Convergence of EM Scheme for SDEs with Irregular Coefficients

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  - bounded Dini-continuous drift
  - unbounded Dini-continuous drift
- Convergence of EM scheme for SPDEs with multiplicative noise and Dini-continuous drift

Motivations

Consider an SDE on $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad t > 0, \quad X_0 = x \in \mathbb{R}^n. \tag{1}$$

Herein, $b : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,\infty)\times\mathbb{R}^n\otimes\mathbb{R}^m \to \mathbb{R}^n$, and $(W_t)_{t\geq 0}$ is an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Discrete time Euler-Maruyama (EM) scheme:

$$\overline{Y}_{(k+1)\delta} = \overline{Y}_{k\delta} + b(k\delta, \overline{Y}_{k\delta})\delta + \sigma(k\delta, \overline{Y}_{k\delta})\triangle W_{k\delta}, \;\; k \geq 0,$$

with $Y_0 = X_0 = x$, where $\triangle W_{k\delta} := W_{(k+1)\delta} - W_{k\delta}$.

Continuous time EM scheme:

$$Y_t = Y_0 + \int_0^t b(\eta_s, Y_{\eta_s})\mathrm{d}s + \int_0^t \sigma(\eta_s, Y_{\eta_s})\mathrm{d}W_s,$$

where $\eta_t := \lfloor t/\delta \rfloor \delta$.

Regular coefficients

If

$$|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \le K|x-y|, \quad x,y \in \mathbb{R}^n$$

for some $K > 0$, then

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X_t - Y_{\eta_t}|^p\Big) \lesssim \delta^{p/2}.$$

- The convergence above is called strong convergence;
- The convergence rate is $1/2$.

Irregular coefficients (Gyöngy, I., PA, 98)

Assume that

b satisfies a one-side Lipschitz condition in a domain D in $\mathbb{R}^n$ and σ is Lipschitzian. Then,

$$\sup_{t\le T}|X_t - Y_{\eta_t}| \le \xi\delta^\gamma, \quad \text{a.s.}, \quad \gamma\in(0,1/4),$$

where and ξ is a finite random variable.

Remark: The equation involved admits an invariance set.

Irregular coefficients (Yan, L.-Q., AOP, 2002)

Assuem that there exist $c > 0$, $0 \le \alpha \le 1/2$, $0 \le \beta_1, \beta_2 \le 1$ such that

$$|b(t,x) - b(s,y)| \lesssim |x-y| + |t-s|^{\beta_1},$$
$$|\sigma(t,x) - \sigma(s,y)| \lesssim |x-y|^{1/2+\alpha} + |t-s|^{\beta_2}.$$

Then,

$$\mathbb{E}|X_t - Y_{\eta_t}| \lesssim \delta^{\gamma},$$

where $\gamma := \beta \wedge \alpha \wedge \frac{4\alpha\beta_2}{1+2\alpha}$.

Tools: Meyer-Tanaka formula & estimates for local time.

Irregular coefficients (Gyöngy & Rásonyi, SPA, 2011)

Let $b = f + g$, where g is monotone decreasing and assume further that there exist $\alpha \in [0, 1/2]$ and $\gamma \in (0, 1)$ such that

$$|f(t,x) - f(t,y)| \lesssim |x-y|, \qquad |g(t,x) - g(t,y)| \lesssim |x-y|^{\gamma},$$
$$|\sigma(t,x) - \sigma(t,y)| \lesssim |x-y|^{1/2+\alpha}.$$

Then,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X_t - Y_{\eta_t}|\Big) \lesssim \begin{cases} \frac{1}{(\log \delta^{-1})^{1/2}}, & \alpha = 0, \\ \delta^{2\alpha^2} + \delta^{\alpha\gamma}, & \alpha \in (0, 1/2]. \end{cases}$$

Approach: Yamada-Watanabe approximation approach.

Irregular coefficients(Ngo & Taguchi, Math. Comp., 2016)

Assume that

- $\langle x-y, b(t,x)-b(t,y)\rangle \lesssim |x-y|^2$;
- $\langle (\sigma\sigma^*)(t,x)\xi, \xi\rangle \asymp |\xi|^2$;
- $|\sigma(t,x)-\sigma(t,y)| \lesssim |x-y|^{1/2+\alpha}, \alpha \in [0,1/2]$;
- $|b(t,x)-b(s,x)| + |\sigma(t,x)-\sigma(s,x)| \lesssim |t-s|^\beta, \beta \geq 1/2$;
- $b^{(i)} \in \mathcal{A}$ (the set, roughly speaking, of bounded variation with respect to a Gaussian measure on $\mathbb{R}^n$). Then,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |X_t - Y_{\eta_t}|\Big) \lesssim \begin{cases} \frac{1}{(\log \delta^{-1})^{1/2}}, & \alpha = 0, \\ \delta^{2\alpha^2}, & \alpha \in (0,1/2]. \end{cases}$$

Key tools: Yamada-Watanabe approach and heat kernel estimate.

Irregular coeff. (Pamen & Taguchi, arXiv1508.07513v1)

Consider an SDE $\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}L_t, \quad t > 0, \quad X_0 = x \in \mathbb{R}^n$, where b is bounded and

$$|b(t,x)-b(t,y)| \lesssim |x-y|^\beta,\ \beta \in (0,1),\ \ |b(t,x)-b(s,x)| \lesssim |x-y|^\eta,\ \eta \in [1/2, 1$$

- Then, $\mathbb{E}\Big(\sup_{0\le t\le T} |X_t - Y_{\eta_t}|^p\Big) \lesssim \delta^{\frac{p\beta}{2}}$ whenever $L =$ Wiener process.
- Moreover,

$$\mathbb{E}\Big(\sup_{0\le t\le T} |X_t - Y_{\eta_t}|^p\Big) \lesssim \begin{cases} \delta, & p \ge 2, p\beta \ge 2, \\ \delta^{\frac{p\beta}{2}}, & p \ge 2, 1 \le p\beta < 2 \text{ or } p \in [1,2) \end{cases}$$

whenever $L =$ truncated symmetric α-stable process with $\alpha \in (1,2)$ and $\alpha + \beta > 2$.

Dini-continuity

The mapping $\phi : E \mapsto \mathbb{R}_+$ is Dini-continuous if $\int_0^1 \frac{\omega_\phi(t)}{t}\mathrm{d}t < \infty$, where

$$\omega_\phi(t) := \sup_{|x-y|\le t} \{|\phi(x) - \phi(y)|\},$$

modulus of continuity.

- Every Dini continuous function is continuous;
- Every Lipschitz continuous function is Dini-continuous;
- Every Hölder continuous function is Dini-continuous;
- There are numerous Dini-continuous functions which are not Hölder continuous as the example below shows.

An Example (Dini-continuous but not Hölder continuous)

Let

$$\phi(x) = \begin{cases} 0 & x = 0 \\ (\log(c + x^{-1}))^{-(1+\delta)}, & x > 0 \end{cases} \tag{2}$$

for some constants $\delta > 0$ and $c \geq e^{2+\delta}$. For any $\alpha \in (0,1)$, note that

$$\lim_{x \to 0+} \frac{|\phi(x) - \phi(0)|}{x^\alpha} = \lim_{x \to 0+} \frac{(\log(c + x^{-1}))^{-(1+\delta)}}{x^\alpha} = \infty.$$

So ϕ is not α-Hölder continuous for any $\alpha \in (0,1)$. We can show that, for any $t \geq 0$,

$$0 \leq \phi(y) - \phi(x) \leq \phi(x+t) - \phi(x) = \int_0^t \phi'(x+s)\mathrm{d}s \leq \int_0^t \phi'(s)\mathrm{d}s = \phi(t),$$

where $0 \leq x < y \leq x + t$. Consequently, $\omega_\phi(t) \leq \phi(t)$. As a result, the function ϕ defined in (2) is Dini-continuous.

Setup I

Consider an SDE on $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, |\cdot|)$

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad t > 0, \quad X_0 = x \in \mathbb{R}^n. \tag{3}$$

Introduce the following class

$$\mathscr{D} := \left\{\phi : [0,\infty) \mapsto [0,\infty) \text{ is increasing}, \phi^2 \text{ is concave}, \int_0^1 \frac{\phi(s)}{s}\mathrm{d}s < \infty\right\}.$$

Clearly, ϕ constructed as in (2) above belongs to $\mathscr{D}$.

EM scheme associated with (3) is

$$\mathrm{d}Y_t = b(\eta_t, X_{\eta_t})\mathrm{d}t + \sigma(\eta_t, X_{\eta_t})\mathrm{d}W_t, \quad t > 0, \quad Y_0 = x, \tag{4}$$

where $\eta_t := \lfloor t/\delta \rfloor \delta$.

Setup I

We assume that

(A1) $\sigma \in C^2(\mathbb{R}^n; \mathbb{R}^{n\times m})$ and

$$\|b\|_{T,\infty} + \|\sigma\|_{T,\infty} + \sum_{i=0}^{2} \|\nabla^i \sigma\|_{T,\infty} + \|(\sigma\sigma^*)^{-1}\|_{T,\infty} < \infty.$$

(A2) There exists $\phi \in \mathscr{D}$ such that

$$|b(t,x) - b(t,y)| \le \phi(|x-y|)$$

and

$$|b(s,x) - b(t,x)| + |\sigma(s,x) - \sigma(t,x)| \le \phi(|s-t|), s,t \in [0,T], x,y \in \mathbb{R}^n.$$

SDEs/SPDEs with irregular coefficients

For instance,

- Luo, D., arXiv:1605.02820, 2016.
- Krylov, N. V. & Röckner, M., PTRF, 2005.
- Veretennikov, A. J., Mat. Sbornik, 1980.
- Wang, F.-Y., JDE, 2016.
- Wang, F.-Y. & Zhang, X.-C., SIAM J. Math. Anal., 2016.
- Wang, F.-Y. & Zhang, X.-C.,, IAQP, 2016.
- Zhang, X.-C., SPA, 2005.
- Zvonkin, A. K., Mat. Sbornik, 1974.

Main result I: bounded drift

Theorem

Under **(A1)** and **(A2)**,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_t - Y_t|^2\Big) \lesssim_T \phi(\sqrt{\delta})^2.$$

- The proof is based on Zvonkin's transform and regularity of Kolmogrov equation.
- The convergence rate is half for $\phi(x) = \sqrt{x}$.

Regularity of Kolmogrov equation (1)

For $\lambda > 0$, consider the following partial differential equation:

$$\partial_t u_t^\lambda(x) + L_t u_t^\lambda(x) + b_t(x) + \nabla_{b_t(x)} u_t^\lambda(x) = \lambda u_t^\lambda(x), \quad u_T^\lambda(x) = 0, \quad (5)$$

where

$$L_t := \frac{1}{2} \sum_{i,j} \langle (\sigma_t \sigma_t^*)(\cdot) e_i, e_j \rangle \nabla_{e_i} \nabla_{e_j}.$$

The unique mild solution of (5) is

$$u_s^\lambda(x) = \int_s^T \mathrm{e}^{-\lambda(t-s)} P_{s,t}^0 \{ b(x) - \nabla_{b(x)} u_t^\lambda(x) \} \mathrm{d}t, \quad (6)$$

where $(P_{s,t}^0)_{0 \le t \le r}$ is the semigroup generated by $(X_t^{s,x})$ solving

$$\mathrm{d}X_t^{s,x} = \sigma(X_t^{s,x}) \mathrm{d}W_t, \quad X_s^{s,x} = x.$$

Regularity of Kolmogrov equation (2)

Under **(A1)**,

(i) There exists $\lambda(T) > 0$ such that, for any $\lambda \geq \lambda(T)$, (6) has a unique solution $u^\lambda \in C([0,T]; C_b^1(\mathbb{R}^d; \mathbb{R}^d))$;

(ii) If (**A2**) further holds, then

$$\lim_{\lambda \to \infty} \{\|\nabla u^\lambda\|_{T,\infty} + \|\nabla^2 u^\lambda\|_{T,\infty}\} = 0.$$

(iii) $\theta_t = x + u_t(x)$ is a C^1-diffeomorphism with

$$\lambda_1 \leq \|\nabla \theta\|_{T,\infty} \leq \lambda_2, \qquad \lambda_1, \lambda_2 \in (0,1).$$

Remark: The infinite-dimensional counterpart is due to Wang, F.-Y. (JDE, 2016).

Zvonkin's transform

By Zvonkin's transform, the solutions to (3) and (4) can be rewritten respectively as

- $$X_t + u_t^\lambda(X_t) = x + u_0^\lambda(x) + \lambda \int_0^t u_s^\lambda(X_s)\mathrm{d}s + \int_0^t \{\sigma_s + (\nabla u_s^\lambda)\sigma_s\}(X_s)\mathrm{d}$$

- and

$$\begin{aligned} &Y(t) + u_t^\lambda(Y_t) \\ &= x + u_0^\lambda(x) + \lambda \int_0^t u_s^\lambda(Y_s)\mathrm{d}s + \int_0^t \{I_{n\times n} + \nabla u_s^\lambda(\cdot)\}(Y_s)\sigma_{\eta_s}(Y_{\eta_s})\mathrm{d}W_s \\ &\quad + \int_0^t \{I_{n\times n} + \nabla u_s^\lambda(\cdot)\}(Y_s)\{b_{\eta_s}(Y_{\eta_s}) - b(Y(s))\}\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \sum_{k,j} \langle\{(\sigma_{\eta_s}\sigma_{\eta_s}^*)(Y_{\eta_s}) - (\sigma_s\sigma_s^*)(Y_s)\}e_k, e_j\rangle(\nabla_{e_k}\nabla_{e_j}u_s^\lambda)(Y_s)\mathrm{d}s. \end{aligned}$$

Setup II

We still focus on (3) and (4).

(H1) There exists an increasing function $C_{b,\sigma} : [0,\infty) \to [0,\infty)$ such that

$$|b(t,x)| + \sum_{i=0}^{2} |\nabla^i \sigma(t,x)| + |(\sigma\sigma^*)^{-1}(t,x)| \le C_{b,\sigma}(t)(1+|x|).$$

(H2) For any $m \ge 1$, there exists $\phi_m \in \mathscr{D}$ satisfying $\phi_m \le C_\phi(m)\phi_1$ for a non-negative function $C_\phi(x) = O(\mathrm{e}^{\mathrm{e}^{x^4}})$ on $[0,\infty)$ such that

$$|b_t(x) - b_t(y)| \le \phi_m(|x-y|), \quad t \in [0,m], |x| \vee |y| \le m,$$

and

$$|b_s(x) - b_t(x)| + |\sigma_s(x) - \sigma_t(x)| \le \phi_m(|s-t|), \quad s,t \in [0,m], |x| \le m.$$

Main result II: unbounded drift

Theorem

Under **(H1)** and **(H2)**,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_t - Y_t|^2\Big) \to 0.$$

- The proof is based on the truncation argument.
- Refine the proof of Lemma 2.1 (Wang, F.-Y., JDE, 2016) on the estimates for

$$|\nabla P^0_{s,t}f|(x),\quad \|\nabla^2 P^0_{s,t}f\|_\infty,\quad \|\nabla u\|_{T,\infty},\quad \|\nabla^2 u\|_{T,\infty}.$$

- We have $|X_t - X_t^n|^2 \lesssim |X_t - X_t^{(m)}|^2 + |X_t^{(m)} - X_t^{(m),\delta}|^2 + |X_t^{(m),\delta} - X_t^\delta|^2.$

Setup III

Consider an SPDE on $\mathbb{H}$:

$$\mathrm{d}X_t = \{AX_t + b_t(X_t)\}\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W(t), \quad t > 0, \quad X_0 = x. \tag{7}$$

Assume that

(a1) $(A, \mathscr{D}(A))$ is a negative definite self-adjoint operator on $\mathbb{H}$ such that $(-A)^{\varepsilon-1}$ is of trace class for some $\varepsilon \in (0,1)$;

(a2) $\|b\|_{T,\infty} + \sum_{i=0}^{2} \|\nabla^i \sigma\|_{T,\infty} + \|(\sigma\sigma^*)^{-1}\|_{T,\infty} < \infty$.

(a3) For any $T > 0$, there exist $\phi \in \mathscr{D}$ and $\widetilde{K} > 0$ such that

$$|b_t(x) - b_t(y)| \le \phi(|x-y|),$$

$$|b_s(x) - b_t(x)| + |\sigma_s(x) - \sigma_t(x)| \le \phi(|s-t|).$$

Setup III

(a4) Moreover for any $x \in \mathbb{H}$ and $t \geq 0$,

$$\lim_{n\to\infty} \|\sigma_t(x) - \sigma_t(\pi_n x)\|_{\mathrm{HS}}^2 := \lim_{n\to\infty} \sum_{k\geq 1} |[\sigma_t(x) - \sigma_t(\pi_n x)]\overline{e}_k|^2 = 0$$

(a5) (Initial value) There exists $\alpha \in (0,1]$ such that $X_0 : \Omega \to \mathscr{D}((-A)^\alpha)$ is $\mathscr{F}_0/\mathscr{B}(\mathscr{D}((-A)^\alpha))$ measurable with $\mathbb{E}\|X_0\|_\alpha^4 < \infty$.

- Under **(a1)-(a4)**, (7) has a unique non-explosive mild solution, see Theorem 1.1 due to Wang, F.-Y. (JDE, 2016).
- **(a5)** is imposed just for the convergence of EM scheme.

Exponential Integrator Scheme

The exponential integrator scheme associated with (7) is

$$Y_t = e^{At}Y_0 + \int_0^t e^{A(t-\eta_s)} b_{\eta_s}(Y_{\eta_s})ds + \int_0^t e^{A(t-\eta_s)} \sigma_{\eta_s}(Y_{\eta_s})dW(s).$$

Convergence of numerical scheme for SPDEs with regular coefficients. For instance,

- Hutzenthaler et al., AAP, 2012;
- Jentzen, A., SIAM J. Numer. Anal., 2011;
- Jentzen, A., Röckner, M., Found. Comput. Math., 2015.

Main result III: infinite-dimensional setting

Theorem

Under **(a1)-(a5),**

$$\mathbb{E}|X_t - Y_{\eta_t}|^2 \to 0.$$

- The proof is based on Zvonkin's transform and regularity of Kolmogrov equation.
- Difficulty: A is an unbounded operator.
- The convergence rate can also be revealed.

Zvonkin's transform

$$
\begin{aligned}
X_t &= \mathrm{e}^{At}(x+u_0(x)) - u_t(X_t) + \int_0^t \mathrm{e}^{A(t-s)}(\lambda - A)u_s(X_s)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{A(t-s)}(\sigma_s + (\nabla u_s)\sigma_s)(X_s)\mathrm{d}W(s).
\end{aligned}
$$

and

$$
\begin{aligned}
Y_t &= \mathrm{e}^{At}\theta_0(X_0) - u_t(Y_t) + \int_0^t \mathrm{e}^{A(t-s)}(\lambda - A)u_s(Y_s)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{A(t-s)}(\nabla\theta_s)(Y_s)[\mathrm{e}^{A(s-\eta_s)}b_{\eta_s}(Y_{\eta_s}) - b_s(Y_s)]\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \mathrm{e}^{A(t-s)}\sum_{k=1}^{\infty}(\nabla^2_{\{\mathrm{e}^{A(s-\eta_s)}Q_{\eta_s}(Y_{\eta_s})-Q_s(y_s)\}\bar{e}_k}u_s)(Y_s)\mathrm{d}s \\
&+ \int_0^t \mathrm{e}^{A(t-s)}(\nabla\theta_s)(Y_s)\mathrm{e}^{A(s-\eta_s)}Q_{\eta_s}(Y_{\eta_s})\mathrm{d}W(s).
\end{aligned}
$$

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Thanks A Lot !