The contact process on the regular tree with random vertex weights

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- Background and Motivations
- Model and the Main Results
- Proof of Two Lemmas

Background and Motivations

The contact process is an **interacting particle system**. (不忘初心)

T is the underlying graph, Z^d , T_d

The state space X={0,1}^{T} with the product topology.

$\eta \epsilon X$, η= {η(x), xϵ T}.

The process {η_t, t$\geq$0} is formally defined by a collection of infinitesimal rates

$$c(x, \eta) = 1 \quad \text{if } \eta(x)=1$$

$$\lambda \, \Sigma_{y:y\sim x}\, \eta(y), \quad \text{if } \eta(x)=0$$

where λ is a nonnegative parameter.

Background and Motivations

This process may serve as a model of the spread of an infection.

An individual at $x \in T$ is infected if $\eta(x)=1$ and healthy if $\eta(x)=0$.

A healthy individual x is infected by some infected neighbor y at rate λ.

Infected individuals recover at a constant rate, normalized to be 1.

Background and Motivations

Phase Transition.

- As λ increases, it is more likely to get infected.
- $A_t = \{x \epsilon\ T, \eta_t(x)=1\}$
- Two different limiting behaviors:
- the infection dies out $A_t = \phi$ for some t,
- or persists forever $A_t \neq \phi$ for all t .
- Critical point $\lambda_c = \inf\{\lambda,\ A_t \neq \phi \text{ for all } t\}$
- $0< \lambda_c < \infty$, largely unknown for $T = Z^d$. $\lim_{d\to\infty} d\,\lambda_c(d) = \frac{1}{2}$.

Background and Motivations

If $\mathrm{T} = \mathrm{Z}^{\mathrm{d}}$, $\lim_{t\to\infty} A_t = \phi \iff \lim_{t\to\infty} \eta_t(x) = 0$ for any $x \in T$

If *T* is a regular tree

$\lim_{t\to\infty} A_t = \phi \implies \lim_{t\to\infty} \eta_t(x) = 0$ for any $x \in T$

Two critical points

$\lambda_1 = \inf\{\lambda,\ A_t \neq \phi$ for all $t\}$,

$\lambda_2 = \inf\{\lambda, \limsup_{t\to\infty} \eta_t(x) = 1$ for all $x\}$.

$0 \leq \lambda_1 \leq \lambda_2 \leq \infty, \quad 0 < \lambda_1 < \lambda_2 < \infty,$

Background and Motivations

$0 < \lambda_1 < \lambda_2 < \infty$ verified for regular trees by Pemantle, Liggett & Stacey

infinitely many invariant measures for $\lambda_1 < \lambda < \lambda_2$,

one invariant measure for $\lambda < \lambda_1$

two invariant measures for $\lambda > \lambda_2$

Background and Motivations

Two formulations of random environments.

(1) Birth rate (or death rate) are i.i.d.

sufficient conditions for survival/dying out.

(2) random graph, Galton-Watson tree

For each GW tree T, there are also $0 \leq \lambda_1(T) \leq \lambda_2(T) \leq \infty$

easy to see that $\lambda_1(T)$ and $\lambda_2(T)$ are independent of *T.*

Easy $0 \leq \lambda_1 \leq \lambda_2 \leq \infty$, difficult $0 < \lambda_1 < \lambda_2 < \infty$,

Model and the main results

T^d = regular tree where the root O has degree d and other vertices have degree $d+1$.

omit d for simplicity if there is no confusion.

random weight nonnegative r. v. $\rho(x)$ for each vertex x ϵ T^d,

Assumption $\{\rho(x)$, x ϵ $T\}$ are i.i.d.

$\{\rho(x)$, x ϵ $T\}$ is uniformly bounded by a positive number M.

Model and the main results

This ensures the process is a spin system.

$c(x, \eta) = 1 \quad \text{if } \eta(x)=1$

$\lambda \, \Sigma_{y:y\sim x} \, \rho(x)\rho(y) \, \eta(y), \quad \text{if } \eta(x)=0$

where λ is a nonnegative parameter.

The contact process on the regular tree with random vertex weights.

Graphical construction.

Model and the main results

The random weights $\{\rho(x), x \in T\}$ are defined on some probability space $(\Omega, \mathscr{F}, \mu)$.

E = expectation operator w.r.t. measure μ.

For any $\omega \in \Omega$, P_λ^ω = the quenched law of the contact process on T with vertex weights $\{\rho(x, \omega), x \in T\}$ and infection rate λ.

The expectation operator w.r.t. P_λ^ω is denoted by E_λ^ω

Annealed measure $\mathbf{P}_\lambda(\bullet) = E\ P_\lambda^\omega(\bullet) = \int P_\lambda^\omega(\bullet)\, d\mu$

$\mathbf{E}_\lambda$ = expectation operator w.r.t. $\mathbf{P}_\lambda$.

Model and the main results

$\lambda_1(\omega) = \inf\{\lambda,\ A_t \neq \phi \text{ for all } t\}$ is independent of ω,

Easy to prove. So write $\lambda_1 = \lambda_1(T^d) = \lambda_c(d)$.

Theorem: $\lim_{d\to\infty} d\,\lambda_c(d) = \dfrac{1}{E\rho^2}$

Recall that for T = Z^d, $\lim_{d\to\infty} d\,\lambda_c(d) = \frac{1}{2}$.

Model and the main results

Lemma 1. If $\frac{(1+\lambda M)^2}{\lambda E\rho^2} < d$ for some $\lambda > 0$, then $\lambda_c(d) \leq \lambda$.

Recall that $\mu(\rho<M) = 1$

Let $\lambda_e(d, \omega) = \sup\{\lambda, \limsup_{t\to\infty} \frac{1}{t} \log \mathbf{P}_\lambda (A_t \neq \phi) < 0\}$.

Again $\lambda_e(d, \omega)$ is independent of ω, *so write* $\lambda_e(d)$

$\lambda_e(d) \leq \lambda_c(d)$

Lemma 2. $\lambda_e(d) \geq \left(dE\rho^2 + \frac{M^4}{E\rho^2}\right)^{-1}$.

Proof of Lemma 2

Comparison with a linear system $\{\xi_t\}$ = $\{\xi_t(x), x \epsilon T\}$.

$\xi_0(x) = 1$, $\xi_t(x) \epsilon\{0,1,2,3,\ldots \ldots\}$,

$\mathcal{A}f(\xi)$ =

$\sum_{x,x\epsilon T}|f(\xi_{x\delta}) - f(\xi)| + \sum_{x,x\epsilon T} \sum_{y:y\sim x} \lambda\rho(x)\rho(y)\, |f(\xi_{xy}) - f(\xi)|$

where $\xi_{x\delta}$ (z) = ξ(x) if $z \neq x$, or 0 if z =x;

ξ_{xy} (z) = ξ(x) if $z \neq x$, or ξ(x)+ ξ(y) if z =x

Proof of Lemma 2

the finite contact process A_t starting with one particle on the root,

- self-duality

the infinite contact process η_t *starting with* ***1*** (a particle at every site)

- coupling

The linear system $\{\xi_t\}$ *starting with* ***1***

$$P_\lambda^\omega (A_t \neq \phi) = P_\lambda^\omega (\eta_t(O) = 1) \leq P_\lambda^\omega (\xi_t(O) = 1) \leqq E_\lambda^\omega \xi_t(O),$$

$$\mathbf{P}_\lambda (A_t \neq \phi) = \mathbf{P}_\lambda (\eta_t(O) = 1) \leq \mathbf{E}_\lambda \xi_t(O)$$

Proof of Lemma 2

$\frac{d}{dt}\, \mathrm{E}_\lambda{}^\omega\, \xi_t(\mathrm{x}) = -\, \mathrm{E}_\lambda{}^\omega\, \xi_t(\mathrm{x}) + \sum_{\mathrm{y:y}\sim\mathrm{x}} \lambda \rho(x,\, \omega)\rho(y,\, \omega)\, \mathrm{E}_\lambda{}^\omega\, \xi_t(\mathrm{y}),$

$\frac{d}{dt}\, \mathrm{E}_\lambda{}^\omega\, \xi_t = (\mathrm{G}_\omega - I)\, \mathrm{E}_\lambda{}^\omega\, \xi_t$

where G_ω is a matrix , $\mathrm{G}_\omega(\mathrm{x,y}) = \lambda\rho(x,\, \omega)\rho(y,\, \omega)$ if $\mathrm{x}\sim y$

0 otherwise

$\mathrm{E}_\lambda{}^\omega\, \xi_t(\mathrm{O}) = e^{-t} \sum_{n=0}^{\infty} \sum_{x,x\epsilon T} \frac{t^n \mathrm{G}_\omega{}^{\mathrm{n}}(\mathrm{o,x})}{n!}$

$\mathbf{E}_\lambda\, \xi_t(\mathrm{O}) = e^{-t} \sum_{\mathrm{n}=0}^{\infty} \frac{\mathrm{t^n \lambda^n M^{2n} (d+1)^n}}{\mathrm{n!a}}\ \mathrm{E}\, a^{|Xn|}$ where a = $\mathbf{E}_\lambda \rho(x)^2 /_{M^2}$

Proof of Lemma 2

$\mathrm{E}\, a^{|Xn|} \leq [\lambda\, (dE\rho^2 + \frac{M^4}{E\rho^2}) - 1]^n$

by a martingale argument

exponential convergence to 0 occurs if $\lambda < \left(dE\rho^2 + \frac{M^4}{E\rho^2}\right)^{-1}$.

Therefore $\lambda_e(d) \geq \left(dE\rho^2 + \frac{M^4}{E\rho^2}\right)^{-1}$.

Proof of Lemma 1

SIR model $\{\zeta_t, t \geq 0\}$, $\zeta_t(x) = 0, 1$, or -1

$S_t = \{x \in T, \ \zeta_t(x) = 0\}$ susceptible

$I_t = \{x \in T, \ \zeta_t(x) = 1\}$ infected

$R_t = \{x \in T, \ \zeta_t(x) = -1\}$ removed

Initially $\zeta_0(O) = 1$, $\zeta_0(x) = 0$ for all $x \neq O$

$1 \to -1$ at rate 1.

$0 \to 1$ at rate $\lambda\rho(x, \omega)\rho(y, \omega)$ if $\zeta(x) = 1$, $\zeta(y) = 0$, y is a son of x

forward spreading of disease

Proof of Lemma 1

If initially both in the SIR model and in the contact process, only the root of tree T is infected.

By coupling $I_t \leq A_t$

$I_\infty = \cup\{I_t, \ t \geq 0\}$

$\{I_t \neq \emptyset, \ \text{for all } t \geq 0\} = \{|I_\infty| = \infty\}$

$P_\lambda^\omega(A_t \neq \phi \text{ for all } t \geq 0) \geq P_\lambda^\omega(I_t \neq \emptyset \text{ for all } t \geq 0) = P_\lambda^\omega(|I_\infty| = \infty)$,

$\mathbf{P}_\lambda(A_t \neq \phi \text{ for all } t \geq 0) \geq \mathbf{P}_\lambda(I_t \neq \emptyset \text{ for all } t \geq 0) = \mathbf{P}_\lambda(|I_\infty| = \infty)$,

Proof of Lemma 1

$L_n = \{x \in T, \quad |x| = n, x \in I_\infty \}$

$\{| I_\infty | = \infty\} = \bigcap_{n=0}^{\infty} \{|L_n| > 0\}$

$\mathbf{P}_\lambda (| I_\infty | = \infty) = \mathbf{P}_\lambda (\bigcap_{n=0}^{\infty} \{|L_n| > 0\}) = \lim_{n\to\infty} \mathbf{P}_\lambda (|L_n| > 0)$

Second moment method

$$\mathbf{P}_\lambda (|L_n| > 0) \geq \frac{(\mathbf{E}_\lambda |L_n|)^2}{\mathbf{E}_\lambda |L_n|^2}$$

Proof of Lemma 1

Suffices to prove $\liminf_{n\to\infty} \frac{(\mathbf{E}_\lambda |L_n|)^2}{\mathbf{E}_\lambda |L_n|^2} > 0$

Or equivalently $\limsup_{n\to\infty} \frac{\mathbf{E}_\lambda |L_n|^2}{(\mathbf{E}_\lambda |L_n|)^2} < \infty$

To compute $\mathbf{E}_\lambda |L_n|$

$\mathbf{E}_\lambda |L_n| = \mathbf{E}_\lambda \sum_{x:|x|=n} 1\{x \in I_\infty\} = \sum_{x:|x|=n} \boldsymbol{P}_\lambda(x \in I_\infty)$

$= d^n \boldsymbol{P}_\lambda(x \in L_n) = d^n \mathrm{E} \prod_{i=0}^{n-1} \frac{\lambda\rho(xi, \omega)\rho(x_{i+1}, \omega)}{1+\lambda\rho(xi, \omega)\rho(x_{i+1}, \omega)}$

Proof of Lemma 1

To compute $\mathbf{E}_\lambda\, |L_n|^2$

$\mathbf{E}_\lambda\, |L_n|^2 = \sum_{x:|x|=n} \sum_{y:|y|=n} \mathbf{P}_\lambda(x \in I_\infty, y \in I_\infty)$

$= \sum_{k=0}^{n} \sum_{x,y} \mathbf{P}_\lambda(x \in L_n, y \in L_n, |x \wedge y| = k)$

$= \sum_{x:|x|=n} \mathbf{P}_\lambda(x \in I_\infty) + \sum_{k=0}^{n-1} \sum_{x,y} \mathbf{P}_\lambda(x \in L_n, y \in L_n, |x \wedge y| = k)$

$= d^n \mathbf{P}_\lambda(x \in L_n) + \sum_{k=0}^{n-1} d^k \sum_{x,y} \mathbf{P}_\lambda(x \in L_{n-k}, y \in L_{n-k}, |x \wedge y| = k)$

$= d^n \mathbf{P}_\lambda(x \in L_n)$

$+ \sum_{k=0}^{n-1} d^k\, d^{n-k} d^{n-k-1}\, (d-1) \mathbf{P}_\lambda(x \in L_k) [\mathbf{P}_\lambda(x \in L_{n-k})]^2$

$= d^n \mathbf{P}_\lambda(x \in L_n) + \sum_{k=0}^{n-1} d^{2n-k-1}\, (d-1) \mathbf{P}_\lambda(x \in Ln) \mathbf{P}_\lambda(x \in L_{n-k})$

Proof of Lemma 1

$$\mathbf{E}_\lambda\, |L_n|^2 = \mathbf{E}_\lambda\, |L_n| \times (1 + \sum_{k=0}^{n-1} d^{n-k-1}\, (d-1)\, \mathbf{P}_\lambda(x \in L_{n-k}))$$

$$= \mathbf{E}_\lambda\, |L_n| \times (1 + \sum_{k=0}^{n-1} (1 - \tfrac{1}{d}) \mathbf{E}_\lambda\, |L_{n-k}|)$$

$$\frac{\mathbf{E}_\lambda\, |L_n|^2}{(\mathbf{E}_\lambda\, |L_n|)^2} = \frac{1}{\mathbf{E}_\lambda\, |L_n|} + \sum_{k=0}^{n-1} (1 - \tfrac{1}{d})\, \frac{\mathbf{E}_\lambda\, |L_{n-k}|}{\mathbf{E}_\lambda\, |L_n|}$$

$\frac{(1+\lambda M)^2}{\lambda E\rho^2} < d$ is a sufficient condition to ensure convergence.

Reference

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Thanks for your attention!