Quasi-Stationary Distributions And Their Applications

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Outline of Topics

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QSDs

- What is QSD, and why study it? The first thing is to define Quasi-Stationary Distributions(QSDs).

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- What is QSD, and why study it? The first thing is to define Quasi-Stationary Distributions(QSDs).
- Before we do that, we first recall the concept of Stationary Distribution.
- Suppose that $(X_t)_{t\geq 0}$ is a Markov process, if $(X_t)_{t\geq 0}$ exists a stationary distribution μ. Then,

$$P(X_t \in A) \to \mu(A), \qquad t \to \infty$$

for any Borel set A, where $\mu(A) > 0$ for any nonempty open set A.

QSDs

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- If μ is a stationary distribution, then $P_\mu(X_t \in A) = \mu(A), \quad t \geq 0$ for any Borel set A.
- When we discuss the QSDs, we still assume that $(X_t)_{t\geq 0}$ is a Markov Process and limit distribution exists. i.e.
- $\lim_{t\to\infty} P(X_t \in A) = \mu(A)$ for any Borel set A. But, where

$$\mu(A) = \begin{cases} 1 & 0 \in A, \\ 0 & \text{otherwise.} \end{cases}$$

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QSDs

- A Quasi-Stationary Distribution (in short QSD) for X is a probability measure supported on $(0,\infty)$ satisfying for all $t \geq 0$,

$$\mathbf{P}_{\nu}(\mathbf{X(t)} \in \mathbf{A}|\mathbf{T} > \mathbf{t}) = \nu(\mathbf{A}),\ \forall\ \textbf{borel set}\ \mathbf{A} \subseteq (\mathbf{0},\infty).$$

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- It is not hard to show if such ν exist then

$$\mathbf{P}_\nu(\mathbf{T} > \mathbf{t}) = \mathbf{e}^{-\mu \mathbf{t}}$$

  for some $\mu \in (0, \infty)$.
- A QSD must be infinitely divisible (D.Vere-Jones 1969)

QSDs

- A probability measure π supported on $(0, \infty)$ is a LCD If there exists a probability measure ν on $(0, \infty)$ such that the following limit exists in distribution

$$\lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X}(\mathbf{t}) \in \bullet \mid \mathbf{T} > \mathbf{t}) = \pi(\bullet).$$

We also say that ν is attracted to π or is in the domain of attraction of π or π is a ν-LCD

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- Trivially, any QSD ν is an ν-LCD.
- The ν-LCD is a QSD (Vere-Jones(1969)).

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- (i) determination of all QSD's; and
- (ii) solve the domian of attraction problem, namely, characterize all probability measure ν such that a given QSD M is a ν-LCD.
- (iii) The rate of convergence of the transition probabilities of the conditioned process to their limiting values.

QSDs of Markov Chains

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- For a Birth and death process, Erik A. VAN DOORN (Adv.Appl.Prob.23, 683-700,1991) obtained
  Proposition (i) if $S = \infty$, then either λ_C (Kingman's decay parameter)=0 and there is no QSD, or $\lambda_C > 0$ and there is a one-parameter family of QSDs, Viz,$\{\{q_j(x)\}, 0 < x \leq \lambda_C\}$.
  (ii) If $S < \infty$, then $\lambda_C > 0$ and there is precisely one QSD,Viz,$\{q_j(\lambda_C)\}$

QSDs of Markov Chains

- Our result is
  Theorem 1 For a Birth and death process,
  (i) there is QSD iff $A = \infty$ and $\delta < \infty$ holds;
  When they hold, in addition to $S = \infty$, then there is a one-parameter family of QSDs, Viz, $\{\{q_j(x)\}, 0 < x \leq \lambda_C\}$.
  (ii) If $S < \infty$, then $\lambda_C > 0$ and there is precisely one QSD, Viz, $\{q_j(\lambda_C)\}$. And the unique QSD $\{q_j(\lambda_C)\}$ attracts all initial distributions ν supported in $(0, \infty)$, that is,

$$\lim_{t \to \infty} \mathbf{P}_\nu(\mathbf{X_t} = \mathbf{j} | \mathbf{T} > \mathbf{t}) = \mathbf{q_j}(\lambda_\mathbf{C}).$$

QSDs of Markov Chains

- Let X_t be a continuous-time Markov chain in $I = \{0\} \cup \{1, 2, \ldots\}$ such that 0 is an absorbing state. Let $C \equiv \{1, 2, \ldots\}$. Denote by $Q = (q_{ij})$ the q-matrix (transition rate matrix) and $P(t) = (P_{ij}(t))$ the transition function. X_t is stochastically monotone if and only if $\sum_{j \geq k} P_{ij}(t)$ is a nondecreasing function of i for every fixed $k \in I$ and $t > 0$. We assume that all states other than 0 form an irreducible class and that Q is totally stable, conservative and regular, that is, $q_i = \sum_{i \neq j} q_{ij} < \infty$, and the minimal process $\{X_t\}_{t \geq 0}$corresponding to Q is an honest process. We further define $T = \inf\{t \geq 0 : X_t = 0\}$, the absorption time at 0. So $X_t = 0$ for any $t \geq T$.

QSDs of Markov Chains

- For stochastically monotone Markov chains, we discuss the existence, uniqueness and domain of attraction of QSDs.
  Theorem 2 Assume X_t is stochastically monotone, then
  (i) there exists a QSD if and only if

  $$E_i(e^{\theta T}) < \infty$$

  for some $\theta > 0$ and some $i \in C$ (and hence for all i).
  (ii) there is a unique QSD if and only if

  $$\lim_{i\to\infty} E_i(e^{\theta T}) < \infty$$

  for some $\theta > 0$. Moreover, the unique QSD $\rho = \{\rho_j, j \in C\}$ attracts all initial distributions that supported in C, that is, for any probability measure $\nu = \{\nu_i, i \in C\}$,
  $\rho_j = \lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t} = \mathbf{j} | \mathbf{T} > \mathbf{t}), \quad \mathbf{j} \in \mathbf{C}$.

QSDs of Markov Chains

- For continuous-time general Markov chains, P.A.Ferrari, H.Kesten,S.Martinez, and P.Picco (The Annals of Probability 1995,Vol.23, No.2,501-521.) obtained
  Proposition 2 Assume that

$$\lim_{i\to\infty} P_i(T < t) = 0 \quad \text{for any} \quad t \geq 0$$

  and that $P_i(T < \infty) = 1$ for some (and hence all) i. Then a necessary and sufficient condition for the existence of a QSD is that

$$E_i(e^{\theta T}) < \infty$$

  for some $\theta > 0$ and some $i \in C$ (and hence for all i).

QSDs of Markov Chains

- Our result is
  Theorem 3 (i) Assume that

$$\lim_{i\to\infty} E_i T = \infty,$$

and that $P_i(T < \infty) = 1$ for some (and hence all) i. Then a necessary and sufficient condition for the existence of a QSD is that

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for some $\theta > 0$ and some $i \in C$ (and hence for all i). When it holds, There exists a family of QSDs.

QSDs of Markov Chains

- (ii) If

$$\limsup_{i\to\infty} E_i T < \infty,$$

then there exists unique QSD;Moreover, the unique QSD $\rho = \{\rho_j, j \in C\}$ attracts all initial distributions that supported in C, that is, for any probability measure $\nu = \{\nu_i, i \in C\}$,

$$\rho_j = \lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t} = \mathbf{j} | \mathbf{T} > \mathbf{t}), \quad \mathbf{j} \in \mathbf{C}.$$

QSDs of One-dimensional Diffusion Processes

- A One-dimensional diffusion process will now be taken to mean a continuous Strong Markov process with values in $[0, \infty)$.

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- Consider the solution of SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x > 0,$$

where B is a standard Brownian motion. $\sigma : (0, \infty) \to (0, \infty)$ and $b : R \to R$. We shall assume that both σ and b are locally bounded and measurable.

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- We also assume that the SDE has a unique solution.

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  where B is a standard Brownian motion. $\sigma : (0, \infty) \to (0, \infty)$ and $b : R \to R$. We shall assume that both σ and b are locally bounded and measurable.
- We also assume that the SDE has a unique solution.
- Let $T_z = \inf\{t > 0 : X_t = z\}$ be the hitting time of z. We are mainly interested in the case $z = 0$ and we denote $T = T_0$.

QSDs of One-dimensional Diffusion Processes

- Let P_t the associated sub-markovian semi-group defined by $P_t f(x) = E_x(f(X_t), T > t)$. whose generator is defined by

$$L = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}.$$

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- Set

$$Q(x) = \int_1^x \frac{2b(z)}{\sigma^2(z)}dz \quad and \quad \Lambda(x) = \int_1^x \exp\{-Q(y)\}dy$$

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- $\Lambda(x)$ is called the scale function. $\Lambda(x)$ is a strictly increasing smooth function on $(0, \infty)$ satisfying

$$L\Lambda(x) \equiv 0 \ \ on \ (0, \infty).$$

QSDs of One-dimensional Diffusion Processes

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$$\kappa(x) = \int_1^x \exp\{-Q(y)\} \left(\int_1^y \frac{2}{\sigma^2(z)} \exp\{Q(z)\} dz \right) dy.$$

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$$P_x(T < \infty) = 1$$

  if and only if

$$\Lambda(\infty) = \infty \quad and \quad \kappa(0^+) < \infty.$$

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- It is easy to prove that If $\Lambda(\infty) = \infty$, then $\kappa(\infty) = \infty$.
- ∞ is a natural boundary for X in the sense of Feller boundary if and only if $\kappa(\infty) = \infty, \kappa(0^+) < \infty$ and $S = \infty$, which implies for any $s > 0$

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- ∞ is an entrance boundary for X if and only if $\kappa(\infty) = \infty, \kappa(0^+) < \infty$ and $S < \infty$, which is equivalent to the property that there is $y > 0$ and a time $t > 0$ such that

$$\lim_{x \uparrow \infty} P_x(T_y < t) > 0$$

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- Similarly We may define for boundary point 0.

QSDs of One-dimensional Diffusion Processes

- In the literature there are some works directly related with the problem mentioned above. The first one was published by S. Martínez, J. San Martín, P. Picco (1994,1998) who consider the B.M. with negative constant drift, that is, $dX_t = dB_t - \alpha dt (\alpha > 0)$. They proved that there exits a one-parameter family of QSDs and solve the domain of attraction problem.

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- The second was published by M. Lladser, J. San Martín (2000) who consider the Ornstein-Uhlenbeck process, that is, $dX_t = dB_t - aX_t dt (a > 0)$. They solved the domain of attraction problem.

QSDs of One-dimensional Diffusion Processes

- The third was published by Cattiaux *et al*. (The Annals of Probability 2009,Vol.37,No.5,1926-1969.) who study the existence and uniqueness of the QSD for one-dimensional diffusions killed at 0 and whose drift is allowed to go to $-\infty$ at 0 and the process is allowed to have an entrance boundary at $+\infty$

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- The last one was was published by M. Kolb, D. Steinsaltz (The Annals of Probability 2012,Vol.40,No.1,162-212.) who consider the one-dimensional diffusions with killing. They proved that this process either converges to QSD or escapes to ∞ from a compactly supported distribution.

QSDs of One-dimensional Diffusion Processes

- Moreover, it is worth mentioning that Prof. Jinwen Chen (Tsinghua University) has also made some contribution on this topic (QSD). He proved the existence and uniqueness of both QSD and mean ratio quasi-stationary distribution (mrqsd) for killed Brownian motion by using an eigenfunction expansion for the transition density. In particular, he gived interpretations of the mrqsd from different points of view not only for killed Brownian motion but also for absorbing Markov processes.

QSDs of One-dimensional Diffusion Processes

- Now our results are

  Theorem 4 For the diffusion process, suppose that boundary point 0 is regular, then

  (i) there is QSD iff $\Lambda(\infty)=\infty, \quad \kappa(0^+)<\infty$, and

$$\sup_{x>0}\int_0^x \exp\left(-Q(y)\right)dy \int_x^\infty \frac{2\exp\left(Q(y)\right)}{\sigma^2(y)}dy<\infty.$$

  holds;

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  holds;
- (ii) If in addition to (i) by

  $S:=\int_1^\infty \exp\{-Q(y)\}\left(\int_y^\infty \frac{2}{\sigma^2(z)}\exp\{Q(z)\}dz\right)dy=\infty$,

  then for any $0<\lambda\le\lambda_c$, $d\nu_\lambda=2\lambda\eta_\lambda d\mu$ is a quasi-stationary distribution and all of quasi-stationary distributions for X only have this form.

QSDs of One-dimensional Diffusion Processes

- Theorem 5 For the diffusion process,
  (i) there is exactly one QSD ν_1 iff $S < \infty$.
  (ii) When it holds, the unique quasi-stationary distribution ν_1 attracts all initial distributions ν supported in $(0, \infty)$, that is,

$$\lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t} \in \bullet | \mathbf{T} > \mathbf{t}) = \nu_\mathbf{1}(\bullet).$$

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- Theorem 6 For the one-dimensional diffusions with killing, if the limit inferior of the killing at infinity is larger than the zero, then there is exactly one quasi-stationary distribution ν_1, and that this distribution attracts all initial distributions ν, that is

$$\lim_{t\to\infty} \mathbf{P}_\nu(\mathbf{X_t} \in \bullet | \mathbf{T} > \mathbf{t}) = \nu_\mathbf{1}(\bullet).$$

The Fleming-Viot Type Particle System

- Let Λ be a finite or countable state space. Let $Q = (q(i,j); i,j \in \Lambda \cup 0)$ be the transition rates matrix of an irreducible continuous time Markov jump process on $\Lambda \cup 0$.

The Fleming-Viot Type Particle System

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- System of $N \geq 2$ particles evolving on Λ.
- Each particle moves independently of the others as a continuous time Markov process X with rates Q.
- When it attempts to jump to state 0, it comes back immediately to Λ by jumping to the position of one of the other particles chosen uniformly at random. This operation is called a rebirth.
- We denote by $\tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ the sequence of rebirths times. $\lim_{n\to\infty} \tau_n = +\infty \quad a.s.$

The Fleming-Viot Type Particle System

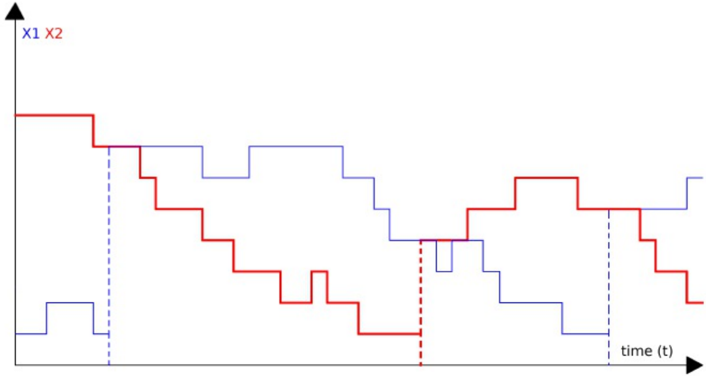

Figure 1: One path of a Fleming-Viot system with two particles.

The Fleming-Viot Type Particle System

- The generator of the Fleming-Viot process acts on functions $f : \Lambda^{(1,2,\dots,N)} \to \mathbb{R}$ as follows

$$\begin{aligned}&\mathcal{L}^N f(\mu)\\ = \;& \sum_{k=1}^{N} \sum_{j\in\Lambda\setminus\{\mu(k)\}} \left[q(\mu(k),j) + \frac{N}{N-1} q(\mu(k),0)\eta(\mu,j) \right]\\ &\cdot (f(\mu^{k,j}) - f(\mu))\end{aligned}$$

where $\mu^{k,j}(n) = j$ for $n = k$ and $\mu^{k,j}(n) = \mu(n)$ otherwise and

$$\eta(\mu,j) := \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}\{\mu(k) = j\}.$$

The Fleming-Viot Type Particle System

- The empirical distribution of the N particles at positions $\mu \in \Lambda^N$ is defined as a function $\eta(\mu, j) : \Lambda \to [0, 1]$ by $\eta(\mu, j)$.

The Fleming-Viot Type Particle System

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- This type of Fleming-Viot process was introduced by Burdzy, Holyst and March (2000) for Brownian motions on a bounded domain.Similar models have been investigated in Villemonais (2014), Asselah et al.(2012), Ferrari et al.(2007).

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- We are interested in the two things:
  (1) if $\eta(\mu, j)$ exists;
  (2) and converges to the minimal quasi-stationary distribution.

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- Fix $N \geq 2$, and let us describe precisely the dynamics of this system with N particles, which we denote by $(X^1, X^2, \ldots, X^N)$. The process starts at a position $(X_0^1, X_0^2, \ldots, X_0^N) \in \Lambda^N$.

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- For any $t \geq 0$, the empirical distribution of $(X^1, X^2, \ldots, X^N)$ at time t is

$$\eta_t^N = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_t^k}.$$

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- A general convergence result obtained in Villemonais (2011) ensures that, if $\eta_0^N \to \eta_0$, then

$$\eta_t^N \to P_{\eta_0}(X_t \in \cdot | t < T), \qquad N \to \infty.$$

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- Villemonais (2014) provides a sufficient criterion ensuring that the empirical stationary distribution of the particle system exists and converges to the minimal quasi-stationary distribution of the underlying birth and death process.

New result

For a general continuous-time Markov chain, we have

Theorem 7 Assume that the Markov chain X admits a unique quasi-stationary distribution ($\lim_{i\to\infty} E_i T < \infty$). Then for any $N \geq 2$, the measure process $(\eta_t^N)_{t\geq 0}$ is ergodic, which means that there exists a random measure $\mathcal{X}^N$ on Λ such that

$$\eta_t^N \to \mathcal{X}^N, \quad t \to \infty.$$

Moreover, if $\lim_{i\to\infty} E_i T < \infty$,

$$\mathcal{X}^N \to \rho, \quad N \to \infty$$

where ρ is the minimal quasi-stationary distribution of X.

Thank you all for your attention!