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A Linear-Quadratic Optimal Control Problem of Forward-Backward Stochastic Differential Equations with Partial Information

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(Based on joint works with Wang and Wu (SICON 2013, IEEE TAC 2014?))

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1. Motivating examples

Example 1: European call option under partial info

d stocks S_t^i, $1 \le i \le d$ and 1 bond S_t^0

$$\begin{cases} dS_t^i = S_t^i \left(X_t^i dt + \sum_{j=1}^m \tilde{\sigma}_t^{ij} d\tilde{W}_t^j \right), & i = 1, 2, \cdots, d, \\ dS_t^0 = S_t^0 X_t^0 dt, \quad t \ge 0, \end{cases} \tag{1.1}$$

where $\tilde{W} := (\tilde{W}^1, \cdots, \tilde{W}^m)^*$ is m-dim B.M. (random factors)

Information available:

$$\mathcal{G}_t := \sigma(S_s^i : \; s \le t, \; i = 0, 1, 2, \cdots, d), \quad t \ge 0.$$

Portfolio:

$$u_t^i = \$ \text{ amount in } i\text{th stock}, \; i = 1, 2, \cdots, d.$$

They should be $\mathcal{G}_t$-measurable.

Let y_t be the wealth process.
Self-finance condition:

$$\begin{aligned} dy_t &= \left(y_t - \sum_{i=1}^{d} u_t^i\right) \frac{dS_t^0}{S_t^0} + \sum_{i=1}^{d} u_t^i \frac{dS_t^i}{S_t^i} \qquad (1.2) \\ &= \left(X_t^0 y_t + \sum_{i=1}^{d} (X_t^i - X_t^0) u_t^i\right) dt + \sum_{i=1}^{d} \sum_{j=1}^{m} \tilde{\sigma}_t^{ij} u_t^i d\tilde{W}_t^j. \end{aligned}$$

Note that $X_t^0 = \frac{d}{dt} \log S_t^0$ is $\mathcal{G}_t$-adapted.
Also, $i = 1, 2, \cdots, d$,

$$\log S_t^i = \log S_0^i + \int_0^t \left(X_s^i - \frac{1}{2} a_s^{ii} \right) ds + \sum_{j=1}^{m} \int_0^t \tilde{\sigma}_s^{ij} d\tilde{W}_s^j, \qquad (1.3)$$

where

$$a_t^{ij} := \sum_{k=1}^{m} \tilde{\sigma}_t^{ik} \tilde{\sigma}_t^{jk}, \quad i, j = 1, 2, \cdots, d.$$

Then,

$$\left\langle \log S^i, \log S^j \right\rangle_t = \int_0^t a_s^{ij} ds$$

is $\mathcal{G}_t$-adapted.

Let

$$\Sigma_t = (\sigma_t^{ij})_{d\times d} = \sqrt{(a_t^{ij})_{d\times d}}.$$

Then, there is d-dim. B.M. W_t s.t.

$$\sum_{j=1}^{m} \int_0^t \tilde{\sigma}_s^{ij} d\tilde{W}_s^j = \sum_{j=1}^{d} \int_0^t \sigma_s^{ij} dW_s^j, \qquad i = 1, \cdots, d. \tag{1.4}$$

Thus,

$$d \log S_t^i = \left(X_t^i - \frac{1}{2} a_t^{ii} \right) dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j, \qquad i = 1, \cdots, d. \tag{1.5}$$

Note that $X_t := (X_t^1, \cdots, X_t^d)^*$ is not necessarily $\mathcal{G}_t$-adapted and hence, its value is not available to the investors.
Let Y_t be defined as

$$dY_t = \Sigma_t^{-1} d \log S_t, \qquad Y_0 = 0.$$

Then

$$Y_t = \int_0^t h_s(X_s) ds + W_t, \tag{1.6}$$

where

$$h_s(x) = \Sigma_s^{-1} \left(x - \frac{1}{2} \tilde{A}_s \right).$$

Suppose the appreciation rate process $X_t = (X_t^1, \cdots, X_t^d)^*$ can be modeled by SDE in $\mathbb{R}^d$ as follows:

$$dX_t = b(X_t)dt + c(X_t)dW_t + \tilde{c}(X_t)dB_t, \quad X_0 = x. \tag{1.7}$$

The wealth process y_t satisfies the following SDE:

$$dy_t = \left(X_t^0 y_t + \sum_{i=1}^{d} (X_t^i - X_t^0) u_t^i \right) dt + \sum_{i,j=1}^{d} \sigma_t^{ij} u_t^i dW_t^j, \tag{1.8}$$

Terminal

$$y_T = (S_T^1 - K)^+.$$

Objective: Minimize

$$J(u.) = y_0.$$

Example 2: Recursive Utility Problem

Cash balance:

$$(\text{Signal}) \quad \begin{cases} dx_t^v = \left(a_t x_t^v + b_t v_t - \bar{b}_t\right) dt + c_t dw_t + \bar{c}_t d\bar{w}_t, \\ x_0^v = e_0, \end{cases}$$

where v is a control strategy of policymaker and may denote the rate of capital injection or withdrawal so as to achieve a certain goal.

Stock price:

$$(\text{Observation}) \quad \begin{cases} dS_t^v = S_t^v \left[\left(f_t x_t^v + g_t\right) dt + h_t dw_t\right], \\ S_0^v = 1. \end{cases}$$

Cost functional:

$$J[v] = \frac{1}{2}\mathbb{E}\left[\int_0^T R_t(v_t - r_t)^2 dt + M(x_T^v - m)^2 - 2y_0^v\right].$$

Recursive utility from v:

$$\begin{cases} -dy_t^v = g(t, y_t^v, z_t^v, \bar{z}_t^v, v_t)dt - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t, \\ \quad y_T^v = x_T^v, \end{cases}$$

where g is concave with respect to $(y, z, \bar{z}, v)$ and satisfies some usual conditions for BSDEs.

Recursive Utility Problem

Find an admissible control v to minimize the cost functional, subject to the cash balance process x^v, the stock price S^v and the recursive utility y^v.

- Suppose that the policymaker can only get information from the stock. Then we are facing a special optimal control problem derived by FBSDEs with partial observations.

2. Problem formulation: Problem (LQC)

Find an admissible control u to minimize

$$J[v] = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[L_t(x_t^v)^2 + O_t(y_t^v)^2 + R_t v_t^2 + 2l_t x_t^v + 2o_t y_t^v + 2r_t v_t\right] dt \right.$$
$$\left. + M(x_T^v)^2 + 2m x_T^v + N(y_0^v)^2 + 2n y_0^v \right\}$$

subject to

$$\begin{cases} dx_t^v = \left(a_t x_t^v + b_t v_t + \bar{b}_t\right) dt + c_t dw_t + \bar{c}_t d\bar{w}_t, \\ -dy_t^v = \left(A_t x_t^v + B_t y_t^v + C_t z_t^v + \bar{C}_t \bar{z}_t^v + D_t v_t + \bar{D}_t\right) dt \\ \qquad - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t, \\ x_0^v = e_0, \quad y_T^v = F x_T^v + G, \end{cases}$$

Observation:

$$\begin{cases} dY_t^v = (f_t x_t^v + g_t)dt + h_t dw_t, \\ Y_0^v = 0. \end{cases} \tag{2.9}$$

Problem formulation: Assumption Conditions

Assumption 1

The coefficients a_t, b_t, $\bar{b}_t$, c_t, $\bar{c}_t$, f_t, g_t, h_t, $1/h_t$, A_t, B_t, C_t, $\bar{C}_t$, D_t and $\bar{D}_t$ are uniformly bounded, deterministic functions. e_0 and F are constants, and $G \in \mathcal{L}^2_{\mathcal{F}_T^{w,\bar{w}}}(\mathbb{R})$.

Assumption 2

$L_t \geq 0$, $O_t \geq 0$, $R_t \geq 0$, l_t, o_t and r_t are uniformly bounded, deterministic functions. $M \geq 0$, $N \geq 0$, m and n are constants.

What is the Difficulty?

Partially observed stochastic control problems are always hard to study:

- Circular dependence between control and observation;
- coupled filtering and control problems;
- How to solve some practical problems;
- $\cdots$

Some Related References

This topic is related to

- Huang, Wang & Xiong [SICON 2009]
  - zero observation coefficient.
  - BSDE state equation.
- Wang & Wu [IEEE TAC 2009], Wu [SCIS, 2010], Xiao [JSSC, 2011]
  - Bounded observation coefficient.
  - Girsanov transformation, variational method.
- Wang, Wu & Xiong [SICON, 2013]
  - Linear observation coefficient.
  - Approximation method.

Their methods, however, are not available to Problem (LQC):

- The observation equation is linear;
- The observation noise is correlated with the signal noise.

3. Decomposition of the Signal and Observation

To solve Problem (LQC), we separate $(x^v, y^v, z^v, \bar{z}^v)$ and Y^v into

$$
\begin{aligned}
(x^v, y^v, z^v, \bar{z}^v) &= \left(x^0, y^0, z^0, \bar{z}^0\right) + \left(x^1, y^1, z^1, \bar{z}^1\right), \\
Y^v &= Y^0 + Y^1,
\end{aligned}
$$

where $\left(x^0, y^0, z^0, \bar{z}^0\right)$ and Y^0 are independent of v.

Define

$$\mathcal{U}_{ad}^0 = \Bigg\{ v | v_t \text{ is an } \mathcal{F}_t^{Y^0}\text{-adapted process with values in R such that } \mathbb{E} \sup_{0\le t\le T} v_t^2 < +\infty \Bigg\}$$

with $\mathcal{F}_t^{Y^v} = \sigma\{Y_s^v; 0 \le s \le t\}$ and $\mathcal{F}_t^{Y^0} = \sigma\left\{Y_s^0; 0 \le s \le t\right\}$.

Definition 2.1

A control v is called admissible, if $v \in \mathcal{U}_{ad}^0$ is $\mathcal{F}_t^{Y^v}$-adapted. The set of all admissible controls is denoted by $\mathcal{U}_{ad}$.

$\mathcal{U}_{ad} \subseteq \mathcal{U}_{ad}^0$.

Proposition 3.1

Under Assumptions 1 and 2,

$$\inf_{v\in\mathcal{U}_{ad}} J[v] = \inf_{v\in\mathcal{U}_{ad}^0} J[v].$$

- Problem (LQC) is equivalent to minimizing $J[v]$ over $v \in \mathcal{U}_{ad}^0$.
- One key point of its proof is that $\mathcal{U}_{ad}$ is dense in $\mathcal{U}_{ad}^0$ under the metric of $\mathcal{L}^2_{\mathcal{F}_0^Y}(0, T; \mathbb{R})$.

4. Maximum Principle for Problem (LQC)

Theorem 3.1

Let Assumptions 1 and 2 hold. Suppose that $(u, x, y, z, \bar{z})$ is the optimal solution. Then the FBSDE

$$\begin{cases} dp_t = (B_t p_t - O_t y_t - o_t)dt + C_t p_t dw_t + \bar{C}_t p_t d\bar{w}_t, \\ -dq_t = (a_t q_t - A_t p_t + L_t x_t + l_t)dt - k_t dw_t - \bar{k}_t d\bar{w}_t, \\ p_0 = -Ny_0 - n, \quad q_T = -Fp_T + Mx_T + m \end{cases}$$

admits a unique solution $\left(p, q, k, \bar{k}\right) \in \mathcal{L}^2_{\mathcal{F}^{w,\bar{w}}}\left(0, T; \mathbb{R}^4\right)$ such that

(Maximum Condition) $\quad R_t u_t - D_t \mathbb{E}\left[p_t \middle| \mathcal{F}_t^Y\right] + b_t \mathbb{E}\left[q_t \middle| \mathcal{F}_t^Y\right] + r_t = 0$

with $\mathcal{F}_t^Y = \sigma\{Y_s^u; 0 \leq s \leq t\}$.

Theorem 3.2

Assume Assumptions 1 and 2 hold. Let $u \in \mathcal{U}_{ad}$ satisfy

$$R_t u_t - D_t \mathbb{E}\left[p_t \middle| \mathcal{F}_t^Y\right] + b_t \mathbb{E}\left[q_t \middle| \mathcal{F}_t^Y\right] + r_t = 0,$$

where $(x, y, z, \bar{z}, p, q, k, \bar{k})$ is a solution to the Hamiltonian system

$$\begin{cases} dx_t = (a_t x_t + b_t u_t + \bar{b}_t)dt + c_t dw_t + \bar{c}_t d\bar{w}_t, \quad x_0 = e_0, \\ -dy_t = (A_t x_t + B_t y_t + C_t z_t + \bar{C}_t \bar{z}_t + D_t u_t + \bar{D}_t)dt - z_t dw_t - \bar{z}_t d\bar{w}_t, \\ dp_t = (B_t p_t - O_t y_t - o_t)dt + C_t p_t dw_t + \bar{C}_t p_t d\bar{w}_t, \ p_0 = -N y_0 - n, \\ -dq_t = (a_t q_t - A_t p_t + L_t x_t + l_t)dt - k_t dw_t - \bar{k}_t d\bar{w}_t, \\ y_T = F x_T + G, \quad q_T = -F p_T + M x_T + m. \end{cases}$$

Then u is an optimal control of Problem (LQC).

Assumption 3

$R_t > 0$ and $1/R_t$ are uniformly bounded and deterministic functions.

Proposition 3.1

Let Assumptions 1, 2 and 3 hold. If u is an optimal control of Problem (LQC), then u is unique.

Optimal Filtering of State Equation

Proposition 3.2

Let Assumption 1 hold. For any $v \in \mathcal{U}_{ad}$, the optimal filtering of $(x_t^v, y_t^v, z_t^v, \bar{z}_t^v)$ with respect to $\mathcal{F}_t^{Y^v}$ satisfies an FBSDE

$$\begin{cases} d\hat{x}_t^v = \left(a_t\hat{x}_t^v + b_t v_t + \bar{b}_t\right) dt + \left(c_t + \dfrac{P_t f_t}{h_t}\right) d\hat{w}_t, \\ -d\hat{y}_t^v = \left(A_t\hat{x}_t^v + B_t\hat{y}_t^v + C_t\hat{z}_t^v + \bar{C}_t\hat{\bar{z}}_t^v + D_t v_t + \bar{D}_t\right) dt - \hat{Z}_t^v d\hat{w}_t, \\ \hat{x}_0^v = e_0, \quad \hat{y}_T^v = F\hat{x}_T^v + \hat{G}, \end{cases} \tag{4.10}$$

where the mean square error P_t of the estimate $\hat{x}_t^v$ is the unique solution of

- A special case of (4.10) is derived originally in Huang, Wang and Xiong [SICON, 2009].

Continuation of Proposition 3.2

$$\begin{cases} \dot{P}_t - 2a_t P_t + \left(c_t + \frac{P_t f_t}{h_t}\right)^2 - (c_t + \bar{c}_t)^2 = 0, \\ P_0 = 0, \end{cases}$$

$$\hat{w}_t = \int_0^t \frac{f_s}{h_s}(x_s^v - \hat{x}_s^v)ds + w_t \tag{4.11}$$

is a standard BM with values in $\mathbb{R}$, and

$$\hat{Z}_t^v = \hat{z}_t^v + \frac{f_t}{h_t}\left(\widehat{x_t^v y_t^v} - \hat{x}_t^v \hat{y}_t^v\right).$$

Proposition 3.3

Let Assumptions 1, 2 and $O_t = 0$ hold. The optimal filtering of (p_t, q_t, k_t) depending on $\mathcal{F}_t^Y$ satisfies an FBSDE

$$\begin{cases} d\hat{p}_t = (B_t\hat{p}_t - o_t)dt + \left[C_t\hat{p}_t + \dfrac{f_t}{h_t}\left(\widehat{x_t p_t} - \hat{x}_t\hat{p}_t\right)\right] d\hat{w}_t, \\ -d\hat{q}_t = (a_t\hat{q}_t - A_t\hat{p}_t + L_t\hat{x}_t + l_t)dt - \hat{K}_t d\hat{w}_t, \\ \hat{p}_0 = \; - N y_0 - n, \quad \hat{q}_T = M\hat{x}_T - F\hat{p}_T + m \end{cases} \tag{4.12}$$

with

$$\hat{K}_t = \hat{k}_t + \frac{f_t}{h_t}\left(\widehat{x_t q_t} - \hat{x}_t\hat{q}_t\right),$$

Continuation of Proposition 3.3

where $(\hat{x}, \hat{y})$, $\hat{w}$ and $\widehat{x^{\mathbf{m}}p}$ satisfy (4.10) with $v = u$, (4.11), and

$$\begin{cases} d\widehat{x_t^{\mathbf{m}}p_t} = \Big[(\mathbf{m}a_t + B_t)\widehat{x_t^{\mathbf{m}}p_t} - o_t\widehat{x_t^{\mathbf{m}}} + \mathbf{m}\left(b_t u_t + \bar{b}_t + c_t C_t + \bar{c}_t \bar{C}_t\right) \widehat{x_t^{\mathbf{m}} \cdots} \\ \qquad + \left[\mathbf{m}c_t\widehat{x_t^{\mathbf{m}-1}p_t} + C_t\widehat{x_t^{\mathbf{m}}p_t} + \dfrac{f_t}{h_t}\left(\widehat{x_t^{\mathbf{m}+1}p_t} - \hat{x}_t\widehat{x_t^{\mathbf{m}}p_t}\right)\right] d\hat{w}_t, \\ \widehat{x_0^{\mathbf{m}}p_0} = -e_0^{\mathbf{m}}(Ny_0 + n), \quad \mathbf{m} = 1, 2, 3, \cdots, \end{cases}$$

respectively.

Set

$$\begin{cases} q_t = \pi_t x_t + \Sigma_t p_t + \theta_t \\ \pi_T = M, \quad \Sigma_T = -F, \quad \theta_T = m. \end{cases}$$

Then,

$$\begin{cases} \dot{\pi}_t + 2a_t\pi_t - \dfrac{1}{R_t} b_t^2 \pi_t^2 + L_t = 0, \\ \pi_T = M, \end{cases} \tag{4.13}$$

$$\begin{cases} \dot{\Sigma}_t + \left(a_t + B_t - \frac{1}{R_t} b_t^2 \pi_t\right) \Sigma_t + \frac{1}{R_t} b_t D_t \pi_t - A_t = 0, \\ \Sigma_T = -F, \end{cases} \tag{4.14}$$

$$\begin{cases} \dot{\theta}_t + \left(a_t - \frac{1}{R_t} b_t^2 \pi_t\right) \theta_t - o_t \Sigma_t - \frac{1}{R_t} b_t r_t \pi_t + \bar{b}_t \pi_t + l_t = 0, \\ \theta_T = m. \end{cases} \tag{4.15}$$

Theorem 3.3

Let Assumptions 1, 2, 3 and $O_t = 0$ hold. If

$$u_t = \frac{1}{R_t}(D_t \hat{p}_t - b_t \hat{q}_t - r_t)$$

is the optimal control of Problem (LQC), then it can be represented as

$$u_t = \frac{1}{R_t}[(D_t - b_t \Sigma_t)\hat{p}_t - b_t \pi_t \hat{x}_t - b_t \theta_t - r_t],$$

where $(\hat{x}, \hat{y}, \hat{z}, \hat{\tilde{z}})$, $\left(\hat{p}, \hat{q}, \hat{k}\right)$, π, Σ and θ are the solutions of (4.10) with $v = u$, (4.12), (4.13), (4.14) and (4.15), respectively.

5. A Special Case of Problem (LQC)

Example 4.1

$$\inf_{v\in\mathcal{U}_{ad}} J[v], \quad J[v] = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[O_t(y_t^v)^2 + R_t v_t^2\right] dt + N(y_0^v)^2 + 2ny_0^v\right\},$$

$$\begin{cases} -dy_t^v = \left(B_t y_t^v + C_t z_t^v + \bar{C}_t \bar{z}_t^v + D_t v_t\right) dt - z_t^v dw_t - \bar{z}_t^v d\bar{w}_t, \\ \quad y_T^v = G. \end{cases}$$

Suppose that w_t is observable at time t. It can be regarded as the case of (2.9) with $f_t = g_t = 0$ and $h_t = 1$.

$$u_t = R_t^{-1} D_t \hat{p}_t$$

where

$$\begin{cases} dp_t = (B_t p_t - O_t y_t)dt + C_t p_t dw_t + \bar{C}_t p_t d\bar{w}_t \\ p_0 = -N y_0 - n \end{cases}$$

and hence,

$$\begin{cases} d\hat{p}_t = (B_t \hat{p}_t - O_t \hat{y}_t)dt + C_t \hat{p}_t dw_t, \\ -d\hat{y}_t = \left(B_t \hat{y}_t + C_t \hat{z}_t + \bar{C}_t \hat{\bar{z}}_t + D_t u_t\right) dt - \hat{z}_t dw_t, \\ \hat{p}_0 = -N y_0 - n, \quad \hat{y}_T = \hat{G}. \end{cases} \quad (5.16)$$

Set

$$p_t = \alpha_t y_t + \beta_t, \quad \alpha_0 = -N, \;\; \beta_0 = -n.$$

Then,

$$\alpha_t z_t = C_t p_t, \quad \alpha_t \bar{z}_t = \bar{C}_t p_t$$

$$\begin{cases} \dot{\alpha}_t - \left(2B_t + C_t^2 + \bar{C}_t^2\right)\alpha_t - \dfrac{1}{R_t} D_t^2 \alpha_t^2 + O_t = 0, \\ \alpha_0 = -N, \end{cases} \tag{5.17}$$

$$\begin{cases} \dot{\beta}_t - \left(B_t + C_t^2 + \bar{C}_t^2 + \dfrac{1}{R_t} D_t^2 \alpha_t\right)\beta_t = 0, \\ \beta_0 = -n. \end{cases}$$

Proposition 4.1

Let Assumptions 1, 2 and 3 hold. The optimal control of Example 4.1 is uniquely denoted by

$$u_t = \frac{1}{R_t} D_t \hat{p}_t,$$

where $\hat{p}$ is the unique solution of

$$\begin{cases} d\hat{p}_t = (B_t \hat{p}_t - O_t \hat{y}_t) dt + C_t \hat{p}_t dw_t, \\ -d\hat{y}_t = \left(B_t \hat{y}_t + C_t \hat{z}_t + \left(\alpha_t^{-1} \bar{C}_t^2 + R_t^{-1} D_t^2 \right) \hat{p}_t \right) dt - \hat{z}_t dw_t, \\ \hat{p}_0 = -N y_0 - n, \quad \hat{y}_T = \hat{G}. \end{cases}$$

6. Summary

- We focus on an LQ optimal control problem of FBSDEs, where observation coefficient is linear with respect to x, and observation noise is correlated with state noise. A backward separation method is introduced. Combining it with variational method and optimal filtering, two optimality conditions and a feedback representation of optimal control are derived. A closed-form optimal solution is obtained in Example 4.1.
- The backward separation method is applicable to some linear stochastic differential games with partial observations.

Thanks!