{A}y = b',\ y \in \mathbf{R}^n$, has at least one solution.

Theorem 1.2

Suppose $\mu(\mathbf{R}^n \backslash \sqrt{A}\mathbf{R}^n) < \infty$. Then, the following three claims are equivalent:

(i) X satisfies (H);

(ii) (S) holds;

(iii) The Kanda-Forst condition $|\mathrm{Im}(\psi)| \leq M(1 + \mathrm{Re}(\psi))$ holds for some positive constant M.

Proposition 1.3

Suppose that X has bounded continuous transition densities, and X and $\tilde{X}$ have the same polar sets. Then X satisfies (H).

Proposition 1.4

Suppose that X is a subordinator. Then ψ can be expressed by

$$\psi(z) = -idz + \int_{(0,\infty)} (1 - e^{izx})\mu(dx), \ z \in \mathbf{R},$$

where the drift coefficient $d \geq 0$ and μ satisfies $\int_{(0,\infty)} (1 \wedge x)\mu(dx) < \infty$.

Proposition 1.4 If X is a subordinator and satisfies (H), then $d = 0$.

X: a Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent (a, A, μ).
μ_1: a finite measure on $\mathbf{R}^n \backslash \{0\}$ such that $\mu_1 \leq \mu$.
$\mu_2 := \mu - \mu_1$
X': Lévy process on $\mathbf{R}^n$ with Lévy-Khintchine exponent (a', A, μ_2), where

$$a' := a + \int_{\{|x|<1\}} x\mu_1(dx).$$

Theorem 2.1 Let X and X' be two Lévy processes defined above. Then
(i) they have the same semipolar sets;
(ii) they have the same essentially polar sets;
(iii) if both X and X' have resolvent densities, then X satisfies (H) if and only if X' satisfies (H).

(H) for subordinators (I): Special subordinators

X is called a special subordinator if $U|_{(0,\infty)}$ has a decreasing density with respect to the Lebesgue measure.

Theorem 2.2 Suppose that X is a special subordinator. Then X satisfies (H) if and only if $d = 0$.

(H) for subordinators (II): Locally (quasi)-stable subordinators

Let X be a subordinator with drift 0 and Lévy measure μ. If there exists a stable subordinator S with Lévy measure μ_S and three positive constants c_1, c_2, δ such that

$$c_1\mu_S - \mu_1 \leq \mu \leq c_2\mu_S + \mu_2 \ \text{ on } \ (0,\delta),$$

where μ_1, μ_2 are two finite measures on $(0,\delta)$, then we call X a locally quasi stable-like subordinator.

Proposition 2.3 Let X be a locally quasi-stable subordinator. Then it satisfies (H).

Extended Kanda-Forst-Rao theorem

Let X be a Lévy process with exponent ψ and set

$$A = 1 + \mathrm{Re}(\psi), \quad B = |1 + \psi|.$$

Theorem 2.4 Assume that all 1-excessive functions are lower semicontinuous. Then (H) holds if the following extended Kanda-Forst-Rao condition ((EKFR) for short) holds:
(EKFR) There are two measurable functions ψ_1 and ψ_2 on $\mathbf{R}^n$ such that $\mathrm{Im}(\psi) = \psi_1 + \psi_2$, and

$$|\psi_1| \le A(z)f(A(z)),$$
$$\int_{\mathbf{R}^n} \frac{|\psi_2(z)|}{(1 + \mathrm{Re}\psi(z))^2 + (\mathrm{Im}\psi(z))^2} dz < \infty, \tag{1}$$

where f is an increasing function on $[1, \infty)$ such that $\int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty$ for any $N \ge 1$.

One corollary of Theorem 2.1 and Theorem 2.4

Corollary 2.5 Let $\gamma > 0$ and X be a Lévy process on $\mathbf{R}$ satisfying

$$\liminf_{|z|\to\infty} \frac{\mathrm{Re}\psi(z)}{|z|\ln^{\gamma}(|z|)} > 0.$$

Then X satisfies (H).

Based on Corollary 2.5, we constructed a class of one-dimensional Lévy processes with enough small jumps which satisfies (H) without any condition on a and A.

Corollary 2.6 Let X be a Lévy process on $\mathbf{R}$. Suppose that

$$\liminf_{|z|\to\infty} \frac{|\psi(z)|}{|z|\ln^{1+\gamma}(|z|)} > 0,$$

for some $\gamma > 0$. Then (H) holds.

Theorem 3.1 (i) X satisfies (H) if the following condition holds:
Condition (C^{log}): For any finite measure ν on $\mathbf{R}^n$ of finite 1-energy, there exist a constant $\varsigma > 1$ and a sequence $\{y_k \uparrow \infty\}$ such that $y_1 > 1$ and

$$\sum_{k=1}^{\infty} \int_{\{y_k \le B(z) < (y_k)^{\varsigma}\}} \frac{1}{B(z)\log(B(z))} |\hat{\nu}(z)|^2 dz < \infty.$$

(ii) Suppose X satisfies (H). Then, for any finite measure ν on on $\mathbf{R}^n$ of finite 1-energy and any $\varsigma > 1$, there exists a sequence $\{y_k \uparrow \infty\}$ such that $y_1 > 1$ and (2) holds.

Theorem 3.2 (i) X satisfies (H) if the following condition holds:

Condition $(C^{\log\log})$: *For any finite measure* ν *on* $\mathbf{R}^n$ *of finite* 1*-energy, there exist a constant* $\varsigma > 1$ *and a sequence of positive numbers* $\{x_k\}$ *such that* $N^{\varsigma}_{x_1} > e,\ x_k + 1 < x_{k+1},\ k \in \mathbf{N},\ \sum_{k=1}^{\infty} \frac{1}{x_k} = \infty,$ *and*

$$\sum_{k=1}^{\infty} \int_{\{N^{\varsigma}_{x_k} \le B(z) < N^{\varsigma}_{x_k+1}\}} \frac{1}{B(z)\log(B(z))[\log\log(B(z))]} |\hat{\nu}(z)|^2 dz < \infty.$$

(ii) Suppose X satisfies (H). Then, for any finite measure ν on $\mathbf{R}^n$ of finite 1-energy and any $\varsigma > 1$, there exists a sequence of positive numbers $\{x_k\}$ such that $N^{\varsigma}_{x_1} > e$, $x_k + 1 < x_{k+1}$, $k \in \mathbf{N}$, $\sum_{k=1}^{\infty} \frac{1}{x_k} = \infty$, and (2) holds.

Proposition 3.3 Condition $(C^{B/A}) \Rightarrow$ Condition $(C^0) \Rightarrow$ (H).

Corollary 3.4 Let X be a Lévy process on $\mathbf{R}$. Suppose that

$$\liminf_{|z|\to\infty} \frac{|\psi(z)|}{|z|(\log\log|z|)^\delta} > 0$$

for some constant $\delta > 0$. Then X satisfies (H).

Corollary 2.6 Let X be a Lévy process on $\mathbf{R}$. Suppose that

$$\liminf_{|z|\to\infty} \frac{|\psi(z)|}{|z| \ln^{1+\gamma}(|z|)} > 0,$$

for some $\gamma > 0$. Then (H) holds.

One picture on (H) for Lévy processes (I)

(ND): Q is non-degenerate.

(KF): It has resolvent densities w.r.t. the Lebesgue measure and the Kanda-Forst condition holds.

(R): It has resolvent densities w.r.t. the Lebesgue measure and Rao's condition holds.

(EKFR): It has resolvent densities w.r.t. the Lebesgue measure and the (EKFR) condition holds.

(SY): It has resolvent densities w.r.t. the Lebesgue measure and is symmetric.

One picture on (H) for Lévy processes (II)

(S): $\mu(\mathbf{R}^n\backslash\sqrt{Q}\mathbf{R}^n) < \infty$, and the solution condition holds, i.e. the equation $\sqrt{Q}y = -a - \int_{\{x\in\mathbf{R}^n\backslash\sqrt{Q}\mathbf{R}^n:\,|x|<1\}} x\mu(dx)$ has at least one solution $y \in \mathbf{R}^n$.

(SP): It has bounded continuous transition densities, and the Lévy process and its symmetrization have the same polar sets.

(C^0): It has resolvent densities w.r.t. the Lebesgue measure and for any finite measure ν on $\mathbf{R}^n$ of finite 1-energy,

$$\int_{\mathbf{R}^n} \frac{1}{B(z)\log(2+B(z))[\log\log(2+B(z))]}|\hat{\nu}(z)|^2 dz < \infty.$$

$(C^{B/A})$: It has resolvent densities w.r.t. the Lebesgue measure and there exists a constant $C > 0$ such that $B(z) \leq CA(z)\log(2+B(z))[\log\log(2+B(z))], \forall z \in \mathbf{R}^n$.

One picture on (H) for Lévy processes (III)

$$
\begin{tikzcd}
 & (SY) \arrow[d, Rightarrow] & & & \\
(ND) \arrow[r, Rightarrow] & (KF) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (C^{B/A}) \arrow[r, Rightarrow] & (C^0) \arrow[d, Rightarrow] & \\
 & (R) \arrow[r, Rightarrow] & (EKFR) \arrow[r, Rightarrow] & (H) & (SP) \arrow[l, Rightarrow] \\
 & & & (S) \arrow[u, Rightarrow] &
\end{tikzcd}
$$

1. Hu-Sun: Hunt's hypothesis (H) and Getoor's conjecture for Levy processes, *SPA*, **122**, 2319-2328 (2012).
2. Hu-Sun-Zhang: New results on Hunt's hypothesis (H) for Lévy processes, *PA*, **42**, 585-605 (2015).
3. Hu-Sun: Further study on Hunt's hypothesis (H) for Lévy processes, Accepted by *SCM* (2016).

(H) for one-dimensional Lévy processes (I)

Let $X = (X_t)_{t \geq 0}$ be a one-dimensional Lévy process with exponent ψ and (a, σ, μ). If $\int (|x| \wedge 1)\mu(dx) < \infty$, we write

$$\psi(z) = ia'z + \frac{1}{2}\sigma z^2 + \int_{\mathbf{R}} \left(1 - e^{i\langle z,x\rangle}\right) \mu(dx).$$

Define

$$C = \{x \in \mathbf{R} : P\{X_t = x \text{ for some } t > 0\} > 0\},$$

and the following cases:

A. $\sigma > 0$.

B. $\sigma = 0$; $\int (|x| \wedge 1)\mu(dx) = +\infty$.

C. $\sigma = 0$; $\int (|x| \wedge 1)\mu(dx) < +\infty$. Now we further decompose it into the following three subcases:

- C_1. $a' = 0$,
- C_2. $a' > 0$, μ's support is $\mathbf{R}^+ = \{x \in \mathbf{R} | x > 0\}$.
- C_3. $a' > 0$, μ charges $\mathbf{R}^- = \{x \in \mathbf{R} | x < 0\}$.

(H) for one-dimensional Lévy processes (II)

Theorem (Bretagnolle, 1971)
(i) For Case A, $C = \mathbf{R}$ and 0 is a regular point of $\{0\}$.
(ii) For Case B, either $C = \emptyset$ or $C = \mathbf{R}$, and if $C = \mathbf{R}$, then 0 is a regular point of $\{0\}$.
(iii) For Case C, suppose that X is not a compound Poisson process, then
(a) for Case C_1, $C = \emptyset$;
(b) for Case C_2, $C = \mathbf{R}^+$, 0 is not a regular point of $\{0\}$;
(c) for Case C_2, $C = \mathbf{R}$, 0 is not a regular point of $\{0\}$.

Remark For Case B, by one result of Kesten, we know that if $\int_0^\infty (x \wedge 1)\mu(dx) < \infty$ or $\int_{-\infty}^0 (|x| \wedge 1)\mu(dx) < \infty$, then $C = \mathbf{R}$, and thus any $x \in \mathbf{R}$ is a regular point of $\{x\}$ and hence (H) holds in this case. As a consequence, all spectrally one sided one-dimensional Lévy process with unbounded variation satisfies (H).

(H) for one-dimensional Lévy processes (III)

Denote by μ_+ and μ_- be the restriction of the Lévy measure μ on $(0,\infty)$ and $(-\infty,0)$, respectively. Define by $\bar{\mu}_-$ the image measure of μ_- under the map

$$x \mapsto -x,\ \forall x \in (-\infty,0).$$

Theorem 4.1. Suppose that $Q=0$ and $\int_0^\infty (x\wedge 1)\mu_+(dx)=\infty$. If there exist $\delta>0$, $\mathbf{k}\in[\mathbf{0},\mathbf{1})$ and one measure ν such that on $(0,\delta)$ (without loss of generality, we can assume that $\delta\le 1$),

$$\bar{\mu}_- \le k\mu_+ + \nu,$$

where ν satisfies $\int_{(0,\delta)} x\nu(dx)<\infty$. Then for X, any point x is a regular point of the set $\{x\}$ and thus X satisfies (H).

Theorem 4.2 Suppose that $Q = 0$ and $\mu_+(dx) = cx^{-\alpha-1}dx$ with $c > 0, \alpha \in (1, 2)$. If there exist two constants $\delta > 0, \mathbf{k} \geq \mathbf{0}$ and one measure ν such that on $(0, \delta)$ (without loss of generality, we can assume that $\delta \leq 1$),

$$\bar{\mu}_- \leq k\mu_+ + \nu, \tag{2}$$

where ν satisfies $\int_{(0,\delta)} x\nu(dx) < \infty$. Then X satisfies (H).

Theorem 4.3. $(S) + (S) \Rightarrow (S)$.

- We give some capacity inequalities for Lévy processes.

- We present other several results on (H) for the sum of two independent Lévy processes.

Hu: Hunt's Hypothesis (H) for Multi-dimensional Lévy Processes, In preparation.

Theorem 5.1 Suppose that X satisfies (H). Then for any nonempty proper subspace A of $\mathbf{R}^n$, the projection process $Y = (Y_t)_{t\geq 0}$ of X on A must satisfy (H).

Proposition 5.2 Let Q be degenerate and μ_1 be the projection of μ on the orthogonal complement space $(\sqrt{Q}\mathbf{R}^n)^\perp$ of $\sqrt{Q}\mathbf{R}^n$. Suppose that

$$\int_{(\sqrt{Q}\mathbf{R}^n)^\perp} (|x| \wedge 1)\mu_1(dx) < \infty. \tag{3}$$

If X satisfies (H), then the drift coefficient of the projection process of X on $(\sqrt{Q}\mathbf{R}^n)^\perp$ must be zero.

- By using Theorem 5.1, we constructed a counterexample.

Outline

Some questions

Question 1: Any subordinator with 0 drift and infinite Lévy measure does satisfies (H) ?

Question 2: Does any pure jump Lévy process satisfy (H)?

Question 3: For one dimensional Lévy processes, if the Gaussian item disappears, then (H) holds under what conditions ?

Question 4: For general finite dimensional Lévy processes, can we give a good (necessary and) sufficient condition for (H) ?

Question 5: For infinite dimensional Lévy processes, can we give a good (necessary and) sufficient condition for (H) ?

Infinite dimensional Brownian motion: René Carmona (1980)

P. J. Fitzsimmons: Gross' Brownian Motion fails to satisfy the polarity principle, Rev. Roumatine Math. Pures Appl. 59 (2014), 87-91.

Question 6: Suppose that X_1 and X_2 are two independent $\mathbf{R}^n$-valued Lévy processes such that both X_1 and X_2 satisfy (H). Does $X_1 + X_2$ satisfy (H)?

Question 7: If for any nonempty proper subspace A of $\mathbf{R}^n$, the projection process $Y = (Y_t)_{t \geq 0}$ of X on A satisfies (H), does X satisfy (H)?

Question 8: If for any one-dimensional subspace A of $\mathbf{R}^n$, the projection process $Y = (Y_t)_{t \geq 0}$ of X on A satisfies (H), does X satisfy (H)?

Question 9: Suppose that X_1 is an $\mathbf{R}^n$-valued Lévy process, X_2 is an $\mathbf{R}^m$-valued Lévy process such that X_1 and X_2 are independent and both of them satisfy (H). Does the $\mathbf{R}^{n+m}$-valued Lévy process $X = (X_1, X_2)$ satisfy (H)?

Thank you for your attention!

$$
\begin{array}{ccccccccc}
 & & (SY) & & & & & & \\
 & & \Downarrow & & & & & & \\
(ND) & \Rightarrow & (KF) & \Rightarrow & (C^{B/A}) & \Rightarrow & (C^0) & & \\
 & & \Downarrow & & & & \Downarrow & & \\
 & & (R) & \Rightarrow & (EKFR) & \Rightarrow & (H) & \Leftarrow & (SP) \\
 & & & & & & \Uparrow & & \\
 & & & & & & (S) & &
\end{array}
$$