THE THRESHOLD FOR RANDOM 3-SAT IS AT LEAST 2.833

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outline

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- Main method

Backgroud

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- The SAT problem has phase transition phenomena.

k-SAT

1. k-SAT:n boolean variables $V = \{x_1, \dots x_n\}$ and the corresponding set of $2n$ literals $L = \{x_1, \bar{x_1} \dots x_n, \bar{x_n}\}$. A k-clause is a disjunction of k literals of distinct underlying variables. k conjunctive normal form $F_k(n, m)$ (k-CNF) is the conjunction of m clauses. k-CNF is called by k-SAT.

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$$F_3(4,4) = (x_1 \vee \bar{x_2} \vee x_3) \wedge (x_2 \vee \bar{x_3} \vee x_4) \wedge (\bar{x_1} \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_4)$$

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3. k-SAT is a The SAT problem and is NP-complete, and it has phase transition phenomena.

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$$\lim_{n\to\infty} T_k(n, r_k - \epsilon) = 1, \text{ and } \lim_{n\to\infty} T_k(n, r_k + \epsilon) = 0$$

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3 V.Chvátal and B.Reed (1992), $r_2 = 1$.

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Corollary: If r is such that $\liminf_{n\to\infty} T_k(n, r - \epsilon) > 0$, then for any $\epsilon > 0$, $\lim_{n\to\infty} T_k(n, r - \epsilon) = 1$. This implies

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5 A.Coja-Oghlan (2013), k large enough,

$$r_k(n) \approx 2^3 \ln 2 - (1 + \ln 2)/2 + o_k(1).$$

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- lower bounds (non-algorithmic)
  best of lower bounds is 2.68 (D.Achlioptas and Y.Peres AMS 2004)(second moment method).

Main result

Theorem *For random 3-SAT, the threshold* $r_3 \geq 2.833$.

Main method

- second moment method

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lemma 1 *For any non-negative random variable X,then*

$$P(X > 0) \geq \frac{E^2(X)}{E(X^2)}.$$

lemma 2 *Let ϕ be any real, positive, twice-differentiable function on $[0,1]$ and let*

$$S_n = \sum_{z=0}^{n} \binom{n}{z} \phi(z/n)^n$$

letting $0^0 \equiv 1$,define g on $[0,1]$ as

$$g(\alpha) = \frac{\phi(\alpha)}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$$

If there exists $\alpha_{max} \in (0,1)$ such that $g(\alpha_{max}) > g(\alpha)$ for all $\alpha \neq \alpha_{max}$, and $g''(\alpha_{max}) < 0$, then there exist constants $B, C > 0$ such that for all sufficiently large n,

$$B \times g(\alpha_{max})^n \leq S_n \leq C \times g(\alpha_{max})^n$$

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$c(\sigma) = v$	$w(v) = w(\sigma, c)$
$(1, 1, 1)$	y_1
$(1, 1, 0)$	y_2
$(1, 0, 1)$	y_3
$(0, 1, 1)$	y_4
$(1, 0, 0)$	y_5
$(0, 1, 0)$	y_6
$(0, 0, 1)$	y_7
$(0, 0, 0)$	0

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Let $f_w(\alpha) = E[w(\sigma, c)w(\tau, c)]$ (a pair of truth assignments σ, τ with overlap $z = \alpha n$).

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Now, take $\phi(\alpha) = 2f_w(\alpha)^r$, then

$$g(1/2) = 2\phi(1/2) = 4f_w(1/2)^r.$$

Clearly,

$$g(\alpha) = g(\alpha; y_1, y_2, \ldots, y_7; r).$$

By Lemma 2, if for some $r > 0$, there exist some positive $y_1, y_2, \ldots, y_7$ such that g satisfies the conditions in Lemma 2 with $\alpha_{\max} = 1/2$, then, for large enough n,

$$r_3(n) \geq r.$$

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By taking

$$y_1 = 1, \; y_2 = y_3 = y_4 = 1.35, \; y_5 = y_6 = y_7 = 1.35,$$

we get our best lower bound $r = 2.553$.

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Now use Lemma 2 for X_+, by taking

$$y_1 = 1, y_2 = y_3 = y_4 = 1.12, y_5 = y_6 = y_7 = 2.12,$$

we get our best lower bound $r = 2.833$.

Thank you for your attention !