Intrinsic Ultracontractivity for Symmetric Jump Processes on Unbounded Open Sets

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Based on a on-going work with Panki Kim and Jian Wang

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Outline

1. **Introduction**
   - IU for Dirichlet Semigroups
   - Some known results
   - Main tools
2. **Main Results**
   - Criterion for the compactness
   - Criterion for IU
   - Estimates for ground state

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Symmetric jump processes killed on exiting a domain

- Let $D \subseteq \mathbb{R}^d$ be a connected domain.
- A symmetric non-local Dirichlet form $(\mathscr{E}_D, \mathscr{D}(\mathscr{E}_D))$

$$\mathscr{E}_D(f,f) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(f(x) - f(y)\big)^2 J(x,y)\,dx\,dy,\ f \in \mathscr{D}(\mathscr{E}_D).$$

- $\mathscr{D}(\mathscr{E}_D) := \overline{C_0^1(D)}^{\|\cdot\|_{\mathscr{E}_{D,1}}}$, $\mathscr{E}_{D,1}(f,f) := \mathscr{E}_D(f,f) + \int_D f^2(x)\,dx$.
- $J(x,y)$ is a nonnegative symmetric measurable function on $\mathbb{R}^d \times \mathbb{R}^d$.

Symmetric jump processes killed on exiting a domain

- When $D = \mathbb{R}^d$, $(\mathscr{E}, \mathscr{D}(\mathscr{E})) := (\mathscr{E}_{\mathbb{R}^d}, \mathscr{D}(\mathscr{E}_{\mathbb{R}^d}))$.
- There exists a Markov process $(X_\cdot, \mathbb{P}^x, x \in \mathbb{R}^d)$ associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.
- Heat kernel $\mathbb{P}^x(X_t \in dy) = p(t,x,y)\,dy = p(t,y,x)\,dy$.
- Suppose for every $t > 0$, $p(t,\cdot,\cdot)$ is continuous, and

$$0 < p(t,x,y) \leqslant c(t), \quad t > 0, x, y \in \mathbb{R}^d.$$

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Symmetric jump processes killed on exiting a domain

- Let $\tau_D := \inf\{t \geqslant 0 : X_t \notin D\}$ and

$$X_t^D := \begin{cases} X_t, & \text{if } t < \tau_D, \\ \partial, & \text{if } t \geqslant \tau_D. \end{cases}$$

- Dirichlet heat kernel

$$p^D(t,x,y) = p(t,x,y) - \mathbb{E}^x\left[p(t-\tau_D, X_{\tau_D}, y)1_{\{t \geqslant \tau_D\}}\right], \quad x, y \in D;$$
$$p^D(t,x,y) = 0, \quad x \notin D \text{ or } y \notin D.$$

- $p^D(t,x,y)$ is continuous and

$$0 < p^D(t,x,y) \leqslant p(t,x,y) \leqslant c(t), \quad t > 0, x, y \in \mathbb{R}^d.$$

Symmetric jump processes killed on exiting a domain

- Dirichlet semigroup

$$T_t^D f(x) = \int_D p^D(t,x,y) f(y)\,dy, \quad t>0, x\in D, f\in L^2(D;dx).$$

- $(X_\cdot^D, \mathbb{P}^x, x\in D)$ is the Markov process associated with $(\mathscr{E}_D, \mathscr{D}(\mathscr{E}_D))$.

Symmetric jump processes killed on exiting a domain

- Suppose $\{T_t^D\}_{t\geqslant 0}$ is compact. Then there exists $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$ with $\lim_{n\to\infty}\lambda_n=+\infty$ and $\{\phi_n\}_{n=1}^{\infty}\subseteq L^2(D;dx)$ such that

$$T_t^D\phi_n(x)=e^{-\lambda_n t}\phi_n(x).$$

- The corresponding eigenfunction ϕ_1 can be chosen to be bounded, continuous and strictly positive on D.
- ϕ_1 is usually called ground state.

Intrinsically Ultracontractivity

- We say $(T_t^D)_{t\geqslant 0}$ is intrinsic ultracontractive(IU), if for every $t>0$, there exist positive constants $c(t)$ and $C(t)$ such that

$$p^D(t,x,y)\leqslant C(t)\phi_1(x)\phi_1(y),\qquad x,y\in D.$$

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- Davies and Simon (1984) for symmetric setting; Kim and Song (2008) for non-symmetric setting:

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Outline

Stable process killed on a bounded domain

- $$J(x,y) = \frac{C(d,\alpha)}{|x-y|^{d+\alpha}}.$$
- (Chen, Z.-Q., Song, R; 1998,2000)
- (Kulczycki, T; 1998)

General jump process killed on a bounded domain

- $(X_t)_{t \geqslant 0}$ is a general (not necessarily symmetric) Lévy process.
- (Kim, P. and Song, R. 2007)
- (Grzywny, T.; 2008)
- (Chen, X. and Wang, J. 2015)

Stable process killed on a unbounded domain (Kwasnicki, M. 2009)

- $$J(x,y) = \frac{C(d,\alpha)}{|x-y|^{d+\alpha}}.$$
- Let
$$g(x) := \int_{D^c} \frac{1}{|x-y|^{d+\alpha}}\,dy.$$
- $(T_t^D)_{t\geqslant 0}$ is compact if
$$\lim_{|x|\to\infty} g(x) = \infty.$$
- $(T_t^D)_{t\geqslant 0}$ is IU if
$$\lim_{|x|\to\infty} \frac{g(x)}{\log|x|} = \infty.$$

Stable process killed on a unbounded domain (Kwasnicki, M. 2009)

- For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, let $\tilde{x} = (x_2, x_3, \ldots, x_d)$. Suppose that a bounded function $f : (0, \infty) \to [0, \infty)$ is lower semi-continuous and that $\lim_{u\to\infty} f(u) = 0$. Then open set $D_f = \{x \in \mathbb{R}^d : x_1 > 0, |\tilde{x}| < f(x_1)\}$ is called the horn-sharped region.
- $(T_t^{D_f})_{t\geqslant 0}$ is IU if and only if

$$\lim_{|x|\to\infty} f(x_1)^{\frac{\alpha}{2}} \log |x| = 0.$$

- If $f \in C^{1,1}$ satisfies some increasing condition, then

$$\phi_1(x) \asymp f(x_1)^{\frac{\alpha}{2}} |x|^{-d-\alpha}.$$

- For Brownian motion case, see Davies, E.B., Simon, B.; 1984 , Banuelos, R., Davis, B.; 1992.

Questions

- (Q1): For more general jumping kernel and unbounded D, could we find some criterion for the compactness of $\{T_t^D\}_{t\geqslant 0}$.
- (Q2): For more general jumping kernel and unbounded D, could we find some sharp criterion for IU of $\{T_t^D\}_{t\geqslant 0}$.
- (Q3): For general jumping kernel and unbounded D, could we find some two-sided estimates for ϕ_1.

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Outline

Functional inequalities

- Intrinsic super Poincaré inequality

$$\mu(f^2) \leqslant s\mathscr{E}(f,f) + \beta(s)\mu(\phi_1|f|)^2,\ s > 0,\ f \in \mathscr{D}(\mathscr{E}),$$

which is equivalent to

$$\mu_{\phi_1}(f^2) \leqslant s\mathscr{E}_{\phi_1}(f,f) + \beta(s)\mu_{\phi_1}(|f|)^2,\ s > 0,\ f \in \mathscr{D}(\mathscr{E}_{\phi_1}),$$

where $\mu_{\phi_1}(dx) := \phi_1^2(x)\mu(dx)$, $\mathscr{E}_{\phi_1}(f,f) := \mathscr{E}(f\phi_1, f\phi_1)$.

- (Wang, F.-Y., Wu, J.-L.; 2008) compactness.
- (Wang, F.-Y. ; 2002, Ouhabaz, E.M., Wang, F.-Y. ; 2007) IU property.

Functional inequalities

- See (Chen, M.-F.; 2005, Wang, F.-Y.; 2005) for overall introduction of functional inequalities.
- Crucial points: suitable estimate of rate function $\beta(\cdot)$ and lower bound estimate for ϕ_1.
- (Chen, X., Wang, J.; 2016) IU for Feyman-Kac semigroups.

Functional inequalities for non-local Dirichlet form

- (Wang, F.-Y., Wang, J.; 2012) Sharp criterion of super Poincaré inequality for (stable-like) non-local Dirichlet form.
- (Chen, X., Wang, J.; 2013) Sharp criterion of super Poincaré inequality for truncated non-local Dirichlet form.
- (Chen, X., Wang, F.-Y., Wang, J.; 2015) Perturbation theory of functional inequalities for non-local Dirichlet form.
- (Chen, X., Wang, J.; 2015) Sharp criterion of weighted Poincaré inequality.
- (Chen, X., Wang, F.-Y., Wang, J.; 2015) Overall review for recent progress on the study of functional inequalities for non-local Dirichlet form.

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Theorem

Define

$$V(x) = \int_{D^c} J(x,y)\,dy, \quad x \in D.$$

If

$$|x \in D : V(x) < r| < \infty, \ \forall\, r > 0,$$

then the semigroup $(T_t^D)_{t \geqslant 0}$ *is compact.*

Corollary

The semigroup $(T_t^D)_{t \geqslant 0}$ *is compact, if*

$$|D| < \infty$$

or

$$\lim_{|x| \to \infty, x \in D} V(x) = \infty.$$

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General Criterion

- Suppose there exist $\alpha_1, \alpha_2 \in (0,2)$ with $\alpha_1 \leqslant \alpha_2$ and positive c_1, c_2 such that

$$c_1|x-y|^{-d-\alpha_1} \leqslant J(x,y) \leqslant c_2|x-y|^{-d-\alpha_2}, \quad 0 < |x-y| \leqslant 1$$

$$\sup_{x\in\mathbb{R}^d} \int_{\{|x-y|>1\}} J(x,y)\,dy < \infty.$$

General Criterion

- Let
$$D_{R,r} := \{x \in D : |x| < R, \delta_D(x) > r\}.$$
- Let
$$\alpha(R,r,s) := \frac{C}{\inf_{x \in D_{R+1,r/2}} \phi_1(x)} (s \wedge r^{\alpha_1} \wedge 1)^{-d/\alpha_1}$$

General Criterion

Theorem

For any $f \in C_0^\infty(D)$ *and* $s > 0$,

$$\int_D f^2(x)\,dx \leqslant s\mathscr{E}^D(f,f) + \beta(s)\left(\int_D |f(z)|\phi_1(z)\,dz\right)^2,$$

where

$$\beta(s) = \inf\left\{\alpha(R,r,s/2) : \frac{1}{\inf_{x\in D\setminus D_{R,r}} V(x)} \leqslant s\right\}.$$

Consequently, if

$$\int_t^\infty \frac{\beta^{-1}(s)}{s}\,ds < \infty, \quad t > \inf\beta,$$

then the semigroup $(T_t^D)_{t\geqslant 0}$ *is IU.*

Horn-Shape region D_f

Proposition

Suppose furthermore for some $0 < \alpha_3 < \alpha_4 < 2$ *and* $0 < \theta_1 < \theta_2$

$$c_4|x-y|^{-d-\alpha_4} \leqslant J(x,y) \leqslant c_3|x-y|^{-d-\alpha_3},\ |x-y| \geqslant 1,$$

and

$$c_4 \log^{-\theta_2} s \leqslant f(s) \leqslant c_3 \log^{-\theta_1} s,\ s > 1,$$

then $\{T_t^{D_f}\}_{t\geqslant 0}$ *is IU if* $\theta_2 > \frac{1}{\alpha_1}$.

Horn-Shape region D_f

Proposition

Suppose furthermore for some $0 < \theta_1 < \theta_2$

$$c_4 s^{-\theta_2} \leqslant f(s) \leqslant c_3 s^{-\theta_1}, \ s > 1,$$

then $\{T_t^{D_f}\}_{t \geqslant 0}$ *is IU if* $\theta_2 > \frac{1}{\alpha_1}$.

Examples

Example

(1) Suppose $f(s) = s^{-\theta}$,

$$J(x,y) = \frac{1}{|x-y|^{d+\alpha}} I_{\{|x-y|\leqslant 1\}} + e^{-c|x-y|^{\gamma}} I_{\{|x-y|>1\}}, \quad \gamma \in [1, \infty]$$

then $\{T_t^{D_f}\}_{t\geqslant 0}$ is IU if and only if $\theta > \frac{1}{\alpha}$

(2) Suppose $f(s) = s^{-\theta}$,

$$J(x,y) = \frac{1}{|x-y|^{d+\alpha}} I_{\{|x-y|\leqslant 1\}} + e^{-c|x-y|^{\gamma}} I_{\{|x-y|>1\}}, \quad \gamma \in (0, 1)$$

then $\{T_t^{D_f}\}_{t\geqslant 0}$ is IU if and only if $\theta > \frac{\gamma}{\alpha}$.

Outline

Estimates for ground state

Theorem

Suppose $f \in C^{1,1}$, IU holds. If

$$J(x,y) = \frac{1}{|x-y|^{d+\alpha}},$$

then

$$\phi_1(x) \asymp \delta_D(x)^{\frac{\alpha}{2}} f(x_1)^{\frac{\alpha}{2}} (|x|^{-d-\alpha} \wedge 1).$$

Estimates for ground state

Theorem

Suppose $f \in C^{1,1}$, IU holds and

$$J(x,y) = \frac{1}{|x-y|^{d+\alpha}} I_{\{|x-y|\leqslant 1\}} + e^{-c|x-y|^{\gamma}} I_{\{|x-y|>1\}}, \quad \gamma \in (0,\infty],$$

then

$$\phi_1(x) \asymp \delta_D(x)^{\frac{\alpha}{2}} f(x_1)^{\frac{\alpha}{2}} \exp\Bigg(-c\bigg(\Big(|x|\log(|x|+1) + \int_0^{[x]+1} |\log f(s)| ds\Big) \wedge c|x|^{\gamma}\bigg)\Bigg)$$

Estimates for ground state

Corollary

Suppose $f \in C^{1,1}$, *IU holds and*

$$J(x,y) = \frac{1}{|x-y|^{d+\alpha}} I_{\{|x-y|\leqslant 1\}} + e^{-c|x-y|^\gamma} I_{\{|x-y|>1\}}, \quad \gamma \in (0,\infty],$$

(i) If $f(s) = s^{-\theta}$, $\gamma \in (1,\infty]$, *then*

$$\phi_1(x) \asymp \delta_D(x)^{\frac{\alpha}{2}} f(x_1)^{\frac{\alpha}{2}} \exp\big(-c|x|\log(|x|+1)\big).$$

(ii)If $f(s) = e^{-s^\theta}$, $\gamma \in (0,\infty]$, *then*

$$\phi_1(x) \asymp \delta_D(x)^{\frac{\alpha}{2}} f(x_1)^{\frac{\alpha}{2}} \exp\big(-c(|x|^{\theta+1} \wedge |x|^\gamma)\big).$$

Thank you for your attention!