Spatial asymptotics for parabolic Anderson equations with white noise

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The main topic

The Parabolic Anderson Model (PAM) is formulated in the form

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + V(t,x)u(t,x) \\ \\ u(0,x) = u_0(x) \end{cases}$$

where $\{V(t,x);\ x \in \mathbb{R}^d\}$ is a random field called potential. In this talk, we are interested in the asymptotic behavior of

$$\max_{|x|\le R} u(t,x) \quad (R \to \infty)$$

in the cases when $V(t,x) = \dot{W}(t,x)$ is a $(1+1)$-white noise.

Association to KPZ equation

A recently hot topic is the study of Kardar-Parisi-Zhang (KPZ) equation in the case $d = 1$:

$$\partial_t h = \frac{1}{2}\Big(\frac{\partial h}{\partial x}\Big)^2 + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \dot{W}(t,x)$$

where $\dot{W}(t,x)$ is the time-space white noise decided by the covariance

$$\mathrm{Cov}\left(\dot{\mathrm{W}}(0,0), \dot{\mathrm{W}}(\mathrm{t},\mathrm{x})\right) = \delta_0(\mathrm{t})\delta_0(\mathrm{x})$$

KPZ equation describes the stochastic growth of the interface. An recent important progress is the mathematical treatment by Martin Hairer (Ann. Math. (2013)).

PAM with white noise

Under the Hopf-Cole transform

$$h(t,x) = \log u(t,x)$$

KPZ equation is formally transformed into the parabolic Anderson equation with $V = \dot{W}$ being a $(1+1)$-white noise. In this setting, the solution $u(t,x)$ of the parabolic Anderson equation is mild in the sense that

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) u(s,x) W(dsdx)$$

where $u_0(x)$ is the initial value, p_t is the density of Brownian semi-group.

PAM with white noise

The parabolic Anderson equation becomes

$$\partial_t u = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \dot{W}(t,x)u$$

The solution $u(t,x)$ can be formally written as

$$u(t,x) = \mathbb{E}_X \exp\left\{ \int_0^t \dot{W}\big(t-s, B(s)\big)ds - \frac{1}{2}t\delta_0(0)\right\}$$

where $B(s)$ is a 1-dimensional Brownian motion independent of W with $B(0) = x$.

PAM with white noise

Theorem (Conus, Joseph and Koshnevisan (2013), Ann. Probab.)

Under the initial condition

$$0 < \inf_{x\in R} u_0(x) \le \sup_{x\in R} u_0(x) < \infty$$

$$0 < \liminf_{R\to\infty} (\log R)^{-2/3} \log \max_{|x|\le R} u(t,x)$$
$$\le \limsup_{R\to\infty} (\log R)^{-2/3} \log \max_{|x|\le R} u(t,x) < \infty \quad a.s.$$

PAM with white noise

This result shows that for the Hopf-Cole solution $h(t,x)$,

$$\max_{|x|\le R} h(t,x) \asymp R^{2/3} \quad (R\to\infty)$$

Question. Does the limit exist? If so, what is the value of the limit?

PAM with white noise

Theorem

Under the same initial condition,

$$\lim_{R\to\infty}(\log R)^{-2/3}\log\max_{|x|\le R}u(t,x)=\frac{3}{4}\sqrt[3]{\frac{2t}{3}}\quad \text{a.s.}$$

Thus, the Hopf-Cole solution $h(t,x)$ *of KPZ equation satisfies*

$$\lim_{R\to\infty}(\log R)^{-2/3}\max_{|x|\le R}h(t,x)=\frac{3}{4}\sqrt[3]{\frac{2t}{3}}\quad \text{a.s.}$$

Idea of the proof

By Borel-Cantelli lemma all we need is to show

$$\sum_k \mathbb{P}\Big\{ \log \max_{|x|\le 2^k} u(t,x) \ge \lambda (\log 2^k)^{2/3} \Big\} < \infty$$

for $\lambda > \frac{3}{4}\sqrt[3]{\frac{2t}{3}}$, and

$$\sum_k \mathbb{P}\Big\{ \log \max_{|x|\le 2^k} u(t,x) \le \lambda (\log 2^k)^{2/3} \Big\} < \infty$$

for $\lambda < \frac{3}{4}\sqrt[3]{\frac{2t}{3}}$.

Idea of the proof

The road map is as follows: First, by comparison we can reduce the problem to the initial condition $u_0(x) = 1$. Consequently, $u(t, x)$ is stationary in x.

In a suitable sense

$$\max_{|x|\le 2^k} u(t, x) \approx \max_{|j|\le 2^k} u(t, j)$$

By stationarity of $u(t, x)$ in x

$$\mathbb{P}\Big\{ \log \max_{|j|\le 2^k} u(t, j) \ge \lambda(\log 2^k)^{2/3} \Big\}$$
$$\le 2^{k+1}\mathbb{P}\Big\{ \log u(t, 0) \ge \lambda(\log 2^k)^{2/3} \Big\}$$

Idea of the proof

On the other hand, $u(t,j)$ ($j = 0, \pm 1, \cdots, 2^k$) are "nearly" i.i.d. Hence,

$$\mathbb{P}\Big\{ \log \max_{|j|\le 2^k} u(t,j) \le \lambda (\log 2^k)^{2/3} \Big\}$$
$$\approx \Big(1 - \mathbb{P}\Big\{ \log u(t,0) \ge \lambda (\log 2^k)^{2/3} \Big\} \Big)^{2^{k+1}}$$

Thus, the problem is to show

$$\mathbb{P}\Big\{ \log u(t,0) \ge \Big(\frac{3}{4} \sqrt[3]{\frac{2t}{3}} \pm \epsilon \Big) (\log 2^k)^{2/3} \Big\}$$
$$\approx \exp\Big\{ -(1 \pm \delta) \log 2^k \Big\}$$

for large k.

Idea of the proof

So the key estimate needed is the large deviation

$$\mathbb{P}\Big\{\log u(t,0) \geq \lambda a\Big\} \quad \text{as } a \to \infty$$

More precisely, we claim that for any $\lambda > 0$

$$\lim_{a\to\infty} a^{-3/2} \log \mathbb{P}\Big\{\log u(t,0) \geq \lambda a\Big\} = -4\left(\frac{6}{t}\right)^{1/2}\left(\frac{\lambda}{3}\right)^{3/2}$$

By Gärter-Ellis theorem, this is reduced to the proof of the high moment asymptotics

$$\lim_{n\to\infty} n^{-3} \log \mathbb{E}\, u(t,0)^n = \frac{t}{24}.$$

Idea of the proof

Our starting point is the moment representation

$$\mathbb{E}\, u(t,0)^n = \mathbb{E}_0 \exp\left\{ \sum_{1\le j<k\le n} \int_0^t \delta_0\big(B_j(s) - B_k(s)\big)ds \right\}$$

where $B_k(s)$ are independent Brownian motions starting at 0, and "E_0" is for the expectation with respect to the Brownian motions. Therefore, the problem is to show for any $\theta > 0$

$$\lim_{n\to\infty} n^{-3} \log \mathbb{E}_0 \exp\left\{\theta \sum_{1\le j<k\le n} \int_0^t \delta_0\big(B_j(s) - B_k(s)\big)ds \right\} = \frac{t\theta^2}{24}$$

The proof we present here is not the one originally given. It adopted from Bertini and Cancrini (1994) where the critical idea is Tanaka's formula.

Lower bounds

Indeed, applying Tanaka formula to the Brownian motion $2^{-1/2}\big(B_j(s) - B_k(s)\big)$ leads to

$$\begin{aligned}&\Big|\frac{B_j(t) - B_k(t)}{\sqrt{2}}\Big|\\&= \frac{1}{\sqrt{2}}\int_0^t \mathrm{sgn}\,\big(B_j(s) - B_k(s)\big)d\big(B_j(s) - B_k(s)\big)\\&+\sqrt{2}\int_0^t \delta_0\big(B_j(s) - B_k(s)\big)ds.\end{aligned}$$

So we have

$$\begin{aligned}&\int_0^t \delta_0\big(B_j(s) - B_k(s)\big)ds\\&\geq \frac{1}{2}\int_0^t \mathrm{sgn}\,\big(B_j(s) - B_k(s)\big)d\big(B_k(s) - B_j(s)\big)\end{aligned}$$

Lower bounds

Summing up, the lower bound follows from

$$
\begin{aligned}
&\sum_{1\le j<k\le n}\int_0^t \delta_0\big(B_j(s)-B_k(s)\big)ds\\
&\ge \frac{1}{2}\sum_{1\le j<k\le n}\int_0^t \mathrm{sgn}\,\big(B_j(s)-B_k(s)\big)d\big(B_k(s)-B_j(s)\big)\\
&= \frac{1}{2}\sum_{j=1}^n\int_0^t\bigg(\sum_{k\neq j}\mathrm{sgn}\,\big(B_j(s)-B_k(s)\big)\bigg)dB_j(s)\\
&\overset{d}{=} \frac{1}{2}B\bigg(t\frac{n(n^2-1)}{3}\bigg)
\end{aligned}
$$

Lower bounds

where the identity in law follows from the time-change Brownian representation for martingale and the fact that for any distinct real numbers $b_1, \cdots, b_n$

$$\sum_{j=1}^{n}\Big(\sum_{k\neq j}\mathrm{sgn}\,(b_j-b_k)\Big)^2=\frac{1}{3}n^2(n-1)$$

and therefore

$$\sum_{j=1}^{n}\int_0^t\Big(\sum_{k\neq j}\mathrm{sgn}\,\big(B_j(s)-B_k(s)\big)\Big)^2ds=\frac{t}{3}n^2(n-1)\quad \text{a.s.}$$

Upper bounds

By Tanaka formula

$$\frac{1}{2}|B_j(t) - B_k(t)|$$
$$= \frac{1}{2}\int_0^t \mathrm{sgn}\,\big(B_j(s) - B_k(s)\big) d\big(B_j(s) - B_k(s)\big)$$
$$+ \int_0^t \delta_0\big(B_j(s) - B_k(s)\big) ds.$$

Applying Skorokhod lemma

$$\int_0^t \delta_0\big(B_j(s) - B_k(s)\big) ds$$
$$= \frac{1}{2}\sup_{s\le t}\int_0^t \mathrm{sgn}\,\big(B_j(s) - B_k(s)\big) d\big(B_k(s) - B_s(s)\big)$$
$$= \frac{1}{\sqrt{2}}\sup_{s\le t} B_{j,k}(s) \quad \text{(say)}.$$

Upper bounds

Summing up

$$\sum_{1\le j<k\le n}\int_0^t \delta_0\big(B_j(s)-B_k(s)\big)ds = \frac{1}{\sqrt{2}}\sum_{1\le j<k\le n}\sup_{s\le t} B_{j,k}(s)$$
$$= \frac{1}{\sqrt{2}}\sup_{\mathbf{s}\in[0,t]^{\mathcal{A}}}\sum_{1\le j<k\le n} B_{j,k}(s_{j,k}) = \frac{1}{\sqrt{2}}\sup_{\mathbf{s}\in[0,t]^{\mathcal{A}}} G(\mathbf{s}) \quad \text{(say)}.$$

Here we adopt the notations

$$\mathcal{A}=\big\{(j,k);\ \ 1\le j<k\le n\big\},\qquad \mathbf{s}=\big(s_{j,k};\ \ 1\le j<k\le n\big).$$

Upper bounds

Let $m \geq 1$ be an integer and let $0 = t_0 < t_1 < \cdots < t_m = t$ be a uniform partition. Set $\Pi_m = \{1, \cdots, m\}^{\mathcal{A}}$ and partition $[0, t]^{\mathcal{A}}$ into $m^{\#\mathcal{A}}$ small boxes B_π labeled by $\pi \in \Pi_m$ such that B_π is a $\mathcal{A}$-product of the intervals of the form $[t_{i-1}, t_i]$ $(i = 1, \cdots, m)$.

$$
\begin{aligned}
&\mathbb{E} \exp\left\{\theta \sum_{1\leq j<k\leq n} \int_0^t \delta_0\big(B_j(s) - B_k(s)\big)ds\right\} \\
&= \mathbb{E} \exp\left\{\frac{\theta}{\sqrt{2}} \max_{\pi\in\Pi_m} \sup_{\mathbf{s}\in B_\pi} G(\mathbf{s})\right\} \\
&\leq \sum_{\pi\in\Pi_m} \mathbb{E} \exp\left\{\frac{\theta}{\sqrt{2}} \sup_{\mathbf{s}\in B_\pi} G(\mathbf{s})\right\}.
\end{aligned}
$$

Upper bounds

Let the box B_π be fixed and write

$$B_\pi = \prod_{(j,k)\in\mathcal{A}} [s_{j,k}, t_{j,k}] = [\mathbf{s}_\pi, \mathbf{t}_\pi].$$

By Hölder inequality,

$$\mathbb{E}\exp\left\{\frac{\theta}{\sqrt{2}}\sup_{\mathbf{s}\in B_\pi} G(\mathbf{s})\right\} \le \left(\mathbb{E}\exp\left\{\frac{p\theta}{\sqrt{2}} G(\mathbf{s}_\pi)\right\}\right)^{1/p}$$

$$\times\left(\mathbb{E}\exp\left\{\frac{q\theta}{\sqrt{2}}\sup_{\mathbf{s}\in B_\pi}\Big(G(\mathbf{s}) - G(\mathbf{s}_\pi)\Big)\right\}\right)^{1/q}$$

where $p, q > 1$ are fixed conjugate numbers with p close to 1.

Upper bounds

Using Hölder inequality again

$$\mathbb{E}\exp\left\{\frac{q\theta}{\sqrt{2}}\sup_{\mathbf{s}\in B_\pi}\Big(G(\mathbf{s})-G(\mathbf{s}_\pi)\Big)\right\}$$

$$\leq \prod_{(j,k)\in\mathcal{A}}\left(\mathbb{E}\exp\left\{\frac{qn(n-1)\theta}{2\sqrt{2}}\sup_{s\in[s_{j,k},t_{j,k}]}\big(B_{j,k}(s)-B_{j,k}(s_{j,k})\big)\right\}\right)^{\frac{2}{n(n-1)}}$$

$$=\mathbb{E}\exp\left\{\frac{qn(n-1)\theta}{2\sqrt{2}}\sqrt{\frac{t}{m}}|B(1)|\right\}\leq 2\exp\left\{\frac{1}{16}(qn(n-1)\theta)^2\frac{t}{m}\right\}.$$

where the second step follows from the fact that $B_{j,k}(s)$ is a Brownian motion.

Upper bounds

In addition, write

$$G(\mathbf{s}_\pi) = \frac{1}{\sqrt{2}} \sum_{1\le j<k\le n} \int_0^{s_{j,k}} \mathrm{sgn}\left(B_k(u) - B_j(u)\right) d\left(B_j(u) - B_k(u)\right)$$

$$= \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \sum_{k\neq j} \int_0^{s_{j,k}} \mathrm{sgn}\left(B_k(u) - B_j(u)\right) dB_j(u)$$

$$= \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \int_0^{t} \left(\sum_{k\neq j} \mathrm{sgn}\left(B_k(u) - B_j(u)\right) 1_{[0,s_{j,k}]}(u) \right) dB_j(u).$$

Here we extend the definition of $s_{j,k}$ as $s_{j,k} = s_{k,j}$ for $j > k$.

Upper bounds

By Ito's formula,

$$\mathbb{E}\ \exp\left\{\frac{p\theta}{\sqrt{2}}G(\mathbf{s}_\pi) - \frac{(p\theta)^2}{8}\sum_{j=1}^n \int_0^t \left(\sum_{k\neq j}\text{sgn}\,\big(B_k(u)-B_j(u)\big)1_{[0,s_{j,k}]}(u)\right)^2 du\right\} = 1.$$

Here we point out that for each $0 \le u \le t$

$$\sum_{j=1}^n \left(\sum_{k\neq j}\text{sgn}\,\big(B_k(u)-B_j(u)\big)1_{[0,s_{j,k}]}(u)\right)^2$$
$$\le \sum_{j=1}^n \left(\sum_{k\neq j}\text{sgn}\,\big(B_k(u)-B_j(u)\big)\right)^2 = \frac{1}{3}n(n^2-1).$$

Upper bounds

Consequently,

$$\mathbb{E}\exp\left\{\frac{p\theta}{\sqrt{2}}G(\mathbf{s}_\pi)\right\}\le\exp\left\{\frac{(p\theta)^2}{24}n(n^2-1)t\right\}.$$

Summarizing our steps,

$$\mathbb{E}\exp\left\{\theta\sum_{1\le j<k\le n}\int_0^t\delta_0\big(B_j(s)-B_k(s)\big)ds\right\}$$
$$\le 2m^{\frac{n(n-1)}{2}}\exp\left\{\frac{1}{16}q\big(n(n-1)\theta\big)^2\frac{t}{m}\right\}\exp\left\{\frac{p\theta^2}{24}n(n^2-1)t\right\}.$$

This leads to the upper bounds with $m=n^2$, $p>1$ is close to 1.

□

Further remark.

By an obvious modification of the above proof, we can have the following moment asymptotics:

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\,u(t,x)^n=\frac{1}{24}n(n^2-1)\quad n=2,3,\cdots$$

which, together with the high moment asymptotics established above, appears as the weak version of the false claim of Bertini and Cancrini (1994).

$$\mathbb{E}\,u(t,x)^n=2\exp\left\{\lambda^4 t\frac{n(n^2-1)}{24}\right\}\Phi\left(\left(t\lambda^2\left(\frac{n(n^2-1)}{12}\right)\right)^{1/2}\right)$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Further remark.

As to the time asymptotics for the parabolic Anderson model, Amir, Corwin and Quastel (2011), who claim that under the initial condition $u_0(x) = \delta_0(x)$

$$t^{-1/3}\Big\{\log u(t,x) + \frac{t}{24}\Big\} \xrightarrow{d} 2^{-1/3}\mathrm{GUE} \qquad (t \to \infty)$$

where GUE represents Tracy-Widom law. So we expect that

$$\lim_{t\to\infty} l(t)^{-1}\mathbb{E}\exp\left\{\theta\Big(\frac{l(t)}{t}\Big)^{1/3}\Big\{\log u(t,x) + \frac{t}{24}\Big\}\right\}$$
$$= \lim_{t\to\infty} l(t)^{-1}\mathbb{E}\exp\left\{\theta\Big(\frac{l(t)}{2}\Big)^{1/3}\mathrm{GUE}\right\} = \frac{\theta^3}{24} \qquad (\theta > 0)$$

for any slow varying function $l(t) \to \infty$.

Further remark.

In the following, we show how it morally link to our moment asymptotics which can be rewritten into

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ n\Big(\log u(t,x) + \frac{t}{24} \Big) \right\} = \frac{n^3}{24}$$

Indeed, if n can be replaced by $\theta(l(t)/t)^{1/3}$,

$$\lim_{t\to\infty} l(t)^{-1} \mathbb{E} \exp\left\{ \theta \Big(\frac{l(t)}{t} \Big)^{1/3} \Big\{ \log u(t,x) + \frac{t}{24} \Big\} \right\} = \frac{\theta^3}{24}$$

Further remark.

By Gärtner-Ellis theorem, we expect some thing like

$$\lim_{t\to\infty} \frac{1}{l(t)} \log \mathbb{P}\left\{ \log u(t,x) + \frac{t}{24} \geq \lambda t^{1/3} l(t)^{2/3} \right\} = -\frac{2\sqrt{8}}{3}\lambda^{2/3}$$

for $\lambda > 0$.

The above analysis shows that the precise Lyapunov exponent may suggest the pattern of the quenched long term asymptotics of the system.

Further remark.

Consider the Parabolic Anderson equation

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + V(t,x)u(t,x) \\ u(0,x) = u_0(x) \end{cases}$$

and assume that the Gaussian potential $V(t,x)$ is white in time and colored in space. More precisely, it has the covariance function

$$\mathrm{Cov}\Big(V(t,x), V(s,y)\Big) = \delta_0(t-s)\gamma(x-y) \quad (t,x),(s,y) \in \mathbb{R}^+ \times \mathbb{R}^d$$

with $\gamma(\cdot) \geq 0$ and the space homogeneity $\gamma(cx) = |c|^{-\alpha}\gamma(x)$ for some $0 < \alpha < 2$.

Further remark.

As a generalization, it has been shown that

$$\lim_{R\to\infty}(\log R)^{-\frac{2}{4-\alpha}}\max_{|x|\le R}u(t,x)=\frac{4-\alpha}{4}\left(\frac{4t\mathcal{E}}{2-\alpha}\right)^{\frac{2-\alpha}{4-\alpha}}d^{\frac{2}{4-\alpha}}\quad \text{a.s.}$$

where $\mathcal{E}$ is the Hartree energy defined as

$$\mathcal{E}=\sup_g\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}$$

and the supremum is over all $g\in\mathcal{L}^2(\mathbb{R}^d)$ with $\nabla g\in\mathcal{L}^2(\mathbb{R}^d)$ and $\|g\|_2=1$.

Further remark.

This result is based on the high moment asymptotics

$$\lim_{n\to\infty} n^{-\frac{4-\alpha}{2-\alpha}} \log \mathbb{E}\, u(t,0)^n = t\Big(\frac{1}{2}\Big)^{\frac{2}{2-\alpha}} \mathcal{E}$$

which follows from a different approach.

As for the Lyapunov exponent, we conjecture that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\, u(t,0)^n = \Big(\frac{1}{2}\Big)^{\frac{2}{2-\alpha}} n\Big(n^{\frac{2}{2-\alpha}} - 1\Big)\mathcal{E}$$

for $n = 1, 2, \cdots$.

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Thank you!