On the occupation times for spectrally negative Lévy processes

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Outline of the Talk

1. Spectrally negative Lévy process (SNLP)
2. Scale functions and the exit problems
3. Scale function identities
4. Laplace transforms on occupation times
   - A new approach
   - Joint occupation times
   - A discounted potential measure

Lévy process

- A Lévy process is a stochastic process with stationary independent increment.
- It is known that

$$X_t - X_0 = \gamma t + \sigma B_t + J_t,$$

where γ is a constant, process B is a Brownian motion, process J is a pure jump process and, B and J are independent.

- X_t is spectrally negative if J_t has no positive jumps.

- For spectrally negative Lévy process X_t with $X_0 = 0$,

$$\mathbb{E}e^{\theta X_t} = e^{t\psi(\theta)},$$

for $\theta, t \geq 0$, where the Laplace exponent

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{0} \left(e^{\theta z} - 1 - \theta z 1_{(-1,0)}(z)\right) \Pi(dz),$$

and the Lévy measure Π is a σ-finite measure on $(-\infty, 0)$ satisfying

$$\int_{-\infty}^{0} (1 \wedge z^2)\Pi(dz) < \infty.$$

- Let Φ be the right inverse of ψ.

Scale function

- The exit problem concerns how the process X first leaves an interval $[a, b]$.
- We often need the scale function to study the exit problem.
- For $q \geq 0$, the q-scale function $W^{(q)}$ of the process X is defined as the function with Laplace transform on $[0, \infty)$ given by

$$\int_0^\infty \mathrm{e}^{-\theta z} W^{(q)}(z)\mathrm{d}z = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Phi(q),$$

  and such that $W^{(q)}(x) = 0$ for $x < 0$.
- We write $W = W^{(0)}$.

Remarks on scale functions

- For SNLP with positive drift,

$$W(x) = \frac{1}{\psi'(0+)} \mathbb{P}_x\{\inf_{t<\infty} X_t \geq 0\}.$$

- Roughly, $W^{(q)}$ is W for process killed at rate q.
- The scale function was initially obtained for diffusion process. It is positive, strictly increasing, continuous and often differentiable.
- The explicit expressions of scale function are not always known.
- In the the study of spectrally negative Lévy processes we often want to express the interested quantities in terms of scale functions.

More scale functions

- $$Z^{(p)}(x) = 1 + p\int_0^x W^{(p)}(y)dy.$$
- $$Z^{(p)}(x,\theta) = e^{\theta x}\left(1 + (p - \psi(\theta))\int_0^x e^{-\theta y} W^{(p)}(y)\mathrm{d}y\right).$$
- Given $p, q \geq 0$, for $0 \leq a$ further define
$$W_a^{(p,q)}(x) := W^{(p)}(x) + (q-p)\int_a^x W^{(q)}(x-y)W^{(p)}(y)dy.$$
$$Z_a^{(p,q)}(x) := Z^{(p)}(x) + (q-p)\int_a^x W^{(q)}(x-y)Z^{(p)}(y)dy$$

Solutions to the exit problems

- Define

$$\tau_b^+ = \inf\{t > 0 \colon X_t > b\} \text{ and } \tau_0^- = \inf\{t > 0 \colon X_t < 0\}.$$

- It is well known that for $0 \leq x \leq b$,

$$\mathbb{E}_x\left[e^{-q\tau_b^+}; \tau_b^+ < \tau_0^-\right] = \frac{W^{(q)}(x)}{W^{(q)}(b)},$$

$$\mathbb{E}_x\left[e^{-q\tau_0^-}; \tau_0^- < \tau_b^+\right] = Z^{(q)}(x) - Z^{(q)}(b)\frac{W^{(q)}(x)}{W^{(q)}(b)}.$$

- For $q, \theta > 0$,

$$\mathbb{E}_x\left[e^{-q\tau_0^- + \theta X_{\tau_0^-}}; \tau_0^- < \infty\right] = Z^{(q)}(x, \theta) - \frac{q - \psi(\theta)}{\Phi(q) - \theta} W^{(q)}(x).$$

Potential measures

Given $q > 0$, for any x, y, the expected total discounted time when process X takes values in $\mathrm{d}x$ is

$$\begin{aligned}
&\int_0^\infty \mathbb{P}\{X_t \in \mathrm{d}x\} e^{-qt} \mathrm{d}t \\
&= q^{-1} \mathbb{P}\{X(e_q) \in \mathrm{d}x\} \\
&= \left(\Phi'(q) e^{-\Phi(q)x} - W^{(q)}(-x) \right) \mathrm{d}x,
\end{aligned}$$

where e_q is an independent exponential random variable with rate q.

Identities on scale functions

For any $p, q > 0$

$$(q-p)\int_0^a W^{(p)}(a-y)W^{(q)}(y)\mathrm{d}y = W^{(q)}(a) - W^{(p)}(a),$$
$$(q-p)\int_0^a W^{(p)}(a-y)Z^{(q)}(y)\mathrm{d}y = Z^{(q)}(a) - Z^{(p)}(a).$$

For any $r > 0$ and $a < z$,

$$(r-q)\int_a^z W^{(r)}(z-x)W_a^{(p,q)}(x)\mathrm{d}x = W_a^{(p,r)}(z) - W_a^{(p,q)}(z),$$

$$(r-q)\int_a^z W^{(r)}(z-x)Z_a^{(p,q)}(x)\mathrm{d}x = Z_a^{(p,r)}(z) - Z_a^{(p,q)}(z).$$

More identities

For any $q \geq 0$ and $0 \leq a < b$, we have

$$\mathbb{E}_x \left[e^{-q\tau_a^-} W^{(p)}(X_{\tau_a^-});\ \tau_a^- < \tau_b^+ \right] = W_a^{(p,q)}(x) - \frac{W^{(q)}(x-a)}{W^{(q)}(b-a)} W_a^{(p,q)}(b)$$

and

$$\mathbb{E}_x \left[e^{-q\tau_a^-} Z^{(p)}(X_{\tau_a^-});\ \tau_a^- < \tau_b^+ \right] = Z_a^{(p,q)}(x) - \frac{Z^{(q)}(x-a)}{Z^{(q)}(b-a)} Z_a^{(p,q)}(b).$$

An observation

We propose a new approach using a property for Poisson process.

- Let N_t be an independent Poisson process with intensity λ. Let $0 < T_1 < T_2 < \dots$ be its arrival times.
- For any subset A of $\mathbb{R}$,

$$\mathbb{E}e^{-\lambda \int_0^t 1_A(X_s)\mathrm{d}s} = \mathbb{P}\{\{T_i\} \cap \{s \le t : X_s \in A\} = \emptyset\}.$$

- By considering the the SNLP observed at discrete Poisson arrival times we get around the problem caused by infinite activity.
- Some fluctuation identities for SNLP observed at Poisson arrival times have been obtained in Albrecher, Ivanovs and Z. (2014).

$$I_x := \mathbb{E}_x e^{-\lambda \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s} = \mathbb{P}\{\{T_i\} \cap \{s \le e_q : X_s > 0\} = \emptyset\}$$

with $I \equiv I_0$. Conditioning on X_{T_1},

$$\begin{aligned} I &= \int_{-\infty}^{0} \mathbb{P}\{T_1 < e_q, X_{T_1} \in \mathrm{d}x\} I_x + \mathbb{P}\{T_1 > e_q\} \\ &= \lambda \int_{-\infty}^{0} \int_0^\infty e^{-(q+\lambda)t} \mathbb{P}\{X_t \in \mathrm{d}x\} (\mathbb{P}_x\{\tau_0^+ < e_q\} I + \mathbb{P}_x\{\tau_0^+ > e_q\}) \mathrm{d}t \\ &\quad + \frac{q}{q+\lambda}. \end{aligned}$$

Solving the equation for I,

$$\begin{aligned} I &= \frac{\frac{q}{q+\lambda} + \lambda \int_{-\infty}^0 (1 - e^{\Phi(q)x}) \int_0^\infty e^{-(q+\lambda)t} \mathbb{P}\{X_t \in \mathrm{d}x\} \mathrm{d}t}{1 - \lambda \int_{-\infty}^0 e^{\Phi(q)x} \int_0^\infty e^{-(q+\lambda)t} \mathbb{P}\{X_t \in \mathrm{d}x\} \mathrm{d}t} \\ &= \cdots = \frac{\Phi(q)}{\Phi(q+\lambda)}. \end{aligned}$$

Joint occupation times

- For $0 < a < b$ and $0 < x < b$ we are interested in

$$\mathbb{E}_x\left[e^{-\lambda_- \int_0^{\tau_0^-} 1_{(0,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{\tau_0^-} 1_{(a,b)}(X_s)\mathrm{d}s};\ \tau_0^- < \tau_b^+\right]. \quad (1)$$

- Note that when $\tau_0^- < \tau_b^+$,

$$\int_0^{\tau_0^-} 1_{(0,a)}(X_s)\mathrm{d}s + \int_0^{\tau_0^-} 1_{(a,b)}(X_s)\mathrm{d}s = \tau_0^-.$$

- We want to associate (1) to Poisson arrival times.

- Let N_- and N_+ be two independent Poisson processes with rates λ_- and λ_+, respectively.
- Let (T_i^-) and (T_i^+) be the respective arrival times.
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$$\mathbb{E}_x\left[e^{-\lambda_- \int_0^{\tau_0^-} 1_{(0,a)}(X_s)\mathrm{d}s-\lambda_+ \int_0^{\tau_0^-} 1_{(a,b)}(X_s)\mathrm{d}s};\ \tau_0^- < \tau_b^+\right]$$

$$= \mathbb{P}_x\left\{\{T_i^-\} \cap \{s : s < \tau_0^- < \tau_b^+, X_s \in (0,a)\}\right.$$
$$= \emptyset$$
$$\left.= \{T_i^+\} \cap \{s : s < \tau_0^- < \tau_b^+, X_s \in (a,b)\}\right\}.$$

Theorem

For any $0 < a < b, 0 \leq x \leq b$ *and* $\lambda_-, \lambda_+ \geq 0$, *we have*

$$\mathbb{E}_x\left[e^{-\lambda_- \int_0^{\tau_b^+} 1_{(0,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{\tau_b^+} 1_{(a,b)}(X_s)\mathrm{d}s}; \tau_b^+ < \tau_0^-\right] = \frac{W_a^{(\lambda_-,\lambda_+)}(x)}{W_a^{(\lambda_-,\lambda_+)}(b)}. \tag{2}$$

$$\mathbb{E}_x\left[e^{-\lambda_- \int_0^{\tau_0^-} 1_{(0,a)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{\tau_0^-} 1_{(a,b)}(X_s)\mathrm{d}s}; \tau_0^- < \tau_b^+\right] = Z_a^{(\lambda_-,\lambda_+)}(x) - \frac{W_a^{(\lambda_-,\lambda_+)}(x) Z_a^{(\lambda_-,\lambda_+)}(b)}{W_a^{(\lambda_-,\lambda_+)}(b)}. \tag{3}$$

By letting either $a \to 0+$, or $a \to b-$, or $p = q$ in (2) we recover

$$\mathbb{E}_x\left[e^{-q\tau_a^+}; \tau_a^+ < \tau_0^-\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

One can also recover

$$\mathbb{E}_x\left[e^{-q\tau_0^-}; \tau_0^- < \tau_a^+\right] = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}$$

similarly from (3).

A discounted potential measure

- Let e_q be an independent exponential random variable with rate q, we are interested in

$$\mathbb{E}\left[e^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s}; X(e_q) \in \mathrm{d}x\right]. \quad (4)$$

- (4) concerns the distributions of

$$\left(\int_0^t 1_{(-\infty,0)}(X_s), \int_0^t 1_{(0,\infty)}(X_s), X_t\right).$$

Theorem

For any $q, \lambda_-, \lambda_+ > 0$ *and* $x \in \mathbb{R}$,

$$\mathbb{E}\left[e^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s}; X(e_q) \in \mathrm{d}x\right]$$
$$= \frac{q}{\lambda_+ - \lambda_-}[\Phi(q+\lambda_+) - \Phi(q+\lambda_-)]Z^{(q+\lambda_-)}(-x, \Phi(q+\lambda_+))\mathrm{d}x$$
$$- qW^{(q+\lambda_-)}(-x)\mathrm{d}x.$$

One more identity

Observe that for $\lambda, q > 0$

$$\mathbb{E}\left[e^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s}; X(e_q) \in \mathrm{d}x\right]$$
$$= \frac{q}{\lambda_+ - \lambda_-}[\Phi(q+\lambda_+) - \Phi(q+\lambda_-)]$$
$$\times \mathbb{E}\left[e^{-(q+\lambda_-)\tau_0^- + \Phi(q+\lambda_+)X_{\tau_0^-}}; \tau_0^- < \infty\right] \mathrm{d}x.$$

Taking an integral on x, we have the following result.

Corollary

For $q, \lambda_-, \lambda_+ > 0$,

$$\mathbb{E}e^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s} = \frac{q\Phi(q+\lambda_-)}{(q+\lambda_-)\Phi(q+\lambda_+)}.$$

Brownian motion

Suppose that B is a standard one-dimensional Brownian motion and $X_t = B_t + \mu t$ for constant $\mu \in \mathbb{R}$. Then

$$\psi(\lambda) = \mu\lambda + \frac{1}{2}\lambda^2,$$

$$\Phi(q) = \sqrt{\mu^2 + 2q} - \mu,$$

$$W^{(q)}(x) = \frac{1}{\sqrt{\mu^2 + 2q}}\left(e^{\left(\sqrt{\mu^2+2q}-\mu\right)x} - e^{-\left(\sqrt{\mu^2+2q}+\mu\right)x}\right),$$

$$Z^{(p)}(x,\theta) = \frac{\sqrt{\mu^2 + 2p} + \sqrt{\mu^2 + 2\theta}}{2\sqrt{\mu^2 + 2p}} e^{-(\sqrt{\mu^2+2p}-\mu)x} + \frac{\sqrt{\mu^2 + 2p} - \sqrt{\mu^2 + 2\theta}}{2\sqrt{\mu^2 + 2p}} e^{(\sqrt{\mu^2+2p}+\mu)x}.$$

For $x \geq 0$,

$$\mathbb{E}\left[e^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s}, X(e_q) \in \mathrm{d}x\right]$$

$$= \frac{q}{\lambda_+ - \lambda_-}\left(\sqrt{\mu^2 + 2(q+\lambda_+)} - \sqrt{\mu^2 + 2(q+\lambda_-)}\right) \times e^{\left(\mu - \sqrt{\mu^2+2(q+\lambda_+)}\right)x}\mathrm{d}x.$$

For $x < 0$,

$$\mathbb{E}\left[e^{-\lambda_- \int_0^{e_q} 1_{(-\infty,0)}(X_s)\mathrm{d}s - \lambda_+ \int_0^{e_q} 1_{(0,\infty)}(X_s)\mathrm{d}s}, X(e_q) \in \mathrm{d}x\right]$$

$$= \frac{q}{\lambda_+ - \lambda_-}\left(\sqrt{\mu^2 + 2(q+\lambda_+)} - \sqrt{\mu^2 + 2(q+\lambda_-)}\right) \times e^{(\mu + \sqrt{\mu^2+2(q+\lambda_-)})x}\mathrm{d}x.$$

Summary

- We use a direct approach to find Laplace transforms on occupation times for spectrally negative Lévy processes.
- It identifies the Laplace transform on occupation time as a fluctuation result on SNLP observed at Poisson arrival times.
- To implement this approach we need to be familiar with fluctuation results on SNLP and identities on scale functions.
- We also need to carry out lengthy computations.

Thank you for your attention!