Asymptotic behavior for a generalized Domany-Kinzel model

Lung-Chi Chen

Department of Mathematical Sciences, National Chengchi University

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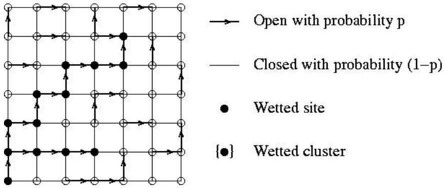
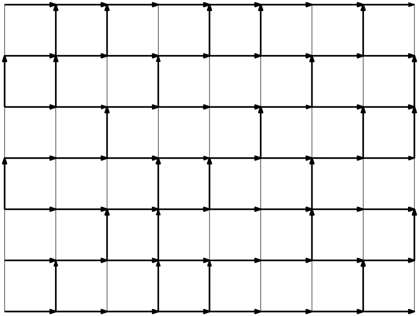
Joint work with Shu-Chiuan Chang and Chien-Hao Huang

The directed percolation (1957) on square lattice.



Domany-Kinzel model (1981)



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Theorem (Domany and Kinzel (1981)) Given any $\alpha > 0$, there is $\alpha_c = q/p := (1-p)/p$ such that

$$\lim_{N\to\infty} \tau(N_\alpha, N) = \begin{cases} 1 & \alpha > \alpha_c \\ \frac{1}{2} & \alpha = \alpha_c \\ 0 & \alpha < \alpha_c \end{cases}.$$

More precisely, for $\alpha < \alpha_c$ and α close to α_c, the scaling theory of critical behavior asserts that the singular part of $\tau(N_\alpha, N)$ varies asymptotically as

$$\tau(N_\alpha, N) \approx \exp\left(\frac{-BN}{(\alpha_c - \alpha)^{-\nu}}\right) ,$$

where $f_{1,\alpha}(N) \approx f_{2,\alpha}(N)$ means that $\lim_{N\to\infty} \log f_{1,\alpha}(N)/\log f_{2,\alpha}(N) = 1$. The constants B depending on p and critical exponent $\nu \in (0,\infty)$ is universal constant.

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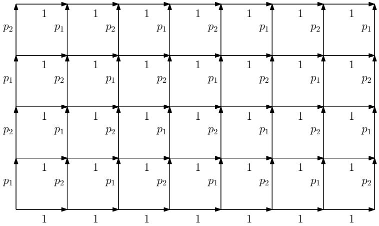
Theorem (Wu and Stanley (1982)) $\nu = 2$ and (Chen (2011)) $B = q/p^2$.

In this talk we consider generalized Domany-Kinzel model as follows:



Main results

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For notation convenience, let us define $q_1 = 1 - p_1$, $q_2 = 1 - p_2$ and

$$\begin{aligned} a &= p_1^2 q_2^2 + p_2^2 q_1^2, \\ b &= p_2 q_1 + p_1 q_2 = p_1 + p_2 - 2p_1 p_2, \\ c &= p_2 q_1 - p_1 q_2 = p_2 - p_1, \\ \sigma^2 &= \frac{4(p_1 q_1 + p_2 q_2)}{(p_1 + p_2)^3}. \end{aligned}$$

Theorem 1. In our model, given $p_1 \in [0,1)$ and $p_2 \in [0,1)$ with $p_1 \vee p_2 > 0$ and the critical aspect ratio $\alpha_c = \frac{q_1+q_2}{p_1+p_2}$, we have

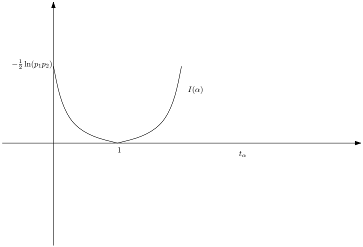
$$\begin{cases} \tau(N_\alpha, N) & \approx \exp\big(-NI(\alpha)\big) \quad \text{for} \quad \alpha < \alpha_c\ , \\ \tau(N_\alpha, N) & = \frac{1}{2} + O(\frac{1}{\sqrt{N}}) \quad \text{for} \quad \alpha = \alpha_c, \\ 1-\tau(N_\alpha, N) & \approx \exp\big(-NI(\alpha)\big) \quad \text{for} \quad \alpha > \alpha_c, \end{cases}$$

where

$$I(\alpha) = \alpha \ln t_\alpha - \ln\Big(\frac{bt_\alpha + \sqrt{c^2 t_\alpha^2 + 4p_1p_2}}{2(1-q_1q_2t_\alpha^2)}\Big),$$

and

$$t_\alpha = \begin{cases} \Big(\frac{2\alpha c^2 - b^2(1+\alpha) + b\sqrt{b^2(\alpha+1)^2 - 4\alpha c^2}}{2q_1q_2c^2(1+\alpha)}\Big)^{\frac{1}{2}} & \text{if} \quad p_1 \neq p_2, \\ \frac{\alpha}{(1-p)(1+\alpha)}. & \text{if} \quad p_1 = p_2 = p. \end{cases}$$


$-\frac{1}{2}\ln(p_1p_2)$
$I(\alpha)$
1
t_α

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1. $I(\alpha)$ is a non-negative, convex function with minimum $\alpha = \alpha_c$. Moreover $t_{\alpha_c} = 1$ and $I(\alpha_c) = 0$.

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2. The model reduces to the Domany-Kinzel model on the square lattice when $p_1 = p_2 = p$ and $q = 1 - p$. We have $t_\alpha = \frac{\alpha}{q(\alpha+1)}$ and $I(\alpha)$ can be simplified as

$$I(\alpha) = \alpha \ln\Big(\frac{\alpha}{q(1+\alpha)}\Big) - \ln\big(p(1+\alpha)\big).$$

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3. The model reduces to the Domany-Kinzel model on the honeycomb lattice as bricks when either $p_1 = 0$, $p_2 = p$ or $p_1 = p$, $p_2 = 0$, and let $q = 1 - p$ with $\alpha > 1$. We obtain $\alpha_c = (q+1)/p$ and $t_\alpha = \sqrt{\frac{\alpha-1}{q(\alpha+1)}}$. Then $I(\alpha)$ can be simplified as

$$I(\alpha) = \Big(\frac{\alpha-1}{2}\Big) \ln\Big(\frac{\alpha-1}{q(\alpha+1)}\Big) - \ln\Big(\frac{p(\alpha+1)}{2}\Big).$$

Using Taylor formula and the first item in Remark 1, we have

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\begin{aligned}
I(\alpha) &= \underbrace{I(\alpha_c)}_{=0} + \underbrace{I'(\alpha_c)}_{=0}(\alpha-\alpha_c) + \frac{I''(\alpha_c)}{2}(\alpha-\alpha_c)^2 + O(\alpha-\alpha_c)^3 \\
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$$\underline{\alpha} = \frac{-1+\sqrt{(2\alpha_c+1)^2-4\sigma^2}}{2}.$$

It is easy to see that $\underline{\alpha} \in [0, \alpha_c)$ and particularly $\underline{\alpha} = 0$ iff $p_1 = p_2$. Furthermore, define

$$\mathcal{U}_{p_1 p_2} = \begin{cases} \frac{1}{\sqrt{p_1 p_2}} & \text{if } p_1 p_2 \neq 0 \\ \infty & \text{if } p_1 p_2 = 0 \end{cases} .$$

Theorem 2 Let $p_1 \in [0,1)$ and $p_2 \in [0,1)$ with $p_1 \vee p_2 > 0$ and $p_1 \wedge p_2 < 1$. We have

$$\frac{\frac{1}{2\sigma^2}(\alpha_c - \alpha)^2}{1 + \frac{2}{\sigma^2}(\alpha_c - \alpha)} \leq I(\alpha) \leq \frac{\frac{1}{2\sigma^2}(\alpha_c - \alpha)^2}{1 - \frac{1}{\sigma^2}(\alpha_c + \alpha + 1)(\alpha_c - \alpha)}$$

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In particular, $I(\alpha) \ge I(0) = \ln(\mathcal{U}_{p_1 p_2})$ for $\alpha \in (0, \alpha_c)$.

Remark 2 Theorem 2 leads to the following information:
1. Our result shows that $\tau(N_\alpha, N)$ with $\alpha < \alpha_c$ and $1 - \tau(N_\alpha, N)$ with $\alpha > \alpha_c$ both decay exponentially to zero. Furthermore, we obtain the critical exponent $\nu = 2$ and $B = \frac{1}{2\sigma^2}$ for $\alpha < \alpha_c$.

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2. One can consider another version of directed bond percolation on the square lattice (see Fig. 2). This model is much easier and the method of steepest decent can be used to get $\alpha_c = (q_1 p_1 + q_2 p_2)/2p_1 p_2$ instantly. Although this model looks similar to the model we consider here, α_c is clearly different.

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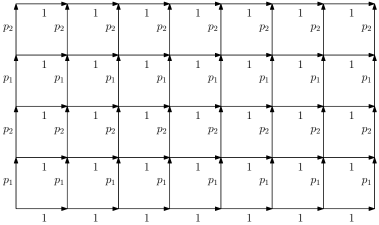
3. Although α_c can also be obtained by applying the Theorem 5.9 in "Markov renewal theory: a survey. Management Science, Vol. **21**, No. 7", Theory Series, pp. 727-752 (1975), the variance σ^2 was not discussed there.

Figure 2



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Theorem 3.

Given $\rho \in (0, \infty)$ and a positive regularly varying sequence $\{\ell_n\}_{n=1}^{\infty}$. Let $\alpha_N^- = \alpha_c - \sigma N^{-\rho}\ell_N/\sqrt{2}$ and $\alpha_N^+ = \alpha_c + \sigma N^{-\rho}\ell_N/\sqrt{2}$. Then both

$$\tau(N_{\alpha_N^-}, N), 1 - \tau(N_{\alpha_N^+}, N)$$

$$\begin{cases} \approx \exp(-N^{-2\rho+1}\ell_N^2) & \text{if} \quad \rho \in (0, \frac{1}{2}), \\ \approx \exp(-\ell_N^2) & \text{if } \rho = \frac{1}{2},\ \ell_N \to \infty \\ = \Psi(\ell) + O(1)\max\{\frac{1}{\sqrt{N}}, |\ell - \ell_N|\} & \text{if} \quad \rho = \frac{1}{2},\ \ell_N \to \ell \in [0, \infty \\ = \frac{1}{2} + O(1)N^{-\rho+\frac{1}{2}}\ell_N & \text{if} \quad \rho \in (\frac{1}{2}, 1)\ , \\ = \frac{1}{2} + O(\frac{1}{\sqrt{N}}) & \text{if} \quad \rho \in [1, \infty)\ , \end{cases}$$

Note that $\rho = \frac{1}{2}$ is a critical value and we have the following corollary.

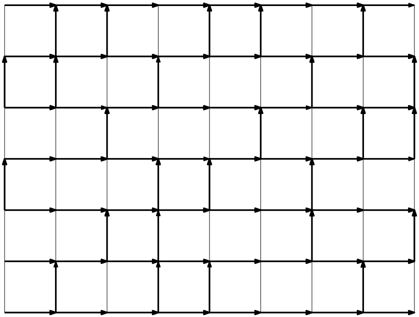
Corollary. Under the same assumptions of Theorem 3, we have

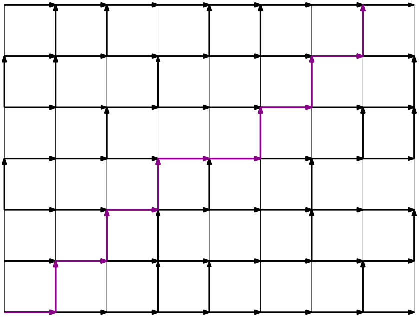
$$\lim_{N\to\infty} \tau(N_{\alpha_N^-}, N) = \lim_{N\to\infty} \Big(1 - \tau(N_{\alpha_N^+}, N)\Big) = \begin{cases} 0 & \text{if} \quad \rho \in (0, \frac{1}{2}) \ , \\ \frac{1}{2} & \text{if} \quad \rho \in (\frac{1}{2}, \infty) \ . \end{cases}$$

When $\rho = 1/2$ and $\ell_N \to \ell \in [0, \infty]$ we have

$$\lim_{N\to\infty} \tau(N_{\alpha_N^-}, N) = \exp(-\ell^2) \ , \quad \lim_{N\to\infty} \tau(N_{\alpha_N^+}, N) = 1 - \exp(-\ell^2) \ .$$

Idea of the proof





Proof of Theorem 1

Let $S_n = X_1 + \cdots X_n$. (The process is actually called Markov renewal process or Semi-Markov Chain). By law of large number we have

$$\frac{S_n - \text{Exp.}(S_n)}{n} \to 0 \quad \text{a.s. when } n \to \infty .$$

Let $C_n(m) = P(S_n = m)$ for $m, n \in \mathbb{Z}_+$ and let

$$\alpha_c := \lim_{n\to\infty} \frac{\text{Exp.}(S_n)}{n} = \lim_{n\to\infty} \frac{\text{Exp.}(S_{2n})}{2n} = \lim_{n\to\infty} \frac{\sum_{m=1}^{\infty} mC_{2n}(m)}{2n}.$$

We have

$$\frac{S_n}{n} \to \alpha_c \quad \text{a.s. when } n \to \infty .$$

Let the variance of S_n denote by $\sigma^2(S_n)$.

By Berry-Esseen theorem (non-identically distributed summands)

$$\Big|Prob.(\frac{S_N-\alpha_c N}{\sqrt{\sigma^2(S_N)}}\le\frac{N(\alpha-\alpha_c)}{\sqrt{\sigma^2(S_N)}})-\int_{-\infty}^{\frac{N(\alpha-\alpha_c)}{\sqrt{\sigma^2(S_N)}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\;du\Big|$$
$$\le\quad O(\frac{1}{\sqrt{\sigma^2(S_N)}}).$$

We have

$$\begin{aligned}\sigma^2(S_N) &= \sum_{m=1}^{\infty}m^2C_N(m)^2-\Big(\sum_{m=1}^{\infty}mC_N(m)^2\Big)^2\\ &= N\Big[\frac{\sum_{m=1}^{\infty}m^2C_N(m)^2-\big(\sum_{m=1}^{\infty}mC_N(m)^2\big)^2}{N}\Big]\\ &= N\Big[\frac{\sigma^2(S_N)}{N}\Big].\end{aligned}$$

Let

$$\sigma^2:=\lim_{N\to\infty}\frac{\sigma^2(S_N)}{N}.$$

Setting $\alpha = \alpha_c$, we have

$$\tau(N_{\alpha_c}, N) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\ du + O(\frac{1}{\sqrt{N}}) = \frac{1}{2} + O(\frac{1}{\sqrt{N}})\ ,$$

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We need find the value of α_c and σ^2.

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Note that

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Let

$$C_n(m) = C_n^o(m) + C_n^e(m),$$

where

$$C_n^o(m) = \begin{cases} C_n(m) & \text{if } m \text{ is odd,} \\ 0 & \text{others,} \end{cases}$$

and

$$C_n^e(m) = \begin{cases} C_n(m) & \text{if } m \text{ is even} \cup \{0\}, \\ 0 & \text{others.} \end{cases} ,$$

We have

$$C^e_{2n+1}(m) = \sum_{j=0}^{m} \Big(C^e_{2n}(m-j) D_{ee}(j) + C^o_{2n}(m-j) D_{oe}(j) \Big), \qquad (1)$$

and

$$C^o_{2n+1}(m) = \sum_{j=0}^{m} \Big(C^e_{2n}(m-j) D_{eo}(j) + C^o_{2n}(m-j) D_{oo}(j) \Big), \qquad (2)$$

where

$$D_{ee}(j) = \begin{cases} (q_1 q_2)^i p_1, & j = 2i+1, \\ 0, & j = 2i, \end{cases}$$

$$D_{oe}(j) = \begin{cases} (q_1 q_2)^i q_2 p_1, & j = 2i, \\ 0, & j = 2i+1, \end{cases}$$

and

$$D_{eo}(j) = \begin{cases} (q_1 q_2)^i q_1 p_2, & j = 2i, \\ 0, & j = 2i+1, \end{cases}$$

Similarly,

$$C_{2n}^{e}(m) = \sum_{j=0}^{m} \Big(C_{2n-1}^{e}(m-j) D'_{ee}(j) + C_{2n-1}^{o}(m-j) D'_{oe}(j) \Big),$$

and

$$C_{2n}^{o}(m) = \sum_{j=0}^{m} \Big(C_{2n-1}^{e}(m-j) D'_{eo}(j) + C_{2n-1}^{o}(m-j) D'_{oo}(j) \Big),$$

where

$$D'_{ee}(j) = D_{oo}(j),\ D'_{oo}(j) = D_{ee}(j),\ D'_{eo}(j) = D_{oe}(j),\ D'_{oe}(j) = D_{eo}(j).$$

Define the generated function for $f : \mathbb{Z}_+ \to \mathbb{R}_+$ as follows

$$\hat{f}(t) = \sum_{n=0}^{\infty} f(n)t^n, \quad |t| \leq 1.$$

Therefore

$$\hat{D}_{ee}(t) = \frac{p_1}{1 - q_1 q_2 t^2}, \quad \hat{D}_{oo}(t) = \frac{p_2}{1 - q_1 q_2 t^2},$$
$$\hat{D}_{eo}(t) = \frac{q_1 p_2 t}{1 - q_1 q_2 t^2}, \quad \hat{D}_{oe}(t) = \frac{q_2 p_1 t}{1 - q_1 q_2 t^2}$$

Let

$$A_1(t) = \begin{pmatrix} \hat{D}_{ee}(t) & \hat{D}_{oe}(t) \\ \hat{D}_{eo}(t) & \hat{D}_{oo}(t) \end{pmatrix}, \quad A_2(t) = \begin{pmatrix} \hat{D}'_{ee}(t) & \hat{D}'_{oe}(t) \\ \hat{D}'_{eo}(t) & \hat{D}'_{oo}(t) \end{pmatrix}.$$

We have

$$\begin{pmatrix} \hat{C}^e_{2n+1}(t) \\ \hat{C}^o_{2n+1}(t) \end{pmatrix} = A_1(t) \begin{pmatrix} \hat{C}^e_{2n}(t) \\ \hat{C}^o_{2n}(t) \end{pmatrix},$$

and for $n \in \mathbb{N}$, we have

$$\begin{pmatrix} \hat{C}^e_{2n}(t) \\ \hat{C}^o_{2n}(t) \end{pmatrix} = A_2(t) \begin{pmatrix} \hat{C}^e_{2n-1}(t) \\ \hat{C}^o_{2n-1}(t) \end{pmatrix}.$$

Let

$$\begin{aligned} A(t) &= A_2(t)A_1(t) \\ &= \frac{1}{(1-q_1q_2t^2)^2} \begin{pmatrix} p_1p_2 + p_1^2q_1^2t^2 & (p_1p_2q_2 + p_2^2q_1)t \\ (p_1p_2q_1 + p_1^2q_2)t & p_1p_2 + p_1^2q_2^2t^2 \end{pmatrix}. \end{aligned}$$

By the definition of our model, for $n \in \mathbb{Z}_+$, we have

$$\hat{C}_{2n}(t) = (1,1)A(t)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{C}_{2n+1}(t) = (1,1)A_1(t)A(t)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Recall that

$$a = p_1^2 q_2^2 + p_2^2 q_1^2, \quad b = p_1 q_2 + p_2 q_1 \quad \text{and} \quad c = p_2 - p_1,$$

we have

$$A(t) = Q(t)D(t)Q(t)^{-1},$$

where

$$D(t) = \begin{pmatrix} \frac{at^2+2p_1p_2+bt\sqrt{c^2t^2+4p_1p_2}}{2(1-q_1q_2t^2)^2} & 0 \\ 0 & \frac{at^2+2p_1p_2-bt\sqrt{c^2t^2+4p_1p_2}}{2(1-q_1q_2t^2)^2} \end{pmatrix},$$

and

$$Q(t) = \begin{pmatrix} -p_2 & -p_2 \\ \frac{ct-\sqrt{c^2t^2+4p_1p_2}}{2} & \frac{ct+\sqrt{c^2t^2+4p_1p_2}}{2} \end{pmatrix}.$$

Therefore

$$\begin{aligned}\hat{C}_{2n}(t) &= \lambda_1(t)^n u_1(t) + \lambda_2(t)^n u_2(t),\\ \hat{C}_{2n+1}(t) &= \lambda_1(t)^n v_1(t) + \lambda_2(t)^n v_2(t),\end{aligned}$$

for some $u_j(t)$ and $v_j(t)$, where

$$\begin{aligned}\lambda_1(t) &= \frac{at^2 + 2p_1p_2 + bt\sqrt{c^2t^2 + 4p_1p_2}}{2(1 - q_1q_2t^2)^2},\\ \lambda_2(t) &= \frac{at^2 + 2p_1p_2 - bt\sqrt{c^2t^2 + 4p_1p_2}}{2(1 - q_1q_2t^2)^2}.\end{aligned}$$

We can define the average mean of two point function n-step walk

$$\mu_n = \frac{\sum_{m=0}^{\infty} mC_n(m)}{n} = \frac{\frac{d}{d\,t}\hat{C}_n(t)}{n}\Big|_{t=1} = \frac{\lambda_1'(1)}{2} + O(\frac{1}{n}),$$

and we can define the average variance of n-step walk

$$\sigma_n^2 = \frac{\sum_{m=0}^{\infty} m^2 C_n(m) - \left(\sum_{m=0}^{\infty} m^2 C_n(m)\right)^2}{n}, \quad \sigma^2 = \lim_{n\to\infty} \sigma_n^2.$$

It can be shown that

$$\begin{aligned} \alpha_c &= \lim_{n\to\infty} \mu_{2n} = \frac{\lambda_1'(1)}{2}, \\ \sigma_n^2 &= \sigma^2 + O(\frac{1}{n}). \end{aligned}$$

Original result,

$$\lambda_1'(1) = \frac{p_1^2 q_2^2 + p_2^2 q_1^3 + p_1^2 + p_2^2 - \frac{2p_1p_2(p_1^2+p_2^2)}{p_1+p_2}}{2(1-q_1q_2)^2} + \frac{4q_1q_2(p_1^2+p_2^2-2p_1^2p_2-2p_1p_2^2+2p_1p_2+p_1^2p_2^2)}{(1-q_1q_2)^3}.$$

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Improve (INFORMS (Institute for operations research and the management sciences), 1975)

$$\alpha_c = \frac{p_1 q_2 (2 - p_1) + p_2 q_1 (2 - p_2)}{(1 - q_1 q_2)(p_1 + p_2)}.$$

Original result,

$$\lambda_1'(1) = \frac{p_1^2q_2^2 + p_2^2q_1^3 + p_1^2 + p_2^2 - \frac{2p_1p_2(p_1^2+p_2^2)}{p_1+p_2}}{2(1-q_1q_2)^2} + \frac{4q_1q_2(p_1^2 + p_2^2 - 2p_1^2p_2 - 2p_1p_2^2 + 2p_1p_2 + p_1^2p_2^2)}{(1-q_1q_2)^3}.$$

Improve (INFORMS (Institute for operations research and the management sciences), 1975)

$$\alpha_c = \frac{p_1q_2(2-p_1) + p_2q_1(2-p_2)}{(1-q_1q_2)(p_1+p_2)}.$$

After computing, we can rewrite

$$\lambda_1(t) = \left(\frac{bt + \sqrt{c^2t^2 + 4p_1p_2}}{2(1-q_1q_2t^2)}\right)^2$$

Hence

$$\frac{t\lambda_1'(t)}{2\lambda_1(t)} = \frac{2\sqrt{\lambda_1(t)}}{\sqrt{c^2t^2+4p_1p_2}} - 1.$$

Note that $\lambda_1(1) = 1$ and $c^2 + 4p_1p_2 = (p_1+p_2)^2$, we have

$$\alpha_c = \frac{\lambda_1'(1)}{2} = \frac{2-p_1-p_2}{p_1+p_2} = \frac{q_1+q_2}{p_1+p_2}.$$

Furthermore

$$\frac{\lambda_1'(t)+t\lambda_1''(t)}{\lambda_1(t)}-\frac{t\lambda_1'(t)^2}{\lambda_1(t)^2}=\frac{2\lambda'(t)}{\sqrt{\lambda_1(t)(c^2t^2+4p_1p_2)}}-\frac{4c^2t\sqrt{\lambda_1(t)}}{(c^2t^2+4p_1p_2)^{\frac{3}{2}}}.$$

Put $t=1$ we have

$$\lambda_1''(1)=-\lambda_1'(1)+\lambda_1'(1)^2+\frac{2\lambda_1'(1)}{p_1+p_2}-\frac{4(p_1-p_2)^2}{(p_1+p_2)^3}.$$

Therefore, by the definition of $\hat{C}_{2n}(t)$, we obtain the variance of the two-point function is given by

$$\begin{aligned}\sigma^2 &= \lim_{n\to\infty}\frac{\sum_{m=0}^{\infty}m^2C_{2n}(m)-\left(\sum_{m=0}^{\infty}mC_{2n}(m)\right)^2}{2n}\\ &= \frac{\lambda_1''(1)+\lambda_1'(1)-\lambda_1'(t)^2}{2}\\ &= \frac{\lambda_1'(t)}{p_1+p_2}-\frac{2(p_1-p_2)^2}{(p_1+p_2)^3}\\ &= \frac{4(p_1q_1+p_2q_2)}{(p_1+p_2)^3}.\end{aligned}$$

Estimate $I(\alpha)$

We set $\lambda = -\log t$ for $\alpha < \alpha_c$ and use the Chebyshev inequality to have

$$\begin{aligned}
Prob.(S_N \leq N_\alpha) &\leq Prob.(S_N \leq \alpha N)) \\
&\leq \inf_{\lambda>0} Prob.(e^{-\lambda S_N} \geq e^{-\lambda \alpha N}) \\
&\leq \inf_{\lambda>0} \frac{Exp.(e^{-\lambda S_N})}{e^{-\lambda \alpha N}} \\
&= \inf_{t\in(0,1)} \frac{\hat{S}_N(t)}{t^{\alpha N}} \\
&= \begin{cases} \inf_{t\in(0,1)} \dfrac{(1,1)A(t)^{N'}\begin{pmatrix}1\\0\end{pmatrix}}{t^{2N'\alpha}} & \text{if} \quad N = 2N' \\ \inf_{t\in(0,1)} \dfrac{(1,1)A_1(t)A(t)^{N'}\begin{pmatrix}1\\0\end{pmatrix}}{t^{(2N'+1)\alpha}} & \text{if} \quad N = 2N'+1 \end{cases} \\
&:= e^{-N I_N(\alpha)}.
\end{aligned}$$

When $N = 2N'$, we have

$$I_N(\alpha) = \sup_{t\in(0,1)} \left\{ \alpha \ln t - \frac{1}{2N'} \ln\left(\lambda_1(t)^{N'} u_1(t) + \lambda_2(t)^{N'} u_2(t)\right) \right\},$$

and when $N = 2N' + 1$, we have

$$I_N(\alpha) = \sup_{t\in(0,1)} \left\{ \alpha \ln t - \frac{1}{2N'+1} \ln\left(\lambda_1(t)^{N'} v_1(t) + \lambda_2(t)^{N'} v_2(t)\right) \right\}.$$

It can be shown that for $t \in (0, \infty)$

$$\frac{1}{N'} \ln\left(\lambda_1(t)^{N'} u_1(t) + \lambda_2(t)^{N'} u_2(t)\right) = \frac{\ln \lambda_1(t)}{2} + O(\frac{1}{N}), \quad \text{uniformly.}$$

Similarly for $N = 2N' + 1$ we also have the the uniform bounded.

Hence

$$\begin{aligned} Prob.(S_N \leq N_\alpha) &\leq \sup_{t\in(0,1)} \left\{ \alpha \ln t - \frac{\ln \lambda_1(t)}{2} \right\} + O(\frac{1}{N}) \\ &:= \alpha \ln t_\alpha - \frac{\ln \lambda_1(t_\alpha)}{2} + O(\frac{1}{N}), \end{aligned}$$

where t_α is a function of α.
Let

$$I(\alpha) = \alpha \ln t_\alpha - \frac{\ln \lambda_1(t_\alpha)}{2} \ .$$

We have

$$I_N(\alpha) = I(\alpha) + O(\frac{1}{N}).$$

Therefore, for $\alpha < \alpha_c$ we have

$$Prob.(S_N \leq N_\alpha) \quad \leq \quad O(1)e^{-NI(\alpha)},$$

Therefore, for $\alpha < \alpha_c$ we have

$$Prob.(S_N \leq N_\alpha) \quad \leq \quad O(1)e^{-NI(\alpha)},$$

By the same way, for $\alpha > \alpha_c$ we have

$$Prob.(S_N > N_\alpha) \quad \leq \quad O(1)e^{-NI(\alpha)}.$$

Therefore, for $\alpha < \alpha_c$ we have

$$Prob.(S_N \leq N_\alpha) \quad \leq \quad O(1)e^{-NI(\alpha)},$$

By the same way, for $\alpha > \alpha_c$ we have

$$Prob.(S_N > N_\alpha) \quad \leq \quad O(1)e^{-NI(\alpha)}.$$

The derivatives of $I(\alpha)$ satisfy following relations.

$$I''(\alpha) > 0 \quad \text{for} \quad \alpha \in (0, \infty),$$
$$I''(\alpha_c) = \frac{1}{\sigma^2},$$
$$-(2\alpha + 1)I''(\alpha)^2 \leq I'''(\alpha) \leq 2I''(\alpha)^2 \quad \text{for} \quad \alpha \in (0, \infty).$$

Assume above inequalities holds, by Taylor formula, we have

$$
\begin{aligned}
I(\alpha) &\geq \frac{I''(\alpha_c)}{2}(\alpha - \alpha_c)^2 = \frac{1}{\sigma^2}(\alpha_c - \alpha)^2 \quad \text{for} \quad \alpha \in (0, \alpha_c) \\
I(\alpha) &\leq \frac{I''(\alpha_c)}{2}(\alpha - \alpha_c)^2 = \frac{1}{\sigma^2}(\alpha_c - \alpha)^2 \quad \text{for} \quad \alpha > \alpha_c
\end{aligned}
$$

which yields the lower bound.

For $\alpha \in (\underline{\alpha}, \alpha_c)$, integrate α in $-(2\alpha+1)I''(\alpha)^2 \le I'''(\alpha) \le 2I''(\alpha)^2$ from α to α_c to give

$$-(\alpha_c-\alpha)(\alpha_c+\alpha+1) \le \frac{1}{I''(\alpha)} - \frac{1}{I''(\alpha_c)} \le 2(\alpha_c-\alpha).$$

It follows that

$$\frac{I''(\alpha_c)}{1+2I''(\alpha_c)(\alpha_c-\alpha)} \le I''(\alpha) \le \frac{I''(\alpha_c)}{1-I''(\alpha_c)(\alpha_c+\alpha+1)(\alpha_c-\alpha)}\ .$$

For $\alpha \in (\underline{\alpha}, \alpha_c)$, integrate α in $-(2\alpha+1)I''(\alpha)^2 \leq I'''(\alpha) \leq 2I''(\alpha)^2$ from α to α_c to give

$$-(\alpha_c - \alpha)(\alpha_c + \alpha + 1) \leq \frac{1}{I''(\alpha)} - \frac{1}{I''(\alpha_c)} \leq 2(\alpha_c - \alpha).$$

It follows that

$$\frac{I''(\alpha_c)}{1 + 2I''(\alpha_c)(\alpha_c - \alpha)} \leq I''(\alpha) \leq \frac{I''(\alpha_c)}{1 - I''(\alpha_c)(\alpha_c + \alpha + 1)(\alpha_c - \alpha)} .$$

Then by Taylor formula for any $\alpha \in (\underline{\alpha}, \alpha_c)$, there exists a certain $\xi \in (\alpha, \alpha_c)$ such that

$$\begin{aligned} I(\alpha) &= I(\alpha_c) + I'(\alpha_c)(\alpha - \alpha_c) + \frac{I''(\xi)}{2}(\alpha - \alpha_c)^2 \\ &\leq \frac{1}{2}\left(\frac{I''(\alpha_c)}{1 - I''(\alpha_c)(\alpha_c + \xi + 1)(\alpha_c - \xi)}\right)(\alpha_c - \alpha)^2 \\ &\leq \frac{1}{2}\left(\frac{I''(\alpha_c)}{1 - I''(\alpha_c)(\alpha_c + \alpha + 1)(\alpha_c - \alpha)}\right)(\alpha_c - \alpha)^2. \end{aligned}$$

Similarly, we have

$$I(\alpha) \geq \frac{1}{2}(\frac{I''(\alpha_c)}{1 + 2I''(\alpha_c)(\alpha_c - \alpha)})(\alpha_c - \alpha)^2 \ .$$

Using $I''(\alpha_c) = 1/\sigma^2$ we obtain the lower and upper bounds.

Thank You