Moderate deviations of additive functionals for lattice gas models

Fuqing Gao

Wuhan University

The 12th Workshop on Markov Processes and Related Topics
July 13-17, 2016; BNU and JSNU

Outline

Let X_t, $t \geq 0$ be an ergodic Markov process with generator $\mathcal{L}$ and invariant measure μ.

- Empirical distribution: $\Gamma_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$.
- Large deviations:

$$\mathbb{P}_\mu(\Gamma_t \in A) \approx \exp\left\{-t \inf_{\nu \in A} J(\nu)\right\}. \quad (1.1)$$

- Moderate deviations:

$$\mathbb{P}_\mu\left(\frac{t}{a(t)}(\Gamma_t - \mu) \in A\right) \approx \exp\left\{-\frac{a^2(t)}{t} \inf_{\nu \in A} Q(\nu)\right\} \quad (1.2)$$

where $a(t) > 0$, $t \geq 0$ such that

$$\lim_{t\to\infty} a(t)/\sqrt{t} = \infty, \qquad \lim_{t\to\infty} a(t)/t = 0. \quad (1.3)$$

- If the process is reversible, then

$$J(\nu) = \mathcal{E}(\sqrt{d\nu/d\mu}), \quad Q(\nu) = \frac{1}{4}\mathcal{E}(d\nu/d\mu), \quad (1.4)$$

where $\mathcal{E}(f) = -\mathbb{E}_\mu(f\mathcal{L}f)$.

- LDP for Markov processes:
  See Donsker, Varadhan(CPAM,CMP 1975-1984), Deuschel, Stroock(Book, 1989),de Acost(AOP,1988), Jain(AOP,1990), Wu(JFA, 2000)).
- MDP for Markov processes:
  geometric ergodicity, spectral gap (cf. Acosta, Chen (JTP,1998), Gao (SPA, 1996), Wu (AOP,1995)).

- If the Markov process is reversible, (1.1) is roughly equivalent to

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_\mu \left(\exp \left\{ \int_0^t V(X_s) ds \right\} \right) = \sup \left\{ \mathbb{E}_\mu (Vf) - \mathcal{E}(\sqrt{f}) \right\}$$

and (1.2) is roughly equivalent to

$$\lim_{t\to\infty} \frac{t}{a^2(t)} \log \mathbb{E}_\mu \left(\exp \left\{ \frac{a(t)}{t} \int_0^t V(X_s) ds \right\} \right)$$
$$= \sup_{\|f\|_2 = 1} \left\{ \mathbb{E}_\mu (Vf) - \mathcal{E}(f)/4 \right\}.$$

- If $\mathcal{L}$ has a spectral gap, then from the standard Rayleigh-Schrödinger perturbation theory,

$$\lim_{t\to\infty} \frac{t^2}{a^2(t)} \lambda \left(\frac{a(t)}{t} V \right) = \langle V, -\mathcal{L}^{-1} V \rangle.$$

where $\lambda(V)$ is the largest eigenvalue of $\mathcal{L} + V$.

- Our motivation is to consider some models which do not exist spectral gap, such as some conservative particle systems (lattice gas model, zero range process, etc.)

- The lattice gas model with interaction rate $c(x, y, \eta)$ is defined as the Markov process $\{\eta_t; t \geq 0\}$ with state space $E = \{0, 1\}^{\mathbb{Z}^d}$ and generated by the operator $\mathcal{L}$,

$$\mathcal{L}f(\eta) = \frac{1}{2d} \sum_{|x-y|=1} c(x, y, \eta) \nabla_{x,y} f(\eta).$$

where

$$\nabla_{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta),$$

$$\eta^{x,y}(z) = \begin{cases} \eta(y), & z = x \\ \eta(x), & z = y \\ \eta(z), & z \neq x, y, \end{cases}$$

- $c(x, y, \cdot) : E \to \mathbb{R}$, $x, y \in \mathbb{Z}^d$ are a family of local functions and let these functions be symmetric and nearest-neighbor, that is,

$$c(x, y, \eta) = c(y, x, \eta), \qquad \text{and } c(x, y, \eta) = 0 \text{ for all } |x-y| \neq 1.$$

(H1) *Translation invariance*. For any $x, y \in \mathbb{Z}^d$, $\eta \in E$,

$$c(x, y, \eta) = c(0, y - x, \tau_x \eta),$$

$\tau_x : E \to E$ is defined by $\tau_x \eta(z) = \eta(z - x)$. $\tau_x f(\eta) = f(\tau_x \eta)$.

(H2) Ellipticity. There is a constant $c_0 > 0$ such that for any $\eta \in E$ and $x, y \in \mathbb{Z}^d$,

$$c_0 \leq c(x, y, \eta) \leq c_0^{-1}.$$

(H3) Detailed balance condition. For any $\eta \in E$, $x, y \in \mathbb{Z}^d$,

$$c(x, y, \eta) = c(x, y, \eta^{x,y}).$$

(H4) Gradient condition. There exist local functions $h^{(i)}$, $i = 1, \dots, d$ such that

$$c(0, e_i, \eta)(\eta(e_i) - \eta(0)) = h^{(i)}(\eta) - \tau_{e_i} h^{(i)}(\eta).$$

where $\{e_1, \cdots, e_d\}$ is the canonical basis in $\mathbb{R}^d$.

- For each $\rho \in [0, 1]$, let ν_ρ denote the product Bernoulli measure in E of density ρ:

$$\nu_\rho\{\eta(x) = 1 \text{ for any } x \in A\} = \rho^{|A|}$$

- Denote by $\mathbb{P}_\rho$ the distribution in $\mathcal{D}([0, \infty), E)$ of the process $\{\eta_t; t \geq 0\}$ with initial distribution ν_ρ, and we denote by $\mathbb{E}_\rho$ the expectation with respect to $\mathbb{P}_\rho$.
- If $c(x, y, \eta) = \eta(x)(1 - \eta(y))$, then $\{\eta_t; t \geq 0\}$ is symmetric exclusion process (SEP). In this case, we can write

$$\mathcal{L}f(\eta) = \frac{1}{2d} \sum_{|x-y|=1} \eta(x)(1 - \eta(y))\nabla_{x,y} f(\eta).$$

- Let $V : E \to \mathbb{R}$ be a local function. For each $\rho \in [0, 1]$, define $\varphi_V(\rho) = \int V d\nu_\rho$.
- The centered additive functional associated to V is defined by

$$\Gamma_T^V(t) = \frac{1}{T} \int_0^{Tt} \left(V(\eta_s) - \varphi_V(\rho) \right) ds, \quad t \in [0, 1]. \tag{1.5}$$

Occupation time:

$$\Gamma_T(t) = \frac{1}{T} \int_0^{Tt} (\eta_s(0) - \rho) ds, \quad t \in [0, 1]. \tag{1.6}$$

- CLT for occupation time of SEP. Kipnis (AIHP,1987). Martingale method.

$$\frac{1}{\beta(d,t)}\int_0^t(\eta_s(0)-\rho)ds \longrightarrow N(0,\sigma^2(\rho,d)) \tag{1.7}$$

where

$$\beta(d,t)=\begin{cases} t^{1/2}, & \text{if } d\geq 3,\\ (t\log t)^{1/2}, & \text{if } d=2,\\ t^{3/4}, & \text{if } d=1. \end{cases} \tag{1.8}$$

- CLT. Sethuraman (AOP, 2000). Martingale method.
- CLT for 1-dim lattice gas model. Gonçalves and Jara (CPAM, 2013). Scaling limit and spectral gap.

- LDP for occupation times of SEP: $d=3$ and $d=1$. Landim (AOP, 1992). $d=3$: (Donsker-Varadhan Theory).

$$\frac{1}{t}\log \mathbb{P}_\rho\left(\frac{1}{t}\int_0^t \eta_s(0)ds \in B\right), \quad d \geq 3.$$

$d=1$. (Scaling limit)

$$\frac{1}{\sqrt{t}}\log \mathbb{P}_\rho\left(\frac{1}{t}\int_0^t \eta_s(0)ds \in B\right)$$

- LDP for occupation times of SEP: $d=2$. Chang, Landim, Lee (AOP, 2004). (Scaling limit).

$$\frac{\log t}{t}\log \mathbb{P}_\rho\left(\frac{1}{t}\int_0^t \eta_s(0)ds \in B\right)$$

$$\sim -\inf_{\beta\in B}\frac{\pi}{2}\left(\sin^{-1}(2\beta-1)-\sin^{-1}(2\rho-1)\right)^2.$$

- Set $c_0(\eta) = c(0, 1, \eta)$ and $c_x = \tau_x c_0$, then we can write

$$\mathcal{L}f(\eta) = \sum_{x \in \mathbb{Z}} c_x(\eta) \nabla_{x,x+1} f(\eta).$$

- For $\rho \in (0,1)$,

$$D(\rho) = \frac{\nu_\rho(c_0(\eta)(\eta(0)-\eta(1))^2)}{\chi(\rho)}, \tag{2.1}$$

where $\chi(\rho) = \rho(1-\rho)$.

- $$\sigma^2(\rho) = \frac{8\chi(\rho)\theta^2}{3\sqrt{2D(\rho)\pi}}. \tag{2.2}$$

- $$I(x) = \frac{1}{2\sigma^2(\rho)} x^2. \tag{2.3}$$

- $$J_\rho(w) = \sup_{\lambda \in C[0,1]^*} \left\{ \langle \lambda, w \rangle - \frac{\sigma^2(\rho)}{2} \int_0^1 \int_0^1 \Sigma(s,t)\lambda(ds)\lambda(dt) \right\}, \tag{2.4}$$

where

$$\Sigma(s,t) = |s|^{3/2} + |t|^{3/2} - |s-t|^{3/2}, \quad s,t \in [0,1].$$

Theorem 2.1

For any closed subset $F \subset \mathbb{R}$,

$$\limsup_{T\to\infty} \frac{\sqrt{T}}{a^2(\sqrt{T})} \log \mathbb{P}_\rho \left(\frac{1}{\sqrt{T}a(\sqrt{T})} \int_0^T (\eta_s(0) - \rho) ds \in F \right) \leq - \inf_{x\in F} I(x),$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{\sqrt{T}}{a^2(\sqrt{T})} \log \mathbb{P}_\rho \left(\frac{1}{\sqrt{T}a(\sqrt{T})} \int_0^T (\eta_s(0) - \rho) ds \in G \right) \geq - \inf_{x\in G} I(x).$$

- If we take $a(t) = t^\alpha$ for some $\alpha \in (1/2, 1)$, then MDP for occupation times can be written roughly as

$$\frac{1}{T^{(\alpha-1/2)}} \log \mathbb{P}_\rho \left(\frac{1}{T^{(1+\alpha)/2}} \int_0^T (\eta_s(0) - \rho) ds \in A \right) \approx - \inf_{x \in A} \frac{3\sqrt{2D(\rho)\pi}}{16\chi(\rho)} x^2.$$

- When a reversible Markov process X_t has a gap, for any bounded function V,

$$\frac{1}{T^{(2\alpha-1)}} \log \mathbb{P}_\mu \left(\frac{1}{T^\alpha} \int_0^T (V(X_s) - \mu(V)) ds \in A \right) \approx - \inf_{x \in A} \frac{x^2}{4\langle V, \mathcal{L}^{-1} V \rangle}.$$

Theorem 2.2

If $\varphi_f'(\rho) \neq 0$, then for any closed subset $F \subset C[0,1]$,

$$\limsup_{T\to\infty} \frac{\sqrt{T}}{a^2(\sqrt{T})} \log \mathbb{P}_\rho\left(\frac{\sqrt{T}}{a(\sqrt{T})}\Gamma_T^V \in F\right) \leq -\inf_{w\in F} J_\rho^V(w), \quad (2.5)$$

and for any open subset $G \subset C[0,1]$,

$$\liminf_{T\to\infty} \frac{\sqrt{T}}{a^2(\sqrt{T})} \log \mathbb{P}_\rho\left(\frac{\sqrt{T}}{a(\sqrt{T})}\Gamma_T^V \in G\right) \geq -\inf_{w\in G} J_\rho^V(w), \quad (2.6)$$

where

$$J_\rho^V(w) = \frac{1}{(\varphi_V'(\rho))^2} J_\rho(w). \quad (2.7)$$

- Set $\eta^{(N)} = \{\eta_{N^2 t}, t \geq 0\}$. Then $\eta^{(N)}$ is a Markov process with state space E and generator $\mathcal{L}_N = N^2 \mathcal{L}$.
- Let $\mathbb{P}_\rho^N$ denote the distribution in $\mathcal{D}([0, \infty), E)$ of the process $\{\eta_t^{(N)}; t \geq 0\}$ with initial distribution ν_ρ, and we denote by $\mathbb{E}_\rho^N$ the expectation with respect to $\mathbb{P}_\rho^N$. Then

$$\mathbb{P}_\rho^N \left(\Gamma_1^V \in \cdot \right) = \mathbb{P}_\rho \left(\Gamma_{N^2}^V \in \cdot \right).$$

- Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of test functions and let $\mathcal{S}'(\mathbb{R})$ be the topological dual of $\mathcal{S}(\mathbb{R})$, corresponding to the space of tempered distributions in $\mathbb{R}$.
- For any $T > 0$, for given any path $\eta = \{\eta_s, s \in [0,T]\} \in D([0,T];E)$, the central empirical density $\mu^N(\cdot,\cdot;\eta)$ is defined by

$$\mu^N(s,x;\eta) = \frac{N}{a(N)} \sum_{i\in\mathbb{Z}} (\eta_s(i) - \rho) 1_{[\frac{i}{N},\frac{i+1}{N})}(x). \tag{2.8}$$

-

$$\int_0^t \langle \mu^N(s), \frac{1}{2\varepsilon} I_{[-\varepsilon,\varepsilon]} \rangle ds = \frac{N}{a(N)} \int_0^t \left(\frac{1}{2N\varepsilon} \sum_{i\in[-N\varepsilon,N\varepsilon)} \eta_s(i) - \rho \right) ds. \tag{2.9}$$

- One block estimate. (Guo, Papanicolaou and Varadhan (CMP, 1988), Kipnis,Olla and Varadhan (CPAM, 1989),Landim (AOP, 1992)).

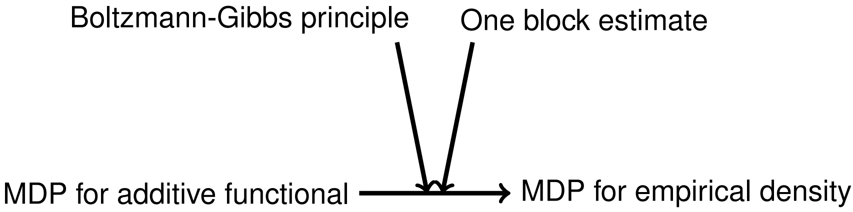
$$\text{To compare } \int_0^T \eta(0)ds$$

$$\text{with } \int_0^T \frac{1}{2l+1} \sum_{|j|\leq l} \eta_s(j)ds.$$

- Boltzmann-Gibbs principle. To extend Boltzmann-Gibbs principle in fluctuations to the moderate deviation case.

$$\text{To compare } \int_0^T (V(\eta) - \varphi_V(\rho))ds$$

$$\text{with } \int_0^T (\eta_s(0) - \rho)ds.$$


Boltzmann-Gibbs principle
One block estimate
MDP for additive functional
MDP for empirical density

Theorem 2.3

For any closed set $C \subset D([0,T], \mathcal{S}'(\mathbb{R}))$,

$$\limsup_{N\to\infty} \frac{N}{a^2(N)} \log \mathbb{P}_\rho^N(\mu^N \in C) \leq -\inf_{\mu\in C} I(\mu) \tag{2.10}$$

and for any open set $O \subset D([0,T], \mathcal{S}'(\mathbb{R}))$,

$$\liminf_{N\to\infty} \frac{N}{a^2(N)} \log \mathbb{P}_\rho^N(\mu^N \in O) \geq -\inf_{\mu\in O} I(\mu) \tag{2.11}$$

where

$$I(\mu) = \sup_{G\in C_K^{2,1}(\mathbb{R}\times[0,T])} \Big\{ l(\mu, G) - \langle \mu_0, G(\cdot, 0)\rangle - \frac{\chi(\rho)}{2}\Big(D(\rho)\int_0^T \int_{\mathbb{R}} \Big|\frac{\partial G}{\partial x}(x,s)\Big|^2 dxds + \int_{\mathbb{R}} |G(x,0)|^2 dx\Big)$$

- Notice that

$$\int_0^1 \langle \mu^N(s), \frac{1}{2\varepsilon} I_{[-\varepsilon,\varepsilon]} \rangle ds \to \frac{1}{a(N)} \int_0^1 (\eta_s(0) - \rho) ds.$$

- From the empirical density to additive functionals.

Approximation method

MDP for empirical density → MDP for additive functional

For two-dimensional symmetric exclusion process, the MDP holds (Gao,Li, Quastel):

- For any closed $F \subset \mathbb{R}$,

$$\limsup_{T\to\infty} \frac{T\log T}{a^2(T)} \log \mathbb{P}_\rho\left(\frac{1}{a(T)}\int_0^T (\eta_s(0)-\rho)ds \in F\right)$$
$$\leq -\inf_{x\in F} \frac{\pi x^2}{4\rho(1-\rho)},$$

- For any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{T\log T}{a^2(T)} \log \mathbb{P}_\rho\left(\frac{1}{a(T)}\int_0^T (\eta_s(0)-\rho)ds \in G\right)$$
$$\geq -\inf_{x\in G} \frac{\pi x^2}{4\rho(1-\rho)},$$

where $\lim_{T\to\infty} \frac{a(T)}{T} = 0, \quad \lim_{T\to\infty} \frac{a(T)}{\sqrt{T\log T}} = \infty.$

Consider an accelerated symmetric simple exclusion process defined by

$$(L_T f)(\eta) = \frac{T}{4} \sum_{\substack{|x-y|=1 \\ x,y\in\mathbb{Z}^2}} \eta(x)(1-\eta(y))(f(\eta^{x,y}) - f(\eta)); \qquad (3.1)$$

and the empirical measures $\mu^T,\ T > 1$ on $\mathbb{R}_+ \times \mathbb{T}_\pi = \mathbb{R}_+ \times [-\pi, \pi)$ in polar coordinates are defined by

$$\mu^T(\eta) = \frac{1}{\log T} \sum_{x\in\mathbb{Z}^2_*} (\eta(x) - \rho) \frac{1}{|x|^2} \delta_{(r_T(x), \Theta(x))},\ T > 1. \qquad (3.2)$$

where $\Theta(v)$ is the angle of v and $r_T :\ \mathbb{Z}^2 \longrightarrow [0, \infty)$ by

$$r_T(0) = 0, \quad \text{and for } x \neq 0,\ r_T(x) = \frac{\log |x|}{\log T}.$$

Let $\mu^{1,T}$ be projection of μ^T on the first coordinate defined by

$$\mu^{1,T}(\eta) = \frac{1}{2\pi \log T} \sum_{x \in \mathbb{Z}_*^2} (\eta(x) - \rho) \frac{1}{|x|^2} \delta_{r_T(x)}. \qquad (3.3)$$

We need to study the LDP for

$$\mathbb{P}_\rho^{\sqrt{T}} \left(\frac{T}{a(T)} \int_0^1 \mu^{1,T}(\eta_s) ds \in \cdot \right).$$

For symmetric exclusion process, if $d \geq 3$, then

Theorem 3.1

For any closed $F \subset \mathbb{R}$,

$$\limsup_{T\to\infty} \frac{T}{a^2(T)} \log \mathbb{P}_\rho \left(\frac{1}{a(T)} \int_0^T (\eta_s(0) - \rho) ds \in F \right)$$

$$\leq - \inf_{x \in F} \frac{x^2}{4\rho(1-\rho)g(0)},$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{T}{a^2(T)} \log \mathbb{P}_\rho \left(\frac{1}{a(T)} \int_0^T (\eta_s(0) - \rho) ds \in G \right)$$

$$\geq - \inf_{x \in G} \frac{x^2}{4\rho(1-\rho)g(0)},$$

Define

$$g(\lambda, x) = \int_0^\infty e^{-\lambda t} p_t(0, x) dt, \quad g(x) = \int_0^\infty p_t(0, x) dt. \tag{3.4}$$

Set

$$M_t(x) = (\eta_t(x) - \alpha) - \int_0^t \mathcal{L}(\eta_s(x) - \alpha) ds, \tag{3.5}$$

and

$$G_\lambda(\eta) = \sum_{x \in \mathbb{Z}^d} g(\lambda, x)(\eta(x) - \rho) = \int_0^\infty e^{-\lambda t}(E_\eta(\eta_t(x)) - \rho) dt. \tag{3.6}$$

Then (cf. Kipnis (AIHP,1987))

$$\int_0^t (\eta_s(0) - \rho) ds = Q_t^\lambda + M_t^\lambda, \tag{3.7}$$

where

$$Q_t^\lambda = -G_\lambda(\eta_t) + G_\lambda(\eta_0) + \int_0^t \lambda G_\lambda(\eta_s) ds, \tag{3.8}$$

and

$$M_t^\lambda = \sum_{x \in \mathbb{Z}^d} g(\lambda, x) M_t(x). \tag{3.9}$$

- For any $\theta \in \mathbb{R}$, $N_t^{\theta,\lambda} := \exp\left\{\theta M_t^\lambda - A_\lambda(\theta, t)\right\}$ is a martingale, where

$$A_\lambda(\theta,t) = \sum_{|x-y|=1} p(x,y)\bigg(e^{\theta(g(\lambda,x)-g(\lambda,y))} \\ - \theta(g(\lambda,x)-g(\lambda,y)) - 1\bigg)\int_0^t \eta_s(x)(1-\eta_s(y))ds.$$

- $\{\sigma_\lambda^2 = \sum_{|x-y|=1} \rho(1-\rho)p(x,y)(g(\lambda,x)-g(\lambda,y))^2, \ \lambda \in (0,1/2]\}$ is bounded.

- For any $\delta > 0$,

$$\limsup_{T \to \infty} \frac{T}{a^2(T)} \log \mathbb{P}_\rho \left(\frac{1}{a(T)} |G_{1/T}| \geq \delta \right) = -\infty.$$

For any closed $F \subset \mathbb{R}$,

$$\limsup_{T\to\infty} \frac{T}{a^2(T)} \log \mathbb{P}_\rho \left(\frac{1}{a(T)} M_T^{1/T} \in F \right) \leq - \inf_{x\in F} \frac{x^2}{4\rho(1-\rho)g(0)},$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{T\to\infty} \frac{T}{a^2(T)} \log \mathbb{P}_\rho \left(\frac{1}{a(T)} M_T^{1/T} \in G \right) \geq - \inf_{x\in G} \frac{x^2}{4\rho(1-\rho)g(0)}.$$

谢谢!