The Nash Equilibrium on an Evolving Random Network

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Outline

1. Motivation

Consider *an evolving random network (graph)* and *a dynamic game* taking place on the *network*.

- Let $G(t) = (V_t, E_t)$, $t = 0, 1, 2, ...$, be a simple graph-valued process.

- Denoted by $D_k(t) = \sharp\{v : \ d_v(t) = k\}$ the number of the vertices with degree k, where $d_v(t)$ is the degree of vertex v in the graph $G(t)$.

- Let $x_v(t)$ denote the state of vertex v at time t.

Now, we present the model in the following, which has been studied by Chung and Lu (2004), Cooper, Frieze and Vera (2004), and Wu, Dong, Liu and Cai (2008).

The Evolving Random Network: We start with G_0 with a single vertex v_0. At each step $t \geq 1$,

(1) with probability p we add a vertex v to V_{t-1} and form j random edges with probability p_j from v to vertex u in V_{t-1} chosen with probability proportional to $w(x_u(t-1))$, where $\sum_{j=1}^{j_0} p_j = 1$ and $w(.)$ is an increasing function.. If $V_{t-1} = \phi$, we do nothing.

(2) with probability $(\alpha - p)\alpha_i$ we add i random edges to existing vertices, where $\sum_{i=1}^{i_0} \alpha_i = 1$. Both endpoints are chosen independently with probability proportional to their states. If $|V_{t-1}| < 2$, we do nothing.

(3) with probability q we delete a randomly chosen vertex v from V_{t-1}. If $V_{t-1} = \phi$, we do nothing.

(4) with probability $(1 - \alpha - q - \beta)q_k$ we delete $\min\{k, |E_{t-1}|\}$ randomly chosen edges from E_{t-1}, where $\sum_{k=1}^{k_0} q_k = 1$.

(5) with probability β we randomly chosen a vertex v from V_{t-1} such that $x_v(t) = x_v(t-1) + \beta h(x_v(t-1), x_u(t-1)|_{u\sim v})$, where $h(.,.)$ is a neighbor mutually interacting function.

Dynamic Game: Let $x(t) = (x_i(t), x_{-i}(t))$ denotes the vector of states on the evolving random network G_t, where

$$x_{-i}(t) = \{x_0(t), ..., x_{i-1}(t), x_{i+1}(t), ...\}$$

Let ψ_k be the utility function of a vertex i with degree k, which is defined as follows

$$\psi_k(x_i, x_{-i}) = f(x_i, x_{-i}) - c(x_i),$$

where $f(x_i, x_{-i})$ is a payoff function and $c(x_i) \geq 0$ is a cost function.

Nash Equilibrium (NE): If $x_i^* = \arg\max_{x_i}\{E(\psi_k(x_i, X_{-i})|X_i = x_i)\} < \infty$, we call $x^* = (x_i^*, x_{-i}^*)$ a NE in the networks.

Obviusly, $Z(t) = (G(t), X(t)), t = 0, 1, 2, ...,$ is a discrete time non-homogenous Markov process.

Three Problems

- **Is there a limit degree distribution as $t \to \infty$?**

- **What kind of the limit degree distribution ?**

- **Is there a NE as $t \to \infty$?**

2. Degree Distribution and Critical Curve

Let $\mu_1 = \sum_{j=1}^{j_0} jp_j$, $\mu_2 = \sum_{i=1}^{i_0} i\alpha_i$ and $\mu_3 = \sum_{k=1}^{k_0} kq_k$.

Assume that

(I) $\alpha \geq p > q \geq 0$

(II) $q \leq \min\{1-\alpha, 1/2\}$

(III) $\alpha\mu_2 + p(\mu_1 - \mu_2) - (1-\alpha-q)\mu_3 > 0$

(IV) $w(x_i) = d_i, \beta = 0$

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Let $a = 2\mu_2 - \mu_1$, $b = q(3\mu_1 - 2\mu_2) - 2\alpha\mu_2 + 2(1-\alpha-q)\mu_3$ and $d = 2q(\alpha\mu_2 - (1-\alpha-q)\mu_3$.

Theorem 1 Assume that the four conditions (I)-(IV) hold. Let $p_c = p(\alpha, q, \mu_1, \mu_2, \mu_3)$ be a solution to the following equation

$$ax^2 + bx + d = 0$$

satisfying $\alpha > p_c > q$. Then the limit degree distribution $\{P_k, k \geq 1\}$ exits and has the power law and the exponential degree distribution respectively for $q < p < p_c$ and $\alpha \geq p > p_c$.

Obviously, p_c is a parabolic curve when $ad \neq 0$, which can be considered as the critical curve.

3. Nash Equilibrium of Network Game

Assumption V. The players' efforts in the network are the same if they have the same degree, they decide independently their own efforts and each player is mainly affected by his (or her) neighbors on the network.

We next discuss on the existence of NE for three game models (a), (b) and (c), respectively.

The model (a) is defined as follows. Let

$$f[x_i,(x_j)_{j\in N_i}] = \sum_{j\in N_i}(x_i - x_j), \qquad c(x_i) = \alpha(x_i)^\beta,$$

where $\alpha > 0$ and $\beta > 0$ are two finite constants. When $X_i = x_i$, the utility function of i with degree k becomes

$$\psi_k[x_i,(X_j)_{j\in N_i}] = \sum_{j\in N_i}(x_i - X_j) - \alpha(x_i)^\beta.$$

The model (b) is defined as follows. Let

$$f[x_i, (x_j)_{j\in N_i}] = x_i \sum_{j\in N_i} x_j, \qquad c(x_i) = \alpha (x_i)^\beta,$$

where $\alpha > 0$ and $\beta > 0$ are two finite constants. The utility function of i with degree k is

$$\psi_k[x_i, (X_j)_{j\in N_i}] = x_i \sum_{j\in N_i} X_j - \alpha (x_i)^\beta.$$

Theorem 2. *Let* $\{P_k, k = 0, 1, 2, \cdots\}$ *be the limit degree distribution and* ψ_k *black be the utility function in the model (b).Suppose that the conditions (I)-(V) hold and let* i *be a vertex in the network with degree* k*. Then*

(1) there is no NE for $\beta = 1$ *;*

(2) there is a NE (not zero and finite) $x^*(k) = x^*(1)k^{1/(\beta-1)}$ *for* $\beta \neq 1$*, and* $\alpha \geq p > p_c$*;*

(3) there is a NE (not zero and finite) $x^*(k) = x^*(1)k^{1/(\beta-1)}$ *if and only if* $0 < \beta < 1 - \frac{1}{\tau-2}$ *and* $\tau > 3$ *when* $q < p < p_c$ *and* $P_k \sim ck^{\tau}$ *for large* k *;*

(4) we have $x^*(k+1) < x^*(k)$ *and* $E\psi_k^* < E\psi_{k+1}^* \leq 0$ *for* $0 < \beta < 1$*.*

Sketch of proof. We first obtain

$$
\begin{aligned}
E\psi_k(x_i, (X_j)_{j\in N_i}) &= x_i \sum_{j\in N_i} EX_j - \alpha(x_i)^\beta \\
&= kx_i EY - \alpha(x_i)^\beta.
\end{aligned}
$$

Then, we have

$$\frac{\partial E\psi_k}{\partial x_i} = kEY - \alpha\beta(x_i)^{\beta-1}.$$

If $\beta = 1$, there is no NE in the network.
If $\beta \neq 1$, at a NE x^*, we have

$$\frac{\partial E\psi_k}{\partial x_i}\mid_{x^*} = \frac{\partial E\psi_k}{\partial x_i}\mid_{(x_i^*, x_{-i}^*)} = 0.$$

Since $x_i^* = x^*(k)$ and $EY \mid_{x^*} = \sum_{l=1}^\infty q_l x^*(l)$, it follows that

$$\alpha\beta(x^*(k))^{\beta-1} = k\sum_{l=1}^{\infty} q_l x^*(l). \tag{2}$$

If $k=1$, we have

$$\alpha\beta(x^*(1))^{\beta-1} = \sum_{l=1}^{\infty} q_l x^*(l). \tag{3}$$

Substituting (3) into (2), we obtain

$$x^*(k) = k^{\frac{1}{\beta-1}} x^*(1), \tag{4}$$

thus, if $\beta > 1$, we have $x^*(k+1) > x^*(k)$, and if $0 < \beta < 1$, we have $x^*(k+1) < x^*(k)$.
Substituting (4) into (3), we have

$$\alpha\beta(x^*(1))^{\beta-1} = \sum_{l=1}^{\infty} q_l l^{\frac{1}{\beta-1}} x^*(1),$$

so

$$\alpha = \frac{1}{\beta} x^*(1)^{2-\beta} \sum_{l=1}^{\infty} l^{\frac{1}{\beta-1}} q_l.$$

The model (c) is defined as follows. Let

$$f[x_i, (x_j)_{j \in N_i}] = x_i \prod_{j \in N_i} x_j, \qquad c(x_i) = \alpha(x_i)^\beta,$$

where $\alpha > 0$ and $\beta > 0$ are two finite constants. The utility function of i with degree k is

$$\psi_k[x_i, (X_j)_{j \in N_i}] \; = \; x_i \prod_{j \in N_i} X_j - \alpha(x_i)^\beta.$$

Theorem 3. *Let* $\{P_k, k = 0, 1, 2, \cdots\}$ *be the limit degree distribution,* ψ_k *be the utility function in the model (c). Suppose that the conditions (I)-(V) hold and let* i *be a vertex in the network with degree* k*. Then*

(1) if $\beta = 1$*, there is no NE in the network*

(2) if $\beta = 2$ *and* $\alpha \geq p > p_c$*, there is a NE (not zero and finite)* $x^*(k) = \alpha^{-1}(\alpha a x^*(1))^k$ *if and only if* $\alpha > \frac{P_1}{\sum_{j=1}^{\infty} k P_k}$*;*

(3) if $q < p < p_c$ *and* $P_k \sim ck^\tau$ *for large* k *, then there is a NE (not zero and finite)* x^* *if and only if* $0 < \beta < 1 - \frac{1}{\tau - 2}$ *and* $\tau > 3$*.*

4. Conclusions and Problems

- The limit degree distribution $\{P_k, k \geq 1\}$ exits and has the power law and the exponential degree distribution respectively for $q < p < p_c$ and $\alpha \geq p > p_c$.

- Present three network game models and give some sufficient conditions for the existence of NE.

- Consider the limit degree distribution when $W(.)$ is not linear function.

- Consider the NE of network game when the player's decisions depend on other's.

References

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Thank You !