On the minimum of a branching random walk

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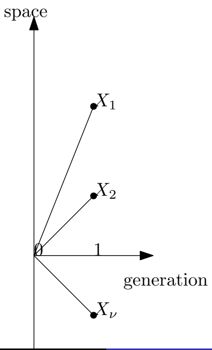
Outline

Branching random walk on $\mathbb{R}$

- At the beginning, there is a single particle located at the origin 0.
- Its children, who form the first generation, are positioned according to a certain point process $\Theta = \sum_{i=1}^{\nu} \delta_{\{X_i\}}$ on $\mathbb{R}$.



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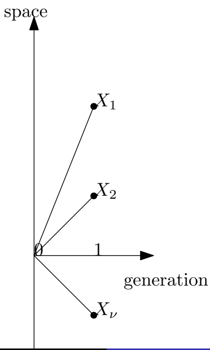

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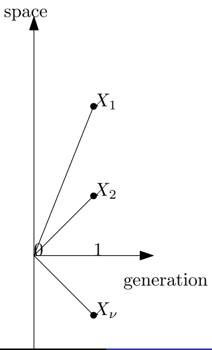
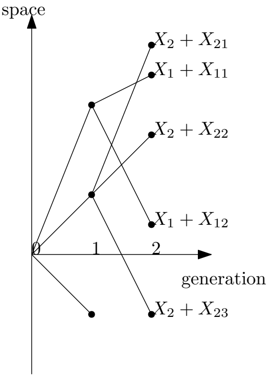


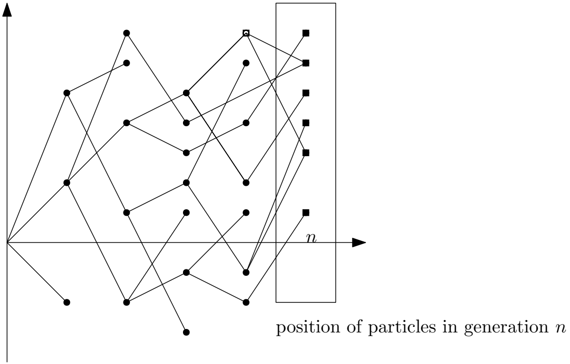
Model

- Each particle i independently gives birth to new particles that are positioned (with respect to X_i) according to a point process with the same law as Θ ; they form the second generation.



Model

- And so on.



Notations

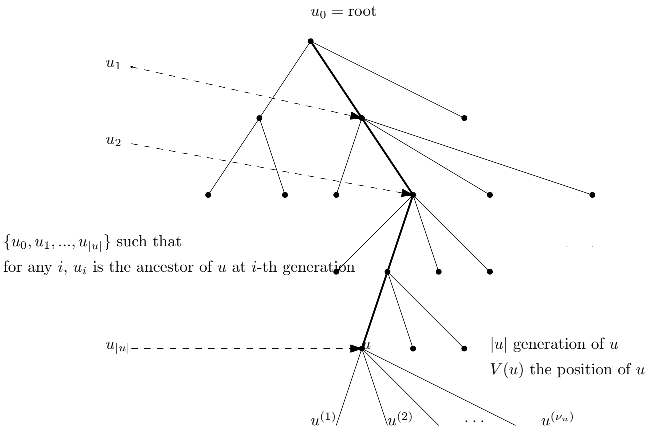


Figure: The particles form a rooted Galton-Watson tree $\mathcal{T}$ of reproduction law ν

- We assume that $\mathbb{E}(\nu) > 1$, so that $\mathcal{T}$ is supercritical.
- Denote by $(V(u))_{|u|=n}$ the positions of particles in the n-th generation.
- We are interested in the minimal position :

$$\mathcal{M}_n := \min_{|u|=n} V(u).$$

- Then $\Theta = \sum_{|u|=1} \delta_{\{V(u)\}} \equiv \sum_{i=1}^{\nu} \delta_{\{X_i\}}$, and $\mathcal{M}_1 = \min_{1\le i\le \nu} X_i$.
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The first order limit :

Theorem of Kingman, Hammersley, Biggins (1974–1976) :

Conditioned on $\{\mathcal{T} = \infty\}$,

$$\lim_{n\to\infty} \frac{\mathcal{M}_n}{n} = c, \qquad \text{a.s.},$$

where the limit $c \in [-\infty, \infty)$ is some deterministic constant, explicitly defined through a variational formula.

Question :

The second order limit of $\mathcal{M}_n$?

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The boundary case :

$$\psi(t) := \log \mathbf{E} \int e^{-tx}\Theta(dx) \equiv \log \mathbf{E} \sum_{i=1}^{\nu} e^{-tX_i}$$

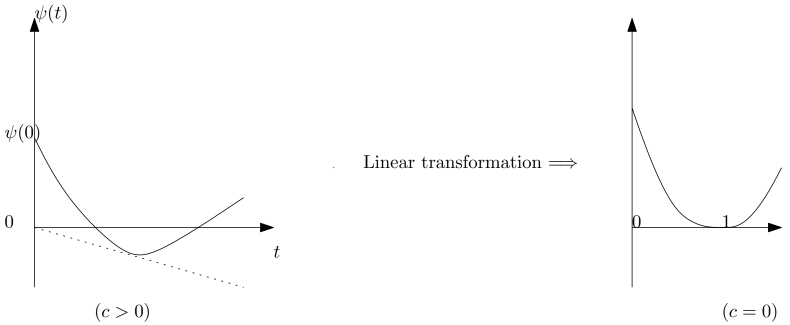

Figure: Transformation to the "boundary case" : $\psi(1) = \psi'(1) = 0$

The boundary case : velocity $c = 0$

- Almost surely, $\mathcal{M}_n \to +\infty$ [Biggins (1977), Lyons (1997)] ;
- Roughly $\mathcal{M}_n \approx \log n$, See Addario-Berry and Reed (2009), Bramson and Zeitouni (2009) [concentration around its expectation], H. and Shi (2009) [a.s. limit] ;
- Aïdékon's theorem (2013) :
  $\mathcal{M}_n - \frac{3}{2} \log n$ converges in law.
- Extremal process : $\sum_{|u|=n} \delta_{\{V(u)-\mathcal{M}_n\}}$ converges in law. See Madaule (2014+), a (weaker) discrete analogue of Aïdékon, Berestycki, Brunet and Shi (2013), Arguin, Bovier and Kistler (2013)' theorem on the branching Brownian motion.

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Objective

Two problems :

- Precise almost sure limits : Law of the iterated logarithm for $\mathcal{M}_n$.
- Moderate deviations for $\mathcal{M}_n$.

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Some a.s. limit theorems

Under some extra integrability conditions, H. & Shi (2009) proved that

$$\limsup_{n\to\infty} \frac{\mathcal{M}_n}{\log n} = \frac{3}{2}, \qquad \mathbb{P}^*\text{-a.s.},$$
$$\liminf_{n\to\infty} \frac{\mathcal{M}_n}{\log n} = \frac{1}{2}, \qquad \mathbb{P}^*\text{-a.s.},$$

where here and in the sequel $\mathbb{P}^*(\cdot) := \mathbb{P}(\cdot|\text{ survival})$.

How big and how small can the minimum $\mathcal{M}_n$ be ?

Question 1 :

How does $\mathcal{M}_n$ approach its lower limits $\frac{1}{2}\log n$? or How small can $\mathcal{M}_n$ be ?

Question 2 :

How does $\mathcal{M}_n$ approach its upper limits $\frac{3}{2}\log n$? or How big can $\mathcal{M}_n$ be ?

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Why should the lower limit be $\frac{1}{2}\log n$?

Many-to-one formula (Chauvin, Rouault and Wakolbinger (1991), Lyons, Pemantle and Peres (1995), Biggins and Kyprianou (2004))

There exists a centered real-valued random walk $\{S_n, n \geq 0\}$ such that for any $n \geq 1$ and $f : \mathbb{R}^n \to \mathbb{R}_+$ measurable,

$$\mathbb{E}\Big[\sum_{|u|=n} e^{-V(u)} f(V(u_1), ..., V(u_n))\Big] = \mathbb{E}\left[f(S_1, ..., S_n)\right].$$

Moreover, $\mathbb{E}(S_1) = 0$ and $Var(S_1) = \psi''(1) =: \sigma^2 > 0$.

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Let $\alpha > 0$. Define $\underline{V} := \inf_{u\in\mathbb{T}} V(u)$ and $\underline{V}(u) := \min_{\varnothing\le v\le u} V(v)$ (with $\inf\emptyset = \infty$). Then for any $c > 0$,

$$\begin{aligned} & \mathbb{P}\Big(\mathcal{M}_n \le c\log n,\ \underline{V} \ge -\alpha\Big) \\ \le\ & \mathbb{E}\Big[\sum_{|u|=n} 1_{(V(u)\le c\log n, \underline{V}(u)\ge -\alpha)}\Big] \\ =\ & \mathbb{E}\Big[e^{S_n} 1_{(S_n\le c\log n, \min_{1\le i\le n} S_i \ge -\ \alpha)}\Big] \\ \le\ & n^c\,\mathbb{P}\Big(S_n \le c\log n,\ \min_{1\le i\le n} S_i \ge -\ \alpha\Big). \end{aligned}$$

Excursion probability

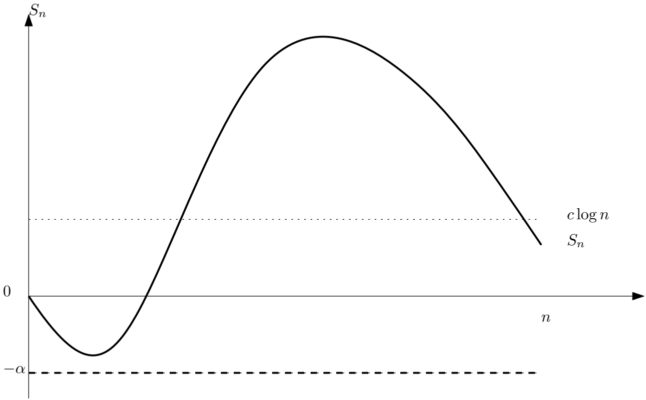

Figure: $\mathbb{P}\Big(S_n \leq c \log n, \min_{1 \leq i \leq n} S_i \geq -\alpha\Big) = n^{-3/2+o(1)}$.

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Then for any fixed $\alpha > 0$,

$$\mathbb{P}\Big(\mathcal{M}_n \leq c \log n,\ \underline{V} \geq -\alpha\Big) \leq n^{c-3/2+o(1)},$$

whose sum converges if $c < 1/2$. The Borel-Cantelli lemma yields the (a.s.) lower limit $\frac{1}{2}\log n$.

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In probability, $\mathcal{M}_n = \frac{3}{2}\log n + O(1)$.

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Question :

Can $\mathcal{M}_n$ be much smaller than its lower limits $\frac{1}{2}\log n$?

Aïdékon and Shi (2014) : Yes !

Under some mild conditions,

$$\liminf_{n\to\infty}(\mathcal{M}_n - \frac{1}{2}\log n) = -\infty, \qquad \mathbb{P}^*\text{-a.s.}.$$

Moreover, they asked whether there exists a deterministic sequence (a_n) such that

$$-\infty < \liminf_{n\to\infty}\frac{1}{a_n}(\mathcal{M}_n - \frac{1}{2}\log n) < 0?$$

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Response to Question 1 : How small is $\mathcal{M}_n$?

Theorem 1 (H. 2015)

For any function $f \uparrow \infty$,

$$\mathbb{P}^*\Big(\mathcal{M}_n - \frac{1}{2}\log n < -f(n), \quad \text{i.o.}\Big) = \begin{cases} 0 \\ 1 \end{cases}$$

$$\Longleftrightarrow \int^{\infty} \frac{dt}{t} e^{-f(t)} \begin{cases} < \infty \\ = \infty \end{cases},$$

where i.o. means infinitely often as the relevant index $n \to \infty$.

Response to Question 1 : How small is $\mathcal{M}_n$?

Corollary

We can choose $a_n = \log\log n$ in the question arised by Aïdékon and Shi :

$$\liminf_{n\to\infty} \frac{1}{\log\log n}(\mathcal{M}_n - \frac{1}{2}\log n) = -1, \qquad \mathbb{P}^*\text{-a.s.}$$

Question 2 : How big is $\mathcal{M}_n$?

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Recalling

$$\limsup_{n\to\infty} \frac{\mathcal{M}_n}{\log n} = \frac{3}{2}, \qquad \mathbb{P}^*\text{-a.s.}$$

How can $\mathcal{M}_n$ approach its upper limits $\frac{3}{2}\log n$?

Theorem 2 (H. 2016)

Assume some integrability conditions. We have

$$\limsup_{n\to\infty} \frac{1}{\log\log\log n}\left(\mathcal{M}_n - \frac{3}{2}\log n\right) = 1, \qquad \mathbb{P}^*\text{-a.s.}$$

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Organization of the rest of the talk :

1. Proof of Theorem 1 by introducing the additive martingale.
2. Proof of Theorem 2 and the moderate deviations of $\mathcal{M}_n$.

The additive and derivative martingales

Two martingale :

$$W_n := \sum_{|u|=n} e^{-V(u)}, \quad D_n := \sum_{|u|=n} V(u)e^{-V(u)}, \quad n \geq 0.$$

Under the same hypothesis as before

- D_n is a real-valued martingale which converges to $D_\infty \geq 0$, a.s., $\{D_\infty > 0\} = \{\text{survival}\}$ by Biggins and Kyprianou (2004), see also Chen (2013+).
- W_n is a nonnegative martingale which converges to 0 a.s. [Biggins (1976), Lyons (1997)] ; What's the Seneta-Heyde norming for W_n ? i.e. find constants c_n such that $\frac{W_n}{c_n}$ converges.
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Aïdékon and Shi (2014)' theorem

Under $\mathbb{P}^*$,

$$\sqrt{n}W_n \overset{(p)}{\to} \sqrt{\frac{2}{\pi\sigma^2}}\, D_\infty.$$

Moreover

$$\limsup_{n\to\infty} \sqrt{n}\, W_n = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Remark : Why the above limsup $= \infty$ *?* Use the inequality

$$\sqrt{n}W_n \geq e^{-(\mathcal{M}_n - \frac{1}{2}\log n)}.$$

The upper limits on W_n

Proposition 3

For any function $f \uparrow \infty$, $\mathbb{P}^*$-almost surely,

$$\limsup_{n\to\infty} \frac{\sqrt{n}\, W_n}{f(n)} = \begin{cases} 0 \\ \infty \end{cases} \iff \int^{\infty} \frac{dt}{tf(t)} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Proofs of Theorem 1 and Proposition 3

Short version :

Read carefully Aïdékon and Shi (2014).

Proofs of Theorem 1 and Proposition 3

Long version :

1. By the elementary inequality : $W_n \geq e^{-\mathcal{M}_n}$, it is enough to prove the convergence part of the integral test for W_n and the divergence part for $\mathcal{M}_n$;
2. Find a maximal inequality for $\mathbb{P}(\max_{n \leq k \leq 2n} W_k > \lambda)$ [for the convergence part on W_n, Proposition 3] ;
3. Use the second moment method for $\mathbb{P}(\min_{n \leq k \leq 2n} \mathcal{M}_k < \lambda)$ [for the divergence part on $\mathcal{M}_n$, Theorem 1].

Estimate $\mathbb{P}(\min_{n\leq k\leq m} \mathcal{M}_k < \lambda)$: Proof of the divergence part in Theorem 1

Lemma (Aïdékon and Shi)

There exist some $K > 1, c(K) > 0$ *such that for all* n*,*

$$\mathbb{P}\Big(\frac{1}{2}\log n \leq \min_{n\leq k\leq 2n} \mathcal{M}_k \leq \frac{1}{2}\log n + K\Big) \geq c(K).$$

Define

$$A(n,\lambda) \;:=\; \Big\{\frac{1}{2}\log n - \lambda \leq \min_{n\leq k\leq 2n} \mathcal{M}_k \leq \frac{1}{2}\log n - \lambda + K\Big\} \cap \Big\{\text{some truncations}\Big\}.$$

A modified version of A&S' lemma

Lemma (Essentially contained in A&S.)

There exists some constant $c > 0$ such that for $0 \le \lambda \le \frac{1}{3}\log n$,

$$c\, e^{-\lambda} \le \mathbb{P}\Big(A(n,\lambda)\Big) \le \frac{1}{c} e^{-\lambda}.$$

Proof : The upper bound by computing the expectation of some quantity (by many-to-one formula) ; the lower bound follows from the second moment computations exactly as in A&S.

Proof of the divergence part in Theorem 1 : The divergence part for $\mathcal{M}_n$

Lemma (Asymptotic independence)

There exists some constant $C > 0$ such that for any $n \geq 2, 0 \leq \lambda \leq \frac{1}{3}\log n$ and $m \geq 4n, 0 \leq \mu \leq \frac{1}{3}\log m$,

$$\mathbb{P}\Big(A(n,\lambda) \cap A(m,\mu)\Big) \leq C\, e^{-\lambda-\mu} + C\, e^{-\mu}\frac{\log n}{\sqrt{n}}.$$

Proof of Lemma. Notice that $A(n,\lambda) \cap A(m,\mu)$ is already a form close to a second moment. Again we use the computations similar to that in Aïdékon and Shi (2014).

Proof of Theorem 1 : the divergence part. Using the above asymptotic independence and the Borel-Cantelli lemma.

Last section : the upper limits of $\mathcal{M}_n$

Recalling

$$\limsup \frac{1}{\log \log \log n} (\mathcal{M}_n - \frac{3}{2} \log n) = 1, \qquad \mathbb{P}^* - \text{a.s.}$$

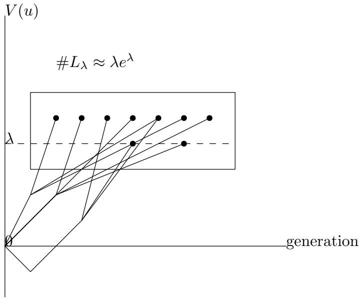
Question : Why $\log \log \log n$ *?*

Proof of Theorem 2 : $\limsup \frac{1}{\log\log\log n}(\mathcal{M}_n - \frac{3}{2}\log n) = 1$

Let

$$L_\lambda := \Big\{ u : \tau_\lambda(u) = |u| \Big\},$$

where $\tau_\lambda(u) := \inf\{0 \le j \le |u| : V(u_j) \ge \lambda\}$. Nerman (1981) [the non-lattice case] and Gatzouras (2000) [the lattice case] say :



Proof of Theorem 2 : $\limsup \frac{1}{\log\log\log n}(\mathcal{M}_n - \frac{3}{2}\log n) = 1$

The strategy to make $\mathcal{M}_n > \frac{3}{2}\log n + \lambda$ is that every particle in L_λ (there are λe^λ particles), evolves normally up to the generation n ; hence

$$\mathbb{P}^*\Big(\mathcal{M}_n > \frac{3}{2}\log n + \lambda,\ \text{good event}\Big) \approx \exp(-c\lambda e^\lambda).$$

A curiosity on the moderate deviations

Moderate deviation of $\mathcal{M}_n$

$$\mathbb{P}^*\left(\mathcal{M}_n > \frac{3}{2}\log n + \lambda\right),$$

for $1 \ll \lambda \ll \log n$.

Multiplicative cascade

Liu (2000)

There is a unique, up to a multiplicative constant, nonnegative nontrivial solution of the cascade equation :

$$D \stackrel{(d)}{=} \sum_{i=1}^{\nu} e^{-X_i} D^{(i)},$$

where $(D^{(i)})_{i \geq 1}$ are i.i.d. copies of D, independent of (X_i) and ν.

Multiplicative cascade

Liu (2001)

Let D_∞ be a solution. Suppose that $\mathbb{P}(\nu = 0) = 0$ and $\mathbb{P}(\nu = 1) \in (0, 1)$. He proved that

$$\mathbb{E}\left[D_\infty^{-a}\right] < \infty \iff a < \gamma,$$

for some positive constant γ.

Remark

We may take D_∞ as the almost sure limit of the so-called derivative martingale in the branching random walk.

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Small deviations for D_∞

Proposition 4a [in the Schröder case]

Assume $\mathbb{P}(\nu \le 1) \in (0,1)$. We have

$$\mathbb{P}\Big(0 < D_\infty < \varepsilon\Big) \asymp \varepsilon^\gamma.$$

Proposition 4b [in the Böttcher case]

Assume $\mathbb{P}(\nu \ge 2) = 1$. There exists some $\beta \in (0,1)$ such that

$$\log \mathbb{P}\Big(D_\infty < \varepsilon\Big) = -\varepsilon^{-\frac{\beta}{1-\beta}+o(1)}.$$

Small deviations for D_∞

Proposition 4a [in the Schröder case]

Assume $\mathbb{P}(\nu \leq 1) \in (0,1)$. We have

$$\mathbb{P}\Big(0 < D_\infty < \varepsilon\Big) \asymp \varepsilon^\gamma.$$

Proposition 4b [in the Böttcher case]

Assume $\mathbb{P}(\nu \geq 2) = 1$. There exists some $\beta \in (0,1)$ such that

$$\log \mathbb{P}\Big(D_\infty < \varepsilon\Big) = -\varepsilon^{-\frac{\beta}{1-\beta}+o(1)}.$$

Moderate deviations for $\mathcal{M}_n$

(in the Schröder case)

We have

$$\mathbb{P}^*\left(\mathcal{M}_n > \frac{3}{2}\log n + \lambda\right) = e^{-(\gamma+o(1))\lambda}, \quad 1 \ll \lambda \ll \log n.$$

(in the Böttcher case)

We have

$$\mathbb{P}\left(\mathcal{M}_n > \frac{3}{2}\log n + \lambda\right) = e^{-e^{(\beta+o(1))\lambda}}, \quad 1 \ll \lambda \ll \log n.$$

Moderate deviations for $\mathcal{M}_n$

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We have

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Questions

- Integral test for the upper limits of $\mathcal{M}_n$?
- Can we remove the $o(1)$ in the moderate deviation for $\mathcal{M}_n$ or in the small deviation for $\mathbb{P}(0 < D_\infty < x)$?
- Last minute information : Liu (2016+) is able to get an exact asymptotic for $\mathbb{P}(0 < D_\infty < x)$ in the polynomial decay case.

Thank you very much !