Stationary Measures for Stochastic Lotka-Volterra Systems with Application to Turbulent Convection

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Outline

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1. Introduction
2. Stochastic Decomposition Formula
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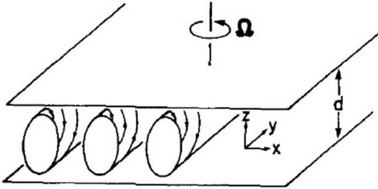
   - The Classification via Stationary Measures
   - Ergodic Properties of Stochastic L-V System

Outline

Motivation: Mathematical Formulation

• Turbulent convection in a fluid layer heated from below and rotating about a vertical axis was studied by Busse et al, (Science, 1980; Nonl. Dyn., 1980).



$$\mathbf{v}_z = F(z, \alpha_c) \sum_{j=-n}^{n} C_j(t) \exp(i\mathbf{k}_j \cdot \mathbf{r}).$$

• The convection model is formulated by the *Navier-Stokes equations* for the velocity vector $\mathbf{v}$ and the heat equation for the deviation θ of the temperature from the static state:

Motivation: Mathematical Formulation

$$\begin{cases} P^{-1}(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\sqrt{T}}{2}\lambda\times\mathbf{v} = -\nabla\pi + \lambda\theta + \nabla^2\mathbf{v} \\ \nabla\cdot\mathbf{v} = 0 \\ (\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla)\theta = R\lambda\cdot\mathbf{v} + \nabla^2\theta \end{cases}, \qquad (1.1)$$

λ: The unit vector in the vertical direction;

$R = \frac{g\gamma(T_2 - T_1)d^3}{\kappa\nu}$: the Rayleigh number;

$T = \frac{4\Omega^2 d^4}{\nu^2}$: the Taylor number; $P = \frac{\nu}{\kappa}$: the Prandtl number.

- Then time-dependent amplitude $C_j(t)$ satisfies L-V equation:

$$M\frac{dC_i}{dt} = C_i\{(R - R_c)K - \frac{1}{2}\sum_{j=-n}^{n} T_{ij} \mid C_j \mid^2\}.$$

Motivation

• It is demonstrated by experiments: when the Rayleigh number R exceeds the critical value R_c depending on the Taylor number T, the static state becomes unstable and convective motions set in.

• Let $n = 3$, seting $S_i = |C_i|^2$.
Special case — standard *symmetric May-Leonard system*:

$$\begin{cases} \frac{dS_1}{dt} = S_1(1 - S_1 - \alpha S_2 - \beta S_3), \\ \frac{dS_2}{dt} = S_2(1 - \beta S_1 - S_2 - \alpha S_3), \\ \frac{dS_3}{dt} = S_3(1 - \alpha S_1 - \beta S_2 - S_3), \end{cases} \tag{1.2}$$

with $\alpha, \beta > 0$.

The possible equilibrium solutions of (1.2) as points: $\diamondsuit$ $O(0,0,0)$; $\heartsuit$ 3 single-species solutions of the form $(1,0,0)$; $\clubsuit$ 3 two-species solutions of the form $(1-\alpha, 1-\beta, 0)/(1-\alpha\beta)$; $\spadesuit$ the three-species equilibrium $(1,1,1)/(1+\alpha+\beta)$.

The eigenvalues λ_i $(i = 1,2,3)$ of the *three-species equilibrium*'s matrix can be written down

$$\lambda_1 = -1-(\alpha+\beta), \quad \lambda_{2,3} = -1+\frac{(\alpha+\beta)}{2} \pm \frac{\sqrt{3}}{2}(\alpha-\beta)i.$$

- This equilibrium is stable if $\alpha+\beta \le 2$;
  It is asymptotical stable if and only if $\alpha+\beta < 2$.
- It is unstable if $\alpha+\beta > 2$.

Stability Properties of (1.2) as a Function of α and β

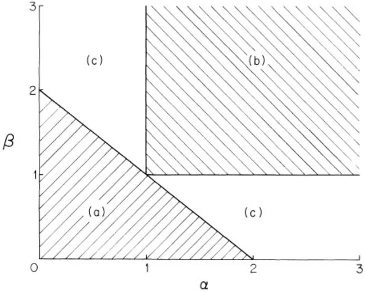


Figure: In the domain (a) the stable equilibrium point is that with all three populations present; in the domain (b) the 3 single-species equilibrium points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are all stable, and which one the system converges to depends on the initial conditions; in the domain (c) there is no asymptotically stable equilibrium point, and periodic solutions, as well.

Periodic Orbit: $\alpha + \beta = 2$

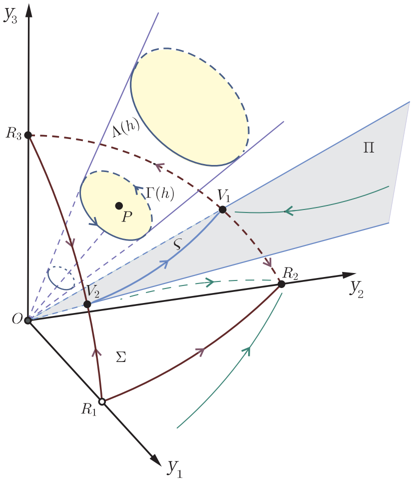


Figure: The global phase portraits for a system (1.2) with $\alpha + \beta = 2$.

Statistical Limit Cycle: $\alpha + \beta > 2$

- Statistical limit cycle occurs ((1.2) with $\alpha + \beta > 2$):

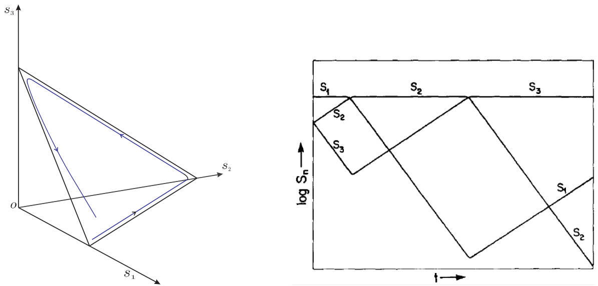


Figure: 2. The development in time of the three roll orientations in the absence of a low level noise source.

Motivation

• Heikes and Busse (Nonl. Dyn., 1980) showed that the randomness occurs for Rayleigh number R close to the critical value for the onset of convection, R_c.

• They expected that the transition from one set of rolls (stationary solutions) to the next becomes nearly periodic, with a transition time which fluctuates statistically about a mean value.

• *This stimulates us to exploit that whether stochastic version of cyclically fluctuating solution (limit cycle) exists when $R - R_c$ is perturbed by a white noise ——*

$$(R - R_c) + \sigma B_t,$$

where B is a Brownian motion.

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Stochastic Lotka-Volterra Systems

We consider the perturbed system

$$(\mathrm{E}_\sigma): dy_i = y_i(r + \sum_{j=1}^{n} a_{ij} y_j)dt + \sigma y_i \circ dB_t, \ \ i = 1, 2, ..., n$$

on the positive orthant $\mathbf{R}^n_+$, where $r = (R - R_c)K$, $a_{ij} = -T_{ij}$ and σ are parameters, $\circ$ denotes Stratonovich stochastic integral.

Stochastic Decomposition Formula

- Auxiliary equation (1-D Stochastic Logistic equation)

$$dg = g(r - rg)dt + \sigma g \circ dB_t. \tag{2.1}$$

The following theorem play an important role to analysis the ergodic properties of equation (E_σ).

Theorem 1 (Stochastic Decomposition Formula)

Let $\Phi(t,\omega,y)$ *and* $\Psi(t,y)$ *be the solutions of* (E_σ) *and* (E_0)*, respectively. Then*

$$\Phi(t,\omega,y) = g(t,\omega,g_0)\Psi(\int_0^t g(s,\omega,g_0)ds, \frac{y}{g_0}),\ \ y \in \mathbf{R}_+^n,\ \ g_0 > 0, \tag{2.2}$$

where $g(t,\omega,g_0)$ *is a positive solution of the Logistic equation (2.1).*

• Of course, the same conclusion (2.2) remains true, if we understand the stochastic equation (E_σ) and (2.1) in the *Itô sense*. Also, the result given above remain true for all $a_{ij} \in \mathbf{R}$.

• In following, we only pay attention to Stratonovitch stochastic integral, since Stratonovitch stochastic integral has some simplifications in formulas. Also, there is a simple relation between the Itô and Stratonovich cases.

• (Proof) It can be checked by Itô's extension formula.

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Intuition

Stochastic Chaos in Trajectory.

From the Stochastic Decomposition Formula it follows that

$$\Phi(t, \theta_{-t}\omega, y) = g(t, \theta_{-t}\omega, 1)\Psi(\int_0^t g(s, \theta_{-t}\omega, 1)ds, y).$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$u(\omega) \qquad\qquad \text{chaos}$$

Here u is an an *equilibrium* (or *stationary solution*) of RDS generated by (2.1).

• Roughly speaking, that *complexity* of the deterministic case which describes the stochastic case.

The Complete Classification for 3-Dim Stochastic L-V System

In this section we focus on three dimensional stochastic L-V system:

$$\begin{aligned} dy_1 &= y_1(r - a_{11}y_1 - a_{12}y_2 - a_{13}y_3)dt + \sigma y_1 \circ dB_t, \\ dy_2 &= y_2(r - a_{21}y_1 - a_{22}y_2 - a_{23}y_3)dt + \sigma y_2 \circ dB_t, \quad (3.0)_\sigma \\ dy_3 &= y_3(r - a_{31}y_1 - a_{32}y_2 - a_{33}y_3)dt + \sigma y_3 \circ dB_t. \end{aligned}$$

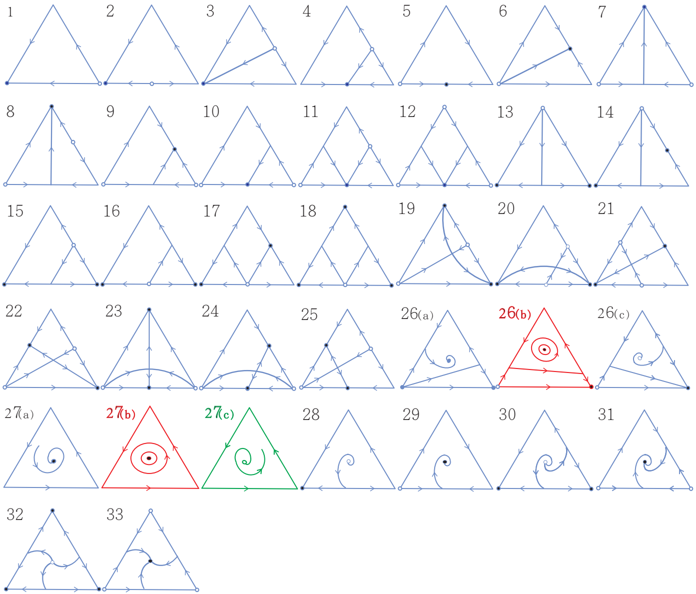
Here $r > 0, a_{ij} > 0,\ i, j = 1, 2, 3.$

- We deal with $a_{ij} > 0$ only, in this case the system $(3.0)_\sigma$ is called *competitive*. However several results given below remain true for all $a_{ij} \in \mathbf{R}$.
- When $\sigma = 0$, the above system $(3.0)_0$ becomes **deterministic** *competitive L-V system*, which can be classified by the parameters a_{ij}.

Theorem 2 (Chen, Jiang and Niu, SIADS, 2015)

There are exactly 37 dynamical classes in 33 stable nullcline classes for deterministic system $(3.0)_0$.
(1) All trajectories tend to equilibria for classes 1-25, 26 a), 26 c), 27 a) *and* 28-33;
(2) a center on Σ *only occurs in* 26 b) *and* 27 b);
(3) the heteroclinic cycle attracts all orbits except ray-$L(P)$ *in class* 27 c).

All are depicted on Σ (called carrying simplex—see, Hirsch, 1988) and presented in following Figure.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- $L(P) := \{\lambda P : \lambda \geq 0\}$ for any $P \in \mathbf{R}^3_+ \setminus \{O\}$;
- $\mathcal{E}$: the equilibrium set for $(3.0)_0$;
- $\mathcal{A}(Q)$: the attracting domain for an equilibrium $Q \in \mathcal{E}$;
- $P(t, y, A)$: the transition probability function is defined by $P(t, y, A) := \mathbb{P}(\Phi(t, \omega, y) \in A)$.

Theorem 3

Let $Q \in \mathcal{E} \setminus \{O\}$, then $\mu^\sigma_Q(A) = \mathbb{P}(U \in A)$ is a **stationary measure** *of semigroup $\{P_t\}_{t\geq 0}$, where $U(\omega) := u(\omega)Q$. Furthermore, (i) for each $y \in \mathcal{A}(Q)$, $P(t, y, \cdot) \xrightarrow{w} \mu^\sigma_Q$ as $t \to \infty$, and*

$$\lim_{t\to\infty} P(t, y, A) = \mu^\sigma_Q(A), \text{ for any } A \in \mathcal{B}(\mathbf{R}^3_+). \tag{3.1}$$

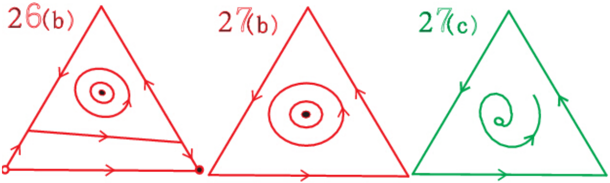
Hence, it is **ergodic** *when the system is restricted on $\mathcal{A}(Q)$. (ii) $\mu^\sigma_Q(\cdot) \xrightarrow{w} \delta_Q(\cdot)$ as $\sigma \to 0$.*
These results are available for classes 1-25, 26 a), 26 c), 27 a) and 28-33 when we restrict the state space in its stable manifold.

The Complete Classification via Stationary Measures

Theorem 4

Suppose that $(3.0)_0$ *is one of systems in classes* 1-25, 26 a), 26 c), 27 a) *and* 28-33. *Then*
(1) all its stationary measures are the convex combinations of ergodic stationary measures $\{\mu_Q^\sigma : Q \in \mathcal{E}\}$;
(2) as $\sigma \to 0$, *all their limiting measures are the convex combinations of the Dirac measures* $\{\delta_Q(\cdot) : Q \in \mathcal{E}\}$.

Recall the following dynamical classes (red—Periodic case and green—Heterclinic case).



Theorem 5

Suppose that $(3.0)_0$ *is one of systems in classes 26 b) and 27 b).*
(1) Then there exists a unique **ergodic** *nontrivial stationary measure* ν_h^σ *supporting on the cone*

$$\Lambda(h) : V(y) := y_1^\mu y_2^\nu y_3^\omega(\beta_2\alpha_3 y_1 + \alpha_1\alpha_3 y_2 + \beta_1\beta_2 y_3) \equiv h \in I, \quad (3.2)$$

where $\mu = -\beta_2\beta_3/D$, $\nu = -\alpha_1\alpha_3/D$, $\omega = -\alpha_1\beta_2/D$,
$D = (\beta_2\beta_3 + \beta_2\alpha_1 + \alpha_1\alpha_3)$, α_i, β_i *are expressed by* a_{ij}, I *is the feasible image interval for* V *and* $\Gamma(h)$ *is the closed orbit, with initial data* $y_0 \in \Gamma(h)$.
(2) ν_h^σ *converges weakly to the Haar measure on the closed orbit* $\Gamma(h)$ *as* $\sigma \to 0$.

Sketch of Proof

Define $\psi: \ \Lambda(h) \setminus \{O\} \to \mathbf{R} \times S$ by

$$\psi(y) := \Big(\ln \lambda,\ \varphi(z)\Big),\ y \in \Lambda(h) \setminus \{O\},$$

where $\varphi(y) = \inf\{t > 0,\ \Psi(t, y_0) = y\}$. Obviously, ψ is a homeomorphism. Set

$$H(t,\omega,H_0) = \ln(g(t,\omega,\lambda)) \text{ and } T(t,\omega,T_0) = \varphi(\Psi(\int_0^t g(s,\omega,\lambda)ds, z).$$

By the definition,

$$\psi(\Phi(t,\omega,y)) = \Big(H(t,\omega,H_0),\ T(t,\omega,T_0)\Big).$$

The ergodicity for Φ on $\Lambda(h) \setminus \{O\}$ is equivalent to that (H,T) is ergodic on $\mathbf{R} \times S$.

Sketch of Proof

We can prove that (H, T) is *strong Feller*(SF) and *irreducible*(I) on $\mathbf{R} \times S$, this is

(SF) For any $t > 0$, and $F \in \mathcal{B}_b(\mathbf{R} \times S)$,

$$(H_0, T_0) \in \mathbf{R} \times S \to \mathbb{E}F(H(t, H_0), T(t, T_0)) \text{ is continuous;}$$

(I) For any $t > 0$, $(H_0, T_0) \in \mathbf{R} \times S$ and open set $A \in \mathcal{B}(\mathbf{R} \times S)$,

$$\mathbb{P}\Big((H(t, H_0), T(t, T_0)) \in A\Big) > 0.$$

Sketch of Proof

This implies that Φ is ergodic on $\Lambda(h) \setminus \{O\}$. Furthermore, Φ is also ergodic on $\Lambda(h)$ and ν_h^σ is an ergodic stationary measure for Φ on $\mathbf{R}_+^3$. Finally, ν_h^σ converges weakly to the Haar measure on the closed orbit $\Gamma(h)$ as $\sigma \to 0$.

The Complete Classification via Stationary Measures

Theorem 6

Suppose that $(3.0)_0$ is one of systems in classes 26 b) and 27 b). Let $\mu^i := \nu_{h^i}^{\sigma^i}$, $i = 1, 2, \cdots$ satisfy $\sigma^i \to 0$ and $\mu^i \xrightarrow{w} \mu$ as $i \to \infty$. Then

(1) if initial data y_0 lies in the interior of the heteroclinic cycle $\mathcal{H}$, then μ is the Haar measure on $\Gamma(y_0)$ for $y_0 \neq P$, or the Dirac measure $\delta_P(\cdot)$ at P for $y_0 = P$;

(2) if initial data $y_0 \in \mathcal{H}$, then

$$\mu(\{E_1, E_2, E_3\}) = 1, \tag{3.3}$$

where E_1, E_2, E_3 are three equilibria of heteroclinic cycle $\mathcal{H}$ in class 26 b) or class 27 b).

The Complete Classification via Stationary Measures

Theorems 4–6 have given all ergodic stationary measures for all classes except class 27 c).

Theorem 7

Assume that $(3.0)_0$ is the system of class 27 c). Then (i) ν_y^σ will support on the three nonnegative axes for any ν_y^σ with $y \in \mathrm{Int}\mathbf{R}_+^3 \backslash L(P)$; (ii) Let $\mu^i := \nu_{y_0^i}^{\sigma^i}$, $i = 1, 2, \cdots$. If $\mu^i \xrightarrow{w} \mu$ as $\sigma^i \to 0$, $i \to \infty$. Then

$$\mu(\{R_1, R_2, R_3\}) = 1, \tag{3.4}$$

where R_1, R_2, R_3 are three axial equilibria for the deterministic system.

Conclusion

- Theorem 7 only describes the support of stationary measures.
- The nonergodicity can be found by the stochastic turbulence. (Why?)
- It is essential reason to reveal that solutions concentrate around R_1, R_2, R_3 very long time (approximately infinite) with probability nearly one.

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- It is essential reason to reveal that solutions concentrate around R_1, R_2, R_3 very long time (approximately infinite) with probability nearly one.

▲ We reveal the reason for this special case (WLOG, let $\alpha = 0.8$ and $\beta = 1.3$ for symmetric May-Leonard system).
Let

$$A_i = \{y = (y_1, y_2, y_3) \in \Sigma : \|y - R_i\| < \frac{1}{2}\}$$

denote the neighborhood of R_i $(i = 1, 2, 3)$. Then $\Psi(t, y)$ will enter and then depart A_i with infinite times. For $n \geq 2$, define

$$\begin{aligned}
T_{\text{in}}^1 &= \inf\{t \geq 0,\ \Psi(t, y) \in A_1\}, & T_{\text{out}}^1 &= \inf\{t \geq T_{\text{in}}^1,\ \Psi(t, y) \notin A_1\}, \\
T_{\text{in}}^n &= \inf\{t \geq T_{\text{out}}^{n-1},\ \Psi(t, y) \in A_1\}, & T_{\text{out}}^n &= \inf\{t \geq T_{\text{in}}^n,\ \Psi(t, y) \notin A_1\}, \\
S_{\text{in}}^1 &= \inf\{t \geq T_{\text{out}}^1,\ \Psi(t, y) \in A_3\}, & S_{\text{out}}^1 &= \inf\{t \geq S_{\text{in}}^1,\ \Psi(t, y) \notin A_3\}, \\
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\end{aligned}$$

Similarly, we denote by τ_{in}^n and τ_{out}^n the time entering and exiting A_2 in n-th spiral cycle (see Fig. 5).

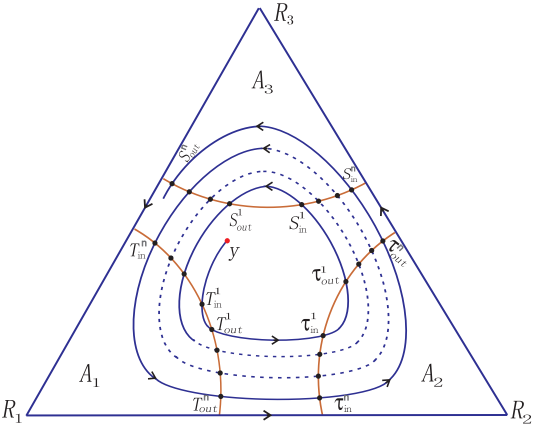

Figure: 5. The phase portrait of Ψ

Deterministic Case

May and Leonard (SIAP, 1975) gave the following estimation:

$$T^n_{\text{out}} - T^n_{\text{in}} \simeq 0.42 T^n_{\text{out}}, \ \tau^n_{\text{out}} - \tau^n_{\text{in}} \simeq 0.42 \tau^n_{\text{out}}, \ S^n_{\text{out}} - S^n_{\text{in}} \simeq 0.42 S^n_{\text{out}}. \quad (3.5)$$

Then

$$\frac{1}{T^n_{\text{out}}} \int_0^{T^n_{\text{out}}} \delta_{\Psi(t,y)}(A_1)dt = \frac{1}{T^n_{\text{out}}} \sum_{i=1}^{n} (T^i_{\text{out}} - T^i_{\text{in}}) \geq \frac{T^n_{\text{out}} - T^n_{\text{in}}}{T^n_{\text{out}}} = 0.42 > 0,$$

$$\frac{1}{S^n_{\text{out}}} \int_0^{S^n_{\text{out}}} \delta_{\Psi(t,y)}(A_1)dt = \frac{1}{S^n_{\text{out}}} \sum_{i=1}^{n} (T^i_{\text{out}} - T^i_{\text{in}}) \leq \frac{T^n_{\text{out}}}{S^n_{\text{out}}} \leq (0.58)^2 \leq 0.34.$$

This implies that the limit of occupation measure of $\Psi(t, y)$ is not unique.

Stochastic Case

For stochastic case, similarly, we analyze

$$\Big\{\frac{1}{T}\int_0^T I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1)ds, y)\Big)dt\Big\}_{T>0} \quad \text{as} \quad T\to\infty.$$

Let $\epsilon = 0.0001$ and $\Omega_T^\epsilon = \{\omega : \sup_{t\in[T,\infty)} |\frac{1}{t}\int_0^t g(s,\omega,1)ds - 1| \le \epsilon\}$. Then $\Omega_T^\epsilon \uparrow$ with respect to T and $\lim_{T\to\infty}\mathbb{P}(\Omega_T^\epsilon) = 1$. Thus for $\eta = 0.9999$, there exists $T_0 > 0$ such that

$$\mathbb{P}(\Omega_T^\epsilon) \ge \eta, \quad \forall T \ge T_0.$$

Stochastic Case

Define

$$t_1^n(\omega) := \tau(\omega, T_{\text{in}}^n) := \inf\{t > 0 : \int_0^t g(s, \omega, g_0)ds > T_{\text{in}}^n\},$$

$$t_2^n(\omega) := \tau(\omega, T_{\text{out}}^n) := \inf\{t > 0 : \int_0^t g(s, \omega, g_0)ds > T_{\text{out}}^n\}.$$

Set $\Omega_{T_0}^n := \{\omega : t_1^n(\omega) \geq T_0\}$. Then $\Omega_{T_0}^n \uparrow$ with respect to n and $\lim_{n\to\infty} \mathbb{P}(\Omega_{T_0}^n) = 1$. Thus there exists an N_0 such that

$$\mathbb{P}(\Omega_{T_0}^n) \geq \eta, \quad \forall n \geq N_0.$$

Stochastic Case

Step 1. Let $T_n = T^n_{\text{out}}$. Consider $\frac{1}{T_n}\int_0^{T_n} I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1)ds, y)\Big)dt$. For any n satisfying $n \geq N_0$ and $T_n \geq T_0$, choosing any $\omega \in \Omega^n_{T_0} \cap \Omega^\epsilon_{T_0}$, we have

- $(1-\epsilon)T^n_{\text{out}} = (1-\epsilon)T_n \leq \int_0^{T_n} g(s,\omega,1)ds \leq (1+\epsilon)T_n = (1+\epsilon)T^n_{\text{out}}$,
- $t^n_2(\omega) \geq t^n_1(\omega) \geq T_0$,
- $(1-\epsilon)t^n_1(\omega) \leq \int_0^{t^n_1(\omega)} g(s,\omega,1)ds = T^n_{\text{in}} \leq (1+\epsilon)t^n_1(\omega)$,
- $(1-\epsilon)t^n_2(\omega) \leq \int_0^{t^n_2(\omega)} g(s,\omega,1)ds = T^n_{\text{out}} \leq (1+\epsilon)t^n_2(\omega)$.

Combining the fact that $T^n_{\text{out}} - T^n_{\text{in}} \simeq 0.42T^n_{\text{out}}$, we have

$$t^n_1(\omega) \leq T^n_{\text{out}} = T_n, \quad t^n_2(\omega) \geq \frac{T^n_{\text{out}}}{1+\epsilon} = \frac{T_n}{1+\epsilon}, \quad \frac{T^n_{\text{in}}}{1-\epsilon} \geq t^n_1(\omega).$$

Stochastic Case

Continued. Therefore

$$
\begin{aligned}
&\frac{1}{T_n}\int_0^{T_n} I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1)ds, y)\Big)dt \\
=&\frac{1}{T_n}\sum_{i=1}^{\infty}\Big(t_2^i(\omega)\bigwedge T_n - t_1^i(\omega)\bigwedge T_n\Big) \\
\geq&\frac{t_2^n(\omega)\bigwedge S_n - t_1^n(\omega)}{S_n} \\
\geq&\frac{\frac{T_{\text{out}}^n}{1+\epsilon}-\frac{T_{\text{in}}^n}{1-\epsilon}}{T_{\text{out}}^n} \geq 0.419.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\underline{\lim}_{n\to\infty}\frac{1}{T_n}\int_0^{T_n}\mathbb{E}I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1)ds, y)\Big)dt \\
\geq\ & 0.419\mathbb{P}(\Omega_{T_0}^{N_0}\cap\Omega_{T_0}^{\epsilon}) \geq 0.419\times 0.9998 \geq 0.41. \qquad (3.6)
\end{aligned}
$$

Stochastic Case

Step 2. Let $S_n = S^n_{\text{out}}$. Consider $\frac{1}{S_n}\int_0^{S_n} I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1),y)\Big)dt$. For any n satisfying $n \geq N_0$ and $S_n \geq T_0$, choosing any $\omega \in \Omega^n_{T_0} \cap \Omega^{\epsilon}_{T_0}$, we have

- $(1-\epsilon)S^n_{\text{out}} = (1-\epsilon)S_n \leq \int_0^{S_n} g(s,\omega,1)ds \leq (1+\epsilon)S_n = (1+\epsilon)S^n_{\text{out}}$,
- $t_2^{n+1}(\omega) \geq t_1^{n+1}(\omega) \geq t_2^n(\omega) \geq t_1^n(\omega) \geq T_0$,
- $(1-\epsilon)t_1^i(\omega) \leq \int_0^{t_1^i(\omega)} g(s,\omega,1)ds = T^i_{\text{in}} \leq (1+\epsilon)t_1^i(\omega), \quad i = n, n+1$,
- $(1-\epsilon)t_2^i(\omega) \leq \int_0^{t_2^i(\omega)} g(s,\omega,1)ds = T^i_{\text{out}} \leq (1+\epsilon)t_2^i(\omega), \quad i = n, n+1$,
- $T^{n+1}_{\text{in}} \simeq S^n_{\text{out}}$, $S^n_{\text{out}} - S^n_{\text{in}} \simeq 0.42S^n_{\text{out}}$, $S^n_{\text{in}} - T^n_{\text{out}} \simeq 0.42S^n_{\text{in}}$.

Stochastic Case

Continued. Hence

- $(1-\epsilon)t_2^n(\omega) \le T_{\text{out}}^n \simeq 0.58 S_{\text{in}}^n \simeq 0.58^2 S_{\text{out}}^n = 0.58^2 S_n \Rightarrow t_2^n(\omega) \le S_n$,
- $$\begin{aligned} & (1-\epsilon)t_1^{n+1}(\omega) \le T_{\text{in}}^{n+1} \simeq S_{\text{out}}^n = S_n \\ \le \quad & (1+\epsilon)t_1^{n+1}(\omega) \le \frac{1+\epsilon}{1-\epsilon} T_{\text{in}}^{n+1} \simeq 0.58 \frac{1+\epsilon}{1-\epsilon} T_{\text{out}}^{n+1} \\ \le \quad & 0.58 \frac{(1+\epsilon)^2}{1-\epsilon} t_2^{n+1}(\omega) < t_2^{n+1}(\omega), \end{aligned}$$

  that is,

  $$(1-\epsilon)t_1^{n+1}(\omega) \le S_n \le (1+\epsilon)t_1^{n+1}(\omega) < t_2^{n+1}(\omega).$$

Stochastic Case

Continued. Therefore

$$
\begin{aligned}
& \frac{1}{S_n}\int_0^{S_n} I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1)ds, y)\Big)dt \\
= \; & \frac{1}{S_n}\sum_{i=1}^{\infty}\Big(t_2^i(\omega)\bigwedge S_n - t_1^i(\omega)\bigwedge S_n\Big) \\
= \; & \frac{1}{S_n}\Big[\sum_{i=1}^{n}\Big(t_2^i(\omega) - t_1^i(\omega)\Big) + \Big(S_n - t_1^{n+1}(\omega)\bigwedge S_n\Big)\Big] \\
\le \; & \frac{t_2^n(\omega) + S_n - t_1^{n+1}(\omega)\bigwedge S_n}{S_n} \\
\le \; & \frac{1}{S_n}\Big(\frac{0.58^2}{1-\epsilon}S_n + S_n - \frac{S_n}{1+\epsilon}\Big) \\
= \; & \frac{0.58^2}{1-\epsilon} + \frac{\epsilon}{1+\epsilon} < 0.34.
\end{aligned}
$$

Stochastic Case

Continued. Then

$$\begin{aligned}&\overline{\lim}_{n\to\infty}\frac{1}{S_n}\int_0^{S_n}\mathbb{E}I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1)ds,y)\Big)dt\\ \le&0.34\mathbb{P}(\Omega_{T_0}^{N_0}\cap\Omega_{T_0}^{\epsilon})+\mathbb{P}\Big((\Omega_{T_0}^{N_0}\cap\Omega_{T_0}^{\epsilon})^c\Big)\le 0.342.\end{aligned} \tag{3.7}$$

Inequalities (3.6) and (3.7) imply that $\frac{1}{T}\int_0^T\mathbb{E}I_{A_1}\Big(\Psi(\int_0^t g(s,\omega,1),y)\Big)dt$ does not have unique limit as $T\to\infty$. Equivalently, $\frac{1}{T}\int_0^T\mathbb{E}I_{\Lambda(A_1)}\Big(\Phi(t,\omega,,y)\Big)dt$ does not have unique limit as $T\to\infty$.

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Thank you!