Density of parabolic Anderson random variable

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jointly with Khoa Le

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Outline

1. Main equation
2. Right tail probability
3. Tail of density
4. Hölder continuity

1. Main equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}, \quad t > 0, x \in \mathbb{R}^d,$$

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$$\begin{aligned} \mathbb{E}(\dot{W}(s,x)\dot{W}(t,y)) &= \gamma(s-t)\Lambda(x-y) \\ \Lambda(x-y) &= \int_{\mathbb{R}^d} e^{-i\xi(x-y)}\mu(d\xi) \end{aligned}$$

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  for some measure μ.
- The product $u\dot{W}$ is in the Skorohod senses.

The relation of the moments of $u(t, x)$ and mean of exponential of “local time”:

$$\mathbb{E}\left[u^k(t,x)\right] = \text{The expectation of the exponential of local time of Brownian motions}\,.$$

Hu, Y. and Nualart, D.

Stochastic heat equation driven by fractional noise and local time.

Prob. Theory and Related Fields, 143 (2009), 285-328.

Feynman-Kac formula for the solution

$$u(t,x) = \mathbb{E}^B \left[u_0(x+B_t) \exp \left(\int_0^t \dot{W}(t-s, B_s+x) ds \right) \right] .$$

Hu, Y.; Nualart, D. and Song, J.

Feynman-Kac formula for heat equation driven by fractional white noise.

The Annals of Probability 39 (2011), no. 1, 291-326.

Fractional Brownian field of Hurst parameters $> 1/2$.

Fractional Brownian field of time Hurst parameters $< 1/2$ but $> 1/4$.

Hu, Y.; Nualart, D. and Lu, F.

Feynman-Kac formula for the heat equation driven by fractional noise with Hurst parameter $H < 1/2$.

Ann. Probab. 40 (2012), no. 3, 1041-1068.

All parameter $H_0 > 0$.

Chen, L.; Hu, Y.; Kalbasi, K. and Nualart, D.

Intermittency for the stochastic heat equation driven by time-fractional Gaussian noise with $H \in (0, 1/2)$.

Submitted.

Hu, Y.; Le, K.

Nonlinear Young integrals and differential systems in Hölder media.

Transaction of American Mathematical Society. In printing.

We assume the following

Hypothesis (on γ)

There exist constants c_0, C_0 and $0 \leq \beta < 1$, such that

$$c_0|t|^{-\beta} \leq \gamma(t) \leq C_0|t|^{-\beta}.$$

Hypothesis (on Λ)

Λ satisfies one of the following conditions.

(i) *There exist positive constants c_1, C_1 and $0 < \kappa < 2$ such that*

$$\begin{cases} d \geq 2\,, \\ c_1|x|^{-\kappa} \leq \Lambda(x) \leq C_1|x|^{-\kappa}\,. \end{cases}$$

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(ii) *There exist positive constants c_1, C_1 and κ_i such that*

$$\begin{cases} 0 < \kappa_i < 1\,, \qquad \sum_{i=1}^d \kappa_i < 2\,, \\ c_1 \prod_{i=1}^d |x_i|^{-\kappa_i} \leq \Lambda(x) \leq C_1 \prod_{i=1}^d |x_i|^{-\kappa_i}. \end{cases}$$

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$$\begin{cases} 0 < \kappa_i < 1\,, \qquad \sum_{i=1}^{d} \kappa_i < 2\,, \\ c_1 \prod_{i=1}^{d} |x_i|^{-\kappa_i} \leq \Lambda(x) \leq C_1 \prod_{i=1}^{d} |x_i|^{-\kappa_i}. \end{cases}$$

(iii)

$$\begin{cases} d = 1\,, \\ \Lambda(x) = \delta(x) \quad (\textit{Dirac delta function}). \end{cases}$$

$$\upsilon = \kappa \text{ (Case (i))}, \qquad \sum_{i=1}^{d} \kappa_i \text{ (Case (ii))}, \qquad 1 \text{ (Case (iii))}.$$

Then we have for all $t \geq 0\,, x \in \mathbb{R}^d\,, k \geq 2$,

$$\exp\left(Ct^{\frac{4-2\beta-\upsilon}{2-\upsilon}} k^{\frac{4-\upsilon}{2-\upsilon}}\right) \leq \mathbb{E}\left[u_{t,x}^k\right] \leq \exp\left(C' t^{\frac{4-2\beta-\upsilon}{2-\upsilon}} k^{\frac{4-\upsilon}{2-\upsilon}}\right)$$

where C, C' are constants independent of t and k.

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If $\Lambda(x) = \delta_0(x)$ and $\dot{W}$ is time independent, then,

$$\exp\left(Ct^3k^3\right) \leq \mathbb{E}\left[u_{t,x}^k\right] \leq \exp\left(C't^3k^3\right)\,,$$

where $C, C' > 0$ are constants independent of t and k.

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Hu, Y.; Huang, J.; Nualart, D. and Tindel, S.

Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency.

Electron. J. Probab. 20 (2015), no. 55, 50 pp.

Amir, G.; Corwin, I. and Quastel, J.

Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions.

Comm. Pure Appl. Math. 64 (2011), no. 4, 466-537.

$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$.

$$F(t,x) = \log\left(\frac{u(t,x)}{p(t,x)}\right)$$

$$F_T(s) = P\left(F(T,x) + \frac{T}{4!} \le s\right) .$$

$$\begin{aligned}
\mathrm{Ai}(x) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt \\
K_\sigma(x,y) &= \int_{-\infty}^\infty \sigma(t)\,\mathrm{Ai}(x+t)\,\mathrm{Ai}(y+t)dt \\
F_T(s) &= \int_{\tilde{C}} \frac{d\tilde{\mu}}{\tilde{\mu}} e^{-\tilde{\mu}} \det\left(I - K_{\sigma_{T,\tilde{\mu}}}\right)_{L^2(K_T^{-1}a,\,\infty)} ,
\end{aligned}$$

where

$$\begin{aligned}
\sigma_{T,\tilde{\mu}} &= \frac{\tilde{\mu}}{\tilde{\mu} - e^{-K_T t}} \\
a &= a(s) = s - \log\sqrt{2\pi T} \\
K_t &= 2^{-1/3} T^{1/3} \\
\tilde{C} &= \left\{e^{i\theta}\right\}_{\frac{\pi}{2} \le \frac{3\pi}{2}} \cup \{x + \pm i\}_{x>0} .
\end{aligned}$$

2. Tail probability

Theorem (Right tail)

Let the initial condition $u_0(x)$ be bounded from above and from below. Then, there are positive constants c_1, c_2, C_1 and C_2 (independent of t and a) such that

$$C_1 \exp\left(-c_1 t^{\frac{2\beta+\upsilon-4}{2}} (\log a)^{\frac{4-\upsilon}{2}}\right) \leq P(u(t,x) \geq a) \leq C_2 \exp\left(-c_2 t^{\frac{2\beta+\upsilon-4}{2}} (\log a)^{\frac{4-\upsilon}{2}}\right)$$

for all sufficiently large a.

Left tail probability

$d = 1$ and $\dot{W}$ is space time white, $u_0(x) \geq c > 0$,

$$P(u(t,x) \leq a) \leq Ce^{-c(\log a)^2}, \quad \text{as } a \to 0\,.$$

Moreno Flores, G. R.

On the (strict) positivity of solutions of the stochastic heat equation.

Ann. Probab. 42 (2014), no. 4, 1635-1643.

Let

$$|q(t,s,x,y)| \leq L_T |t-s|^{-\alpha_0} \prod_{i=1}^{d} |x_i - y_i|^{-\alpha_i} \, ,$$

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We assume the Gaussian noise is given by

$$\dot{W}(t,x) = \int_0^t \int_{\mathbb{R}^d} q(t,s,x,y)\mathbb{W}(ds,dy)\,,$$

where $\mathbb{W}$ is the space time white noise:

$$\mathbb{E}\dot{\mathbb{W}}(t,x)\dot{\mathbb{W}}(s,y) = \delta(t-s)\delta(x-y)\,.$$

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Denote $\beta = 2\alpha_0 - 1$ and $\upsilon = 2\sum_{i=1}^{d} \alpha_i - d$. Then, there are positive constants C, c_1, and c_2 such that for any sufficiently small positive u

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$$\begin{aligned} & P(u(t,x) \le u) \\ & \le C \exp\left\{ - \left(-c_1 \exp\left[-\rho t^{\frac{4-2\beta-\upsilon}{4-2\upsilon}} \right] \log u - c_2 t^{\frac{4-2\beta-\upsilon}{4-2\upsilon}} \right)^2 \right\}\,. \end{aligned}$$

3. Tail of density

Theorem

Let the initial condition $u_0(x)$ be bounded from above and from below. Then, the law of the random variable $u(t,x)$ has a density $\rho(t,x;y)$ with respect to the Lebesgue measure, namely, for any Borel set $A \subset \mathbb{R}$,

$$P(u(t,x) \in A) = \int_A \rho(t,x;y)dy\,.$$

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Moreover, there are positive constants $c, C > 0$ such that for every $t \in (0,T)$ and all sufficiently large y

$$\inf_{x\in\mathbb{R}} \rho(t,x;y) \geq C \exp\left(-ct^{\frac{2\beta+\upsilon-4}{2}}(\log y)^{\frac{4-\upsilon}{2}}\right)\,.$$

If the noise is space time white, then

Theorem

Let the initial condition $u_0(x)$ be bounded from above and from below. Then, the law of the random variable $u(t,x)$ has a density $\rho(t,x;y)$ with respect to the Lebesgue measure. Moreover, there are positive constants $c_1, C_1, c_2, C_2 > 0$ such that for every $t \in (0,T)$ and all sufficiently large y

$$C_1 \exp\left(-c_1 t^{-1/2} (\log y)^{\frac{3}{2}}\right) \le \inf_{x\in\mathbb{R}} \rho(t,x;y) \le \sup_{x\in\mathbb{R}} \rho(t,x;y) \le C_2 \exp\left(-c_2 t^{-1/2} (\log y)^{\frac{3}{2}}\right) .$$

Theorem

Let $\upsilon + 2\beta < 2$. Let the initial condition $u_0(x)$ be bounded from above and from below. Then, the law of the random variable $u(t, x)$ has a density $\rho(t, x; y)$ with respect to the Lebesgue measure. Moreover, for every $t \in (0, \infty)$ and for any sufficiently large y

$$
\begin{aligned}
C_1 \exp\left(-c_1 t^{\frac{2\beta+\upsilon-4}{2}} (\log y)^{\frac{4-\upsilon}{2}}\right) &\leq \inf_{x\in\mathbb{R}} \rho(t, x; y) \\
&\leq \sup_{x\in\mathbb{R}} \rho(t, x; y) \leq C_2 \exp\left(-c_2 t^{\frac{2\beta+\upsilon-4}{2}} (\log y)^{\frac{4-\upsilon}{2}}\right),
\end{aligned}
$$

for some universal positive constants c_1, c_2, C_1 and C_2.

Theorem

Let the initial condition $u_0(x)$ be bounded from above and from below. Assume that $\upsilon + 2\beta < 2$. Then, for all $0 < y < 1$

$$\sup_{x\in\mathbb{R}} \rho(t,x;y) \le Ct^{\frac{\beta}{2}+\frac{\upsilon}{4}-1} \exp\left(-ct^{\frac{4-2\beta-\upsilon}{2}}(-\log y)^{\frac{4-\upsilon}{2}}\right) ,$$

for some universal positive constants c, C.

Theorem

Let

$$|q(t,s,x,y)| \le L_T|t-s|^{-\alpha_0} \prod_{i=1}^{d} |x_i - y_i|^{-\alpha_i} \,,$$

We assume the Gaussian noise is given by

$$\dot{W}(t,x) = \int_0^t \int_{\mathbb{R}^d} q(t,s,x,y) \mathbb{W}(ds,dy) \,.$$

Denote $\beta = 2\alpha_0 - 1$ *and* $\upsilon = 2\sum_{i=1}^{d} \alpha_i - d$*. Then, there are positive constants* C*,* c_1*, and* c_2 *such that for any sufficiently small positive* y

$$\begin{aligned} &\rho(t,x;y) \\ &\le Ct^{\frac{\beta}{2}+\frac{\upsilon}{4}-1} \exp\left\{ -\left(-c_1 \exp\left[-\rho t^{\frac{4-2\beta-\upsilon}{4-2\upsilon}} \right] \log y - c_2 t^{\frac{4-2\beta-\upsilon}{4-2\upsilon}} \right)^2 \right\} . \end{aligned}$$

Theorem

Let the noise be the one dimensional space time white noise. Then, there are positive constants C, c_1, and c_2 such that for any sufficiently small positive y

$$\rho(t,x;y) \leq Ct^{-1/4} \exp\left\{ - \left(-c_1 \exp\left[-\rho t^{1/2} \right] \log y - c_2 t^{1/2} \right)^2 \right\} .$$

The tool to use is Malliavin calculus

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ANALYSIS ON GAUSSIAN SPACES

by **Yaozhong Hu**
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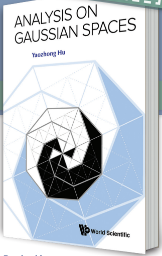

Analysis of functions on the finite dimensional Euclidean space with respect to the Lebesgue measure is fundamental in mathematics. The extension to infinite dimension is a great challenge due to the lack of Lebesgue measure on infinite dimensional space. Instead the most popular measure used in infinite dimensional space is the Gaussian measure, which has been unified under the terminology of "abstract Wiener space".

Out of the large amount of work on this topic, this book presents some fundamental results plus recent progress. We shall present some results on the Gaussian space itself such as the Brunn-Minkowski inequality, Small ball estimates, large tail estimates. The majority part of this book is devoted to the analysis of nonlinear functions on the Gaussian space. Derivative, Sobolev spaces are introduced, while the famous Poincaré inequality, logarithmic inequality, hypercontractive inequality, Meyer's inequality, Littlewood-Paley-Stein-Meyer theory are given in details.

This book includes some basic material that cannot be found elsewhere that the author believes should be an integral part of the subject. For example, the book includes some interesting and important inequalities, the Littlewood-Paley-Stein-Meyer theory, and the Hörmander theorem. The book also includes some recent progress achieved by the author and collaborators on density convergence, numerical solutions, local times.

Readership:

Graduate students and researchers in probability and stochastic processes and functional analysis.

480pp Nov 2016
978-981-3142-17-6
~~US$148 £107~~
US$118.40 £85.60

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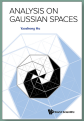
ANALYSIS ON GAUSSIAN SPACES



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CONTENT

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5. Hölder continuity

Theorem

Let the covariance functions of W satisfy $|\gamma(t)| \leq C|t|^{-\beta}$ and $|\hat{\Lambda}(\xi)| \leq C|\xi|^{-\nu}$, where $0 < \beta < 1, \quad d - \nu < 2$. Then, for any α and δ satisfying $4\alpha + 2\delta < \nu - 2\beta + 4 - d$, there is a random constant C such that

$$|u(t,y) - u(s,y) - u(t,x) + u(s,x)| \leq C|t-s|^{\alpha}|x-y|^{\delta} \quad a.s..$$

Moreover, for any δ an α satisfying

$$\delta < (\nu - 2\beta + 4 - d)/2 \qquad \text{and} \qquad \alpha < (\nu - 2\beta + 4 - d)/4$$

there is a random constant C such that

$$|u(t,x) - u(s,x)| + |u(t,y) - u(t,x)| \leq C\left[|t-s|^{\alpha} + |x-y|^{\delta}\right] \quad a.s..$$

When W is a 1-d space time white noise, this corresponds to the case $\nu = 0$, $\beta = 1$. Then the above theorem says if $4\alpha + 2\delta < 1$, then

$$|u_n(t,y) - u_n(s,y) - u_n(t,x) + u_n(s,x)| \leq C|t-s|^{\alpha}|x-y|^{\delta}.$$

This coincides with the optimal Hölder exponent result. On the other hand, the corollary also implies in this case that if $\alpha < 1/4$ and $\beta < 1/2$, Then,

$$|u_n(t,x) - u_n(s,x)| + |u_n(t,y) - u_n(t,x)| \leq C\left[|t-s|^{\alpha} + |x-y|^{\delta}\right].$$

THANKS