Hypercontractivity, Compactness, and Exponential Ergodicity for Functional SDEs

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Outline

Let's start with the following paper:

L. Gross, *Logarithmic Sobolev inequalities and contractivity properties of semigroups,* Lecture Notes in Math. 1563, Springer-Verlag, 1993.

In this paper,suppose that (X,μ) is a probability measure space and that H is a self-adjoint operator on $L^2(X,\mu)$. Assume that H is bounded below. Define

$$\|e^{-tH}\|_{q\to p} = \sup\{\|e^{-tH}f\|_p : f \in L^2 \cap L^q, \|f\|_q \leq 1\}$$

where $\|g\|_p$ denotes the $L^p(\mu)$ norm of g. Consider the following questions concerning the relation between properties of H and the properties of semigroup e^{-tH} which it generates.

Q1: Under what conditions on H is e^{-tH} a contraction semigroup on $L^2(X,\mu)$?
A1: $\|e^{-tH}\|_{2\to 2} \leq 1$ for all $t>0$ if and only if

$$\langle Hf, f\rangle_{L^2(\mu)} \geq 0 \text{ for all } f \text{ in } D(H). \quad (A1)$$

Q2: Assume H is a self-adjoint operator satisfying (A1). Under what condition on H is e^{-tH} a positivity preserving contraction semigroup on $L^p(X,\mu)$ for all $p \geq 1$?
A2: e^{-tH} a positivity preserving and $\|e^{-tH}\|_{p\to p} \leq 1$ if and only if

$$\langle Hf, (f-1)^+\rangle_{L^2(\mu)} \geq 0 \text{ for all } f \text{ in } D(H), \quad (A1)$$

Q3: Assume that H is a self-adjoint operator in $L^2(X,\mu)$ which satisfies both (A1) and (A2). Under what conditions on H is e^{-tH} a contraction from $L^q(\mu)$ to $L^p(\mu)$ for some $t>0$ and some q and p with $1<q<p<\infty$?

A3: $\|e^{-tH}\|_{q\to p}\leq 1$ for some $t>0$ and some q and p with $1<q<p<\infty$ if and only if there is a constant $c>0$ such that

$$c\langle Hf,f\rangle \geq \int_X f^2\log|f|d\mu - \|f\|_2^2\log\|f\|_2^2 \text{ for all } f \text{ in } D(H), \quad (A3)$$

(A3) is generally refereed to as a logarithmic Sobolev inequality.

In the following, the Harnack inequality is used to investigate the logarithmic Sobolev inequality.

F.-Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds,* Probab. Theory Relat. Fields 109 (1997), 417–424.

stochastic functional differential equations

X. Mao, *Stochastic Differential Equations and Application,* Horwood publishing, UK, 2007.

S-E. A. Mohammed, *Stochastic Functional Differential Equations,* Longman Scientific and Technical, 1984.

In many applications, one assumes that the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. Lord Cherwell (see Wright (1961)) has encountered the differential equation

$$dx(t) = -ax(t-1)[1+x(t)].$$

Dunkel(1968) suggested the more general equation

$$dx(t) = -a\left[\int_{-1}^{0} x(t+\theta)d\eta(\theta)\right][1+x(t)].$$

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Volterra (1928) had investigated the equation

$$dx(t) = \left(\varepsilon_1 - \gamma_1 y(t) - \int_{-r}^{0} F_1(\theta)x(t+\theta)d\theta\right)x(t),$$
$$dy(t) = \left(\varepsilon_2 - \gamma_2 x(t) - \int_{-r}^{0} F_2(\theta)y(t+\theta)d\theta\right)y(t)$$

All these equations are special cases of the general functional differential equation

$$dx(t) = f(x_t, t)$$

where $x_t = \{x(t+\theta) : -r \le \theta \le 0\}$ is the past history of the state.

Taking into account the environmental noise, we consider the following SFDEs

$$dx(t) = f(x_t, t)dt + g(x_t, t)dB(t).$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space with the filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual condition, and $B(t)$ is the given BM. Let $\tau > 0$ and denote by $C([-\tau, 0]; \mathbb{R}^d)$ the family of continuous functions φ from $[-\tau, 0]$ to $\mathbb{R}^d$,

$$f : C([-\tau, 0]; \mathbb{R}^d) \times [0, T] \to \mathbb{R}^d$$
$$g : C([-\tau, 0]; \mathbb{R}^d) \times [0, T] \to \mathbb{R}^{d\times m}$$

Define

$$\|\varphi\|_\infty = \sup_{\theta\in[-\tau,0]} |\varphi(\theta)|$$

Assume that there exist two positive constants $\bar{K}, K$ such that
(i) for all $\phi, \varphi \in C([-\tau, 0]; \mathbb{R}^d), t \in [0, T]$

$$|f(\varphi, t) - f(\phi, t)|^2 \vee |f(\varphi, t) - f(\phi, t)|^2 \leq K\|\varphi - \phi\|_\infty^2;$$

(ii) for all $\varphi \in C([-\tau, 0]; \mathbb{R}^d), t \in [0, T]$

$$|f(\varphi, t)|^2 \vee |g(\varphi, t)|^2 \leq \bar{K}(1 + \|\varphi\|_\infty^2).$$

Then there exists a unique solution $x(t)$ to the equation for given initial data.

Harnack inequality

In mathematics, Harnack's inequality is an inequality relating the values of a positive harmonic function at two points, introduced by A. Harnack (1887). J. Serrin (1955) and J. Moser (1961, 1964) generalized Harnack's inequality to solutions of elliptic or parabolic partial differential equations. Perelman's solution of the Poincare conjecture uses a version of the Harnack inequality, found by R. Hamilton (1993), for the Ricci flow. Harnack's inequality is used to prove Harnack's theorem about the convergence of sequences of harmonic functions. Harnack's inequality also implies the regularity of the function in the interior of its domain. (From Wikipedia.)

Deterministic Harnack Inequality

- Harnack's inequality is an inequality relating the values of a positive harmonic function at two points, introduced by A. Harnack (1887).

Theorem

[Harnack, 1887] Let $u : B_R(x_0) \subset \mathbb{R}^d \to \mathbb{R}$ be a harmonic function which is either non-negative or non-positive. Then the value of u at any point in $B_r(x_0)(r < R)$ is bounded from above and below by the quantities

$$u(x_0)\frac{R-r}{R+r}\left(\frac{R}{R+r}\right)^{d-2} \text{ and } u(x_0)\frac{R-r}{R+r}\left(\frac{R}{R-r}\right)^{d-2}.$$

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- J. Serrin (1955) and J. Moser (1961, 1964) generalized Harnack's inequality to solutions of elliptic or parabolic partial differential equations.

Theorem

[Moser, 1964] Let $u \in C^\infty((0,\infty) \times \mathbb{R}^d)$ be a non-negative solution of the heat equation, then

$$u(t,x) \le u(t+s,y) \left(\frac{t+s}{t}\right)^{d-2} \exp\left(\frac{|y-x|^2}{4s}\right),$$

for $x, y \in \mathbb{R}^d, t, s > 0$.

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Harnack inequality of SDEs

F.-Y. Wang, *Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds,* Ann. Probab. 39 (2011), no. 4, 1449–1467,

Let

$$dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt$$

$$P_t f(x) := Ef(X^x(t)), t \geq 0, x \in \mathbb{R}^d, f \in \mathcal{B}_b^+(\mathbb{R}^d).$$

Under some conditions, Wang proved the following Harnack inequality

$$(P_T f(y))^p \leq C P_T f^p(x).$$

C could be explicit.

Example

Consider the following OU process

$$dX(t) = -\lambda X(t)dt + dW(t), \tag{3.1}$$

where $\lambda \in \mathbb{R}$ is a constant and $W(t)$ is a standard Brownian motion on $\mathbb{R}^d$. For initial condition $X(0) = x$, Eq. (3.1) admits explicit solution

$$X(t) = xe^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}dW(s). \tag{3.2}$$

Note that $X(t)$ is a Gaussian process with mean $\mu := xe^{-\lambda t}$ and variance $\sigma^2(t) := \frac{1-e^{-2\lambda t}}{2\lambda}$. Let P_t be the transition semigroup associated with $X(t)$. It can be expressed as

$$P_t f(x) = \int_{\mathbb{R}^d} f(xe^{-\lambda t} + z)d\rho_t(z), x \in \mathbb{R}^d, f \in C_b(\mathbb{R}^d),$$

where $\rho_t = N(0, \sigma^2(t))$.

Example

Note that

$$\rho_t(dz) = (2\pi\sigma^2(t))^{-\frac{d}{2}} \exp\left(-\frac{|z|^2}{2\sigma^2(t)}\right) dz.$$

Let $\alpha, \beta > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and compute by the Hölder inequality

$$\begin{aligned}
& P_t f(x) \\
&= (2\pi\sigma^2(t))^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(xe^{-\lambda t} + z) \exp\left(-\frac{|z|^2}{2\sigma^2(t)}\right) dz \\
&= (2\pi\sigma^2(t))^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(ye^{-\lambda t} + z) \exp\left(-\frac{|(y-x)e^{-\lambda t} + z|^2}{2\sigma^2(t)}\right) dz \\
&= \exp\left(-\frac{e^{-2\lambda t}|y-x|^2}{2\sigma^2(t)}\right) \int_{\mathbb{R}^d} f(ye^{-\lambda t} + z) \exp\left(\frac{e^{-\lambda t}\langle x-y, z\rangle}{\sigma^2(t)}\right) \rho_t(dz
\end{aligned}$$

Example

$$
\begin{aligned}
&\leq \exp\left(-\frac{e^{-2\lambda t}|y-x|^2}{2\sigma^2(t)}\right)\left(\int_{\mathbb{R}^d} f^\alpha(ye^{-\lambda t}+z)\rho_t(dz)\right)^{\frac{1}{\alpha}} \\
&\quad\times\left(\int_{\mathbb{R}^d}\exp\left(\frac{\beta e^{-\lambda t}\langle x-y,z\rangle}{\sigma^2(t)}\right)\rho_t(dz)\right)^{\frac{1}{\beta}} \\
&=\exp\left(-\frac{e^{-2\lambda t}|y-x|^2}{2\sigma^2(t)}\right)(P_tf^\alpha(y))^{\frac{1}{\alpha}}\exp\left(\frac{\beta e^{-2\lambda t}|x-y|^2}{2\sigma^2(t)}\right) \\
&=\exp\left(\frac{(\beta-1)e^{-2\lambda t}|y-x|^2}{2\sigma^2(t)}\right)(P_tf^\alpha(y))^{\frac{1}{\alpha}} \\
&=\exp\left(\frac{\lambda|y-x|^2}{(\alpha-1)(e^{2\lambda t}-1)}\right)(P_tf^\alpha(y))^{\frac{1}{\alpha}}.
\end{aligned}
$$

Example

Therefore: $\forall t > 0, \alpha > 1, f \in C_b^+(\mathbb{R}^d), x, y \in \mathbb{R}^d$,

$$(P_t f(x))^\alpha \le P_t f^\alpha(y) \exp\left(\frac{\lambda|y-x|^2}{(\alpha-1)(e^{2\lambda t}-1)}\right)$$

Harnack inequality for SFDEs

Let $\tau > 0$ be fixed, and let $\mathcal{C} = C([-\tau, 0]; \mathbb{R}^d)$ be equipped with the uniform norm $\|\cdot\|_\infty$. Let $\mathcal{B}_b(\mathcal{C})$ be the set of all bounded measurable functions on $\mathcal{C}$. Let

$$\sigma : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d,$$
$$Z : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d,$$
$$b : [0, \infty) \times \mathcal{C} \times \Omega \to \mathbb{R}^d$$

are progressively measurable, and σ is invertible. Consider the following functional SDE on $\mathbb{R}^d$:

$$dX(t) = \{Z(t, X(t)) + b(t, X_t)\}dt + \sigma(t, X(t))dB(t), \quad X_0 \in \mathcal{C}, \tag{3.3}$$

where $\mathcal{C} := ([-\tau, 0]; \mathbb{R}^d)$

To ensure the existence, uniqueness, non-explosion, and further regular properties of the solution, we make use of the following assumption:

(A) *$Z(t,x)$ is continous in x, and there are constants $K_1, K_2 \geq 0, K_3 > 0$ and $K_4 \in \mathbb{R}$ such that*

(A1) $\left|\sigma(t,\eta(0))^{-1}\{b(t,\xi)-b(t,\eta)\}\right| \leq K_1\|\xi-\eta\|_\infty,\ t \geq 0, \xi, \eta \in \mathcal{C};$

(A2) $\left|(\sigma(t,x)-\sigma(t,y))\right| \leq K_2(|x-y| \wedge 1),\ t \geq 0, x, y \in \mathbb{R}^d;$

(A3) $\left|\sigma(t,x)^{-1}\right| \leq K_3,\ t \geq 0, x \in \mathbb{R}^d;$

(A4) $\|\sigma(t,x)-\sigma(t,y)\|_{HS}^2 + 2\langle x-y, Z(t,x)-Z(t,y)\rangle \leq K_4|x-y|^2,\ t \geq 0, x, y \in \mathbb{R}^d$

hold almost surely.

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hold almost surely.

we aim to establish the Harnack inequality with a power $p > 1$

$$P_T f(\eta) \leq \{P_T f^p(\xi)\}^{1/p} \exp[\Phi_p(T, \xi, \eta)], \quad f \geq 0, T > \tau, \xi, \eta \in \mathcal{C} \tag{3.4}$$

for some positive function Φ_p on $(\tau, \infty) \times \mathcal{C}^2$. Assume that $p > (1 + K_2 K_3)^2$. Let

$$\lambda_p = \frac{1}{2(p-1)^2},$$

$$\Theta_p := \left\{ \varepsilon \in (0,1) : \frac{(1-\varepsilon)^4}{2(1+\varepsilon)^3 K_2^2 K_3^2} \geq \lambda_p \right\}$$

Theorem

Assume **(A)**. *For any* $p > (1 + K_2K_3)^2$ *and* $T > \tau$, *the Harnack inequality* (3.4) *holds for*

$$\Phi_p(T,\xi,\eta) := \frac{p-1}{p} \inf_{\varepsilon\in\Theta_p} \inf_{s\in(0,s_\varepsilon(\lambda_p)\wedge(T-r_0)]} \bigg\{ \frac{\varepsilon}{2(1+\varepsilon)} + \frac{16K_2^2 s^2 W_\varepsilon(\lambda_p)}{1-4K_1K_2 s} + \frac{\lambda_p(1+\varepsilon)^2K_3^2K_4|\xi(0)-\eta(0)|^2}{2\varepsilon(1-\varepsilon)^2(1+2\varepsilon)(1-e^{-K_4 s})} + \left(K_1^2 r_0\lambda_p + 2sW_\varepsilon(\lambda_p)\right)\|\xi-\eta\|_\infty^2 \bigg\}.$$

Harnack inequality for SFDE with additive noise

Let σ be an invertible $d \times d$-matrix, $Z \in C(\mathbb{R}^d; \mathbb{R}^d)$ and $b : \mathcal{C} \to \mathbb{R}^d$ be Lipschitz continuous. Consider the following FSDE on $\mathbb{R}^d$:

$$dX(t) = \{Z(X(t)) + b(X_t)\}dt + \sigma dB(t), \quad X_0 = \xi \in C([-\tau, 0]; \mathbb{R}^d], \tag{3.5}$$

Assume

$$\langle Z(x) - Z(y), x - y\rangle \le -k_1|x - y|^2, \quad x, y \in \mathbb{R}^d, \tag{3.6}$$

$$|b(\xi) - b(\eta)| \le k_2\|\xi - \eta\|_\infty, \quad \xi, \eta \in C([-\tau, 0]; \mathbb{R}^d]. \tag{3.7}$$

Theorem

Let (3.6) *and* (3.7) *hold for some constants* $k_1 \in \mathbb{R}$ *and* $k_2 \geq 0$. *Then, for any* $p > 1, \delta > 0$, *positive* $f \in \mathcal{B}_b(([-\tau, 0]; \mathbb{R}^d))$, *and* $\xi, \eta \in ([-\tau, 0]; \mathbb{R}^d)$,

$$\begin{aligned}&\big(P_{t+\tau}f(\xi)\big)^p\\ &\leq \big(P_{t+\tau}f^p(\eta)\big)\exp\bigg[\frac{p^2\|\sigma^{-1}\|^2(1+\delta)}{2(p-1)}\bigg\{\frac{2k_1|\xi(0)-\eta(0)|^2}{e^{2k_1t}-1}\\ &+\frac{k_2^2}{\delta}\Big(\tau\|\xi-\eta\|_\infty^2+\frac{|\xi(0)-\eta(0)|^2(e^{4k_1t}-1-4k_1te^{2k_1t})}{2k_1(e^{2k_1t}-1)^2}\Big)\bigg\}\bigg].\end{aligned}$$

exponential integrability

Denote $\mathcal{C} := C([-\tau, 0]; \mathbb{R}^d)$. Assume that

$$2\langle Z(\xi(0)) + b(\xi) - Z(\eta(0)) - b(\eta), \xi(0) - \eta(0)\rangle \\ \leq \lambda_2\|\xi - \eta\|^2 - \lambda_1|\xi(0) - \eta(0)|^2, \xi, \eta \in \mathcal{C}.$$

we assume that $\lambda = \lambda_1 - \lambda_2 e^{\tau\lambda_1}$.

Lemma

If $\lambda > 0$, then there exist two constants $c, \varepsilon > 0$ such that

$$\mathbb{E}\, e^{\varepsilon\|X_t^\xi\|_\infty^2} \leq e^{c(1+\|\xi\|_\infty^2)}, \quad t \geq 0, \xi \in \mathcal{C}.$$

Lemma

For any $t \geq 0$ and $\xi, \eta \in \mathcal{C}$, $\|X_t^\xi - X_t^\eta\|_\infty^2 \leq \|\xi - \eta\|_\infty^2 e^{\lambda_1\tau - \lambda t}$.

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Let

$$P_t f(\xi) := \mathbb{E} f(X_t^\xi),\ t \geq 0, f \in \mathcal{B}_b(\mathcal{C}), \xi \in \mathcal{C},$$

where X_t^ξ is the corresponding segment process of $X^\xi(t)$ which solves the equation for $X_0 = \xi$.

Lemma

If $\lambda > 0$, then P_t has a unique invariant probability measure μ such that

$$\lim_{t\to\infty} P_t f(\xi) = \mu(f) := \int_{\mathcal{C}} f d\mu,\ f \in C_b(\mathcal{C}), \xi \in \mathcal{C}.$$

Hypercontractivity

We now state our main result:

Theorem

Assume $\lambda > 0$. Then the following assertions hold.

(1) *P_t is hypercontractive, i.e. $\|P_t\|_{2\to 4} \le 1$ holds for large enough $t > 0$, where $\|\cdot\|_{2\to 4}$ is the operator norm from $L^2(\mu)$ to $L^4(\mu)$.*

(2) *P_t is compact on $L^2(\mu)$ for large enough $t > 0$.*

(3) *There exists a constant $C > 0$ such that*

$$\|P_t - \mu\|_2^2 := \sup_{\mu(f^2)\le 1} \mu\big((P_t f - \mu(f))^2\big) \le Ce^{-\lambda t}, \quad t \ge 0.$$

(4) *There exist two constants $t_0, C > 0$ such that*

$$\|P_t^\xi - P_t^\eta\|_{var}^2 \le C\|\xi - \eta\|_\infty^2 e^{-\lambda t}, t \ge t_0.$$

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Theorem

Assume $\lambda > 0$. Then the following assertions hold.

(1) P_t *is hypercontractive, i.e.* $\|P_t\|_{2\to 4} \leq 1$ *holds for large enough* $t > 0$, *where* $\|\cdot\|_{2\to 4}$ *is the operator norm from* $L^2(\mu)$ *to* $L^4(\mu)$.

(2) P_t *is compact on* $L^2(\mu)$ *for large enough* $t > 0$.

(3) *There exists a constant* $C > 0$ *such that*

$$\|P_t - \mu\|_2^2 := \sup_{\mu(f^2)\leq 1} \mu\big((P_t f - \mu(f))^2\big) \leq Ce^{-\lambda t}, \quad t \geq 0.$$

(4) *There exist two constants* $t_0, C > 0$ *such that*

$$\|P_t^\xi - P_t^\eta\|_{var}^2 \leq C\|\xi - \eta\|_\infty^2 e^{-\lambda t}, t \geq t_0.$$

Hypercontractivity

We now state our main result:

Theorem

Assume $\lambda > 0$. Then the following assertions hold.

(1) *P_t is hypercontractive, i.e. $\|P_t\|_{2\to 4} \le 1$ holds for large enough $t > 0$, where $\|\cdot\|_{2\to 4}$ is the operator norm from $L^2(\mu)$ to $L^4(\mu)$.*

(2) *P_t is compact on $L^2(\mu)$ for large enough $t > 0$.*

(3) *There exists a constant $C > 0$ such that*

$$\|P_t - \mu\|_2^2 := \sup_{\mu(f^2)\le 1} \mu\big((P_t f - \mu(f))^2\big) \le Ce^{-\lambda t}, \quad t \ge 0.$$

(4) *There exist two constants $t_0, C > 0$ such that*

$$\|P_t^\xi - P_t^\eta\|_{var}^2 \le C\|\xi - \eta\|_\infty^2 e^{-\lambda t}, t \ge t_0.$$

Proof of main result

(a) We first prove that $\|P_t\|_{2\to 4} < \infty$ holds for large enough $t > 0$. By the Harnack inequality, for any $t_0 > \tau$ there exists $c_0 > 0$ such that

$$(P_{t_0} f(\xi))^2 \le (P_{t_0} f^2(\eta)) e^{c_0 \|\xi-\eta\|_\infty^2}, \quad \xi, \eta \in \mathcal{C}.$$

By the Markov property and Schwartz's inequality,

$$\begin{aligned} &|P_{t+t_0} f(\xi)|^2 = |\mathbb{E}(P_{t_0} f)(X_t^\xi)|^2 \\ &\le \left(\mathbb{E}\sqrt{(P_{t_0} f^2(X_t^\eta)) \exp[c_0\|X_t^\xi - X_t^\eta\|_\infty^2]}\right)^2 \\ &\le (\mathbb{E}(P_{t_0} f^2(X_t^\eta))\mathbb{E}e^{c_0\|X_t^\xi - X_t^\eta\|_\infty^2} = (P_{t+t_0} f^2(\eta))\mathbb{E}e^{c_0\|X_t^\xi - X_t^\eta\|_\infty^2}. \end{aligned}$$

Proof Cont.

Combining this with Lemma, we obtain

$$|P_{t+t_0} f(\xi)|^2 \le (P_{t+t_0} f^2(\eta)) \exp\left[c_1 e^{-\lambda t} \|\xi - \eta\|_\infty^2\right].$$

Let $r > 0$ such that $\mu(B_r) \ge \frac{1}{2}$, where $B_r := \{\|\cdot\|_\infty < R\}$. Then

$$\begin{aligned}
&|P_{t+t_0} f(\xi)|^2 \exp\left[-c_1 e^{-\lambda t}(\|\xi\|_\infty + r)^2\right] \\
&\le 2|P_{t+t_0} f(\xi)|^2 \int_{B_r} \exp\left[-c_1 e^{-\lambda t}\|\xi - \eta\|_\infty^2\right]\mu(d\eta) \\
&\le 2\int_{\mathcal{C}} P_{t+t_0} f^2(\eta)\mu(d\eta) = 2.
\end{aligned}$$

Proof Cont.

Thus,

$$|P_{t+t_0}f(\xi)|^4 \leq \exp\left[c_2(1+\|\xi\|_\infty^2 e^{-\lambda t})\right], \quad t \geq 0 \qquad (4.1)$$

holds for some constant $c_2 > 0$. On the other hand, by Lemmas (exponential integrability and invariant measure) we have

$$\mu(N \wedge e^{\varepsilon\|\cdot\|_\infty^2}) = \lim_{t\to\infty} \mathbb{E}(N \wedge e^{\varepsilon\|X_t^0\|_\infty^2}) \leq e^c < \infty, \quad N > 0$$

for some constant $c > 0$. Letting $N \to \infty$ we obtain $\mu(e^{\varepsilon\|\cdot\|_\infty^2}) < \infty$. Therefore, (4.1) implies $\|P_{t+t_0}\|_{2\to 4} < \infty$ for large enough $t > 0$.

Idea for the proof of Harnack inequality

- Main tools: Coupling and Girsanov transformation

Fix $T > 0$. Let $b(t, x)$ be an $\mathbb{R}^d$-valued Borel measurable function defined on $[0, T] \times \mathbb{R}^d$. We aim to study Harnack inequality for the transition semigroup P_t

Consider the following coupled stochastic differential equations on $\mathbb{R}^d$

$$
\begin{aligned}
dX(t) &= b(t, X(t))dt + \sigma(t, X(t))dB(t), X(0) = x, \\
dY(t) &= b(t, Y(t))dt + \sigma(t, Y(t))dB(t) \\
&+ \frac{1}{\xi(t)}\sigma(t, Y(t))\sigma(t, X(t))^{-1}(X(t) - Y(t))dt, Y(0) = y.
\end{aligned}
\tag{4.2}
$$

Let

$$d\tilde{B}(t) = dB(t) + \frac{1}{\xi(t)}\sigma(t, X(t))^{-1}(X(t) - Y(t))dt$$

$$R_s := \exp\Big[-\int_0^s \xi(t)^{-1}\langle\sigma(t, X(t))^{-1}(X(t) - Y(t)), dB(t)\rangle - \frac{1}{2}\int_0^s \xi(t)^{-2}|\sigma(t, X(t))^{-1}(X(t) - Y(t))|^2 dt\Big]$$

Rewrite (4.2) as

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))d\tilde{B}(t) - \frac{X(t) - Y(t)}{\xi(t)}, X(0) = x,$$
$$dY(t) = b(t, Y(t))dt + \sigma(t, Y(t))d\tilde{B}(t), Y(0) = y. \tag{4.3}$$

We shall see that the coupling is successful up to time T, so that $X(T) = Y(T)$ under $Q = R_T P$. we then have

$$(P_T f(y))^p = (E_Q[f(Y(T))])^p$$
$$= (E[R_T f(X(T))])^p \leq P_T f^p(x)(E R_T^{p/(p-1)})^{p-1}.$$

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