On the path-independence of Girsanov density for infinite-dimensional stochastic differential equations

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Outline

1. **Characterising path-independence of Girsanov transformation for SDEs on $\mathbb{R}^d$**
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   - The case of $d = 1$
   - Extension to differential manifolds
2. **Further related works**
3. **Extension to SDEs on Hilbert spaces via Galerkin approximation**

Given $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\in[0,\infty)})$. Consider the following SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

where
$b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$,
$\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, and
B_t is d-dimensional $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion.
It is well known that under the usual conditions of linear growth and locally Lipschitz for the coefficients b and σ, there exists a unique solution to the equation with given initial data X_0.

The celebrated Girsanov theorem provides a very powerful tool to solve SDEs under the name of the *Girsanov transformation or the transformation of the drift*. Let $\gamma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the following condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right)\right] < \infty, \quad \forall t > 0.$$

Then, by Girsanov theorem,

$$\exp\left(\int_0^t \gamma(s, X_s) dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right), \quad t \in [0, \infty)$$

is an $\{\mathcal{F}_t\}$-martingale. Furthermore, for $t \geq 0$, we define

$$Q_t := \exp\left(\int_0^t \gamma(s, X_s) dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right) \cdot P$$

or equivalently in terms of the Radon-Nikodym derivative

$$\frac{dQ_t}{dP} = \exp\left(\int_0^t \gamma(s, X_s)dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right).$$

Then, for any $T > 0$,

$$\tilde{B}_t := B_t - \int_0^t \gamma(s, X_s)ds, \quad 0 \leq t \leq T$$

is an $\{\mathcal{F}_t\}$-Brownian motion under the probability Q_T.
Moreover, X_t satisfies

$$dX_t = [b(t, X_t) + \sigma(t, X_t)\gamma(t, X_t)]dt + \sigma(t, X_t)d\tilde{B}_t, \quad t \geq 0.$$

Motivation from economics and finance Now look at

$$\frac{dQ_t}{dP} = \exp\left(\int_0^t \gamma(s, X_s)dB_s - \frac{1}{2}\int_0^t |\gamma(s, X_s)|^2 ds\right)$$

we see that generally $\frac{dQ_t}{dP}$ depends on the "history" of the path up to t (i.e., $\{X_s : 0 \le s \le t\}$)! While in economics and finance studies, in particular towards to the optimal problem for the utility functions in an equilibrium market, it is a necessary requirement that $\frac{dQ_t}{dP}$ depends only on the state X_t, not on the whole "history" $\{X_s : 0 \le s \le t\}$. See, e.g., [1] E. Stein, J.C. Stein: Stock price distributions with stochastic volatility: an analytic approach. *The Review of Financial Studies* **4** (1991), 727-752; [2] S. Hodges, A. Carverhill: Quasi mean reversion in an efficient stock market: the characterisation of Economic equilibria which support Black-Scholes Option pricing. *The Economic Journal* **103** (1993), 395-405.

So mathematically, one requires that the Radon-Nikodym derivative is in the form of

$$Z(t, X_t) - Z(0, X_0) = \ln \frac{dQ_t}{dP}, \quad t \in [0, \infty).$$

We call this the *path-independent property* of the density of the Girsanov transformation. A characterisation of this property for the above SDEs was obtained in

[1] A. Truman, F.-Y. Wang, J.-L. Wu, W. Yang: A link of stochastic differential equations to nonlinear parabolic equations, *SCIENCE CHINA Mathematics* **55** (2012), 1971-1976.
[2] J.-L. Wu, W. Yang: On stochastic differential equations and a generalised Burgers equation, pp 425-435 in *Stochastic Analysis and Its Applications* – Essays in Honor of Prof. Jia-An Yan (eds T S Zhang, X Y Zhou), Interdisciplinary Mathematical Sciences, Vol. 13, World Scientific, Singapore, 2012.

Assumptions:
(i) (*Non-degeneracy*) The coefficient σ satisfies that the matrix $\sigma(t,x)$ is invertible, for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$;

(ii) Specify the function γ by

$$\gamma(t,x) = -(\sigma(t,x))^{-1} b(t,x)$$

so that $b(t,X_t) + \sigma(t,X_t)\gamma(t,X_t) = 0$, and hence we require b and σ satisfy

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |(\sigma(s,X_s))^{-1} b(s,X_s)|^2 ds\right)\right] < \infty, \quad \forall t > 0.$$

Thus the associated probability measure Q_t is determined by

$$\frac{dQ_t}{dP} = \exp\left(-\int_0^t \langle(\sigma(s,X_s))^{-1}b(s,X_s), dB_s\rangle - \frac{1}{2}\int_0^t \left|(\sigma(s,X_s))^{-1}b(s,X_s)\right|^2 ds\right).$$

Now set

$$\hat{Z}_t := -\ln\frac{dQ_t}{dP}$$

that is

$$\hat{Z}_t = \int_0^t \langle(\sigma(s,X_s))^{-1}b(s,X_s), dB_s\rangle + \frac{1}{2}\int_0^t \left|(\sigma(s,X_s))^{-1}b(s,X_s)\right|^2 ds.$$

Clearly, $\hat{Z}_t$ is a one dimensional stochastic process with the stochastic differential form

$$d\hat{Z}_t = \frac{1}{2}\left|(\sigma(t,X_t))^{-1}b(t,X_t)\right|^2 dt + \langle(\sigma(t,X_t))^{-1}b(t,X_t), dB_t\rangle.$$

Theorem 1 (Characterisation Theorem)

Let $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be a scalar function which is C^1 with respect to the first variable and C^2 with respect to the second variable. Then

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \frac{1}{2}\int_0^t \left|(\sigma(s, X_s))^{-1} b(s, X_s)\right|^2 ds \\ &\quad + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle \end{aligned}$$

equivalently,

$$\frac{dQ_t}{dP} = \exp\{v(0, X_0) - v(t, X_t)\}, \quad t \in [0, \infty)$$

holds if and only if

Theorem 1 (cont'd)

$$b(t,x) = (\sigma\sigma^*\nabla v)(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}^d$$

and v satisfies the following time-reversed Burgers-KPZ type equation

$$\frac{\partial}{\partial t} v(t,x) = -\frac{1}{2}\left\{\left[Tr(\sigma\sigma^*\nabla^2 v)\right](t,x) + |\sigma^*\nabla v|^2(t,x)\right\}$$

where $\nabla^2 v$ stands for the Hessian matrix of v with respect to the second variable.

Proof

Necessity Assume that there exists a scalar function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ which is C^1 with respect to the first variable and C^2 with respect to the second variable such that

$$
\begin{aligned}
v(t, X_t) &= v(0, X_0) + \frac{1}{2}\int_0^t \left|(\sigma(s, X_s))^{-1} b(s, X_s)\right|^2 ds \\
&\quad + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle
\end{aligned}
$$

holds, then we have

$$
dv(t, X_t) = \frac{1}{2}\left|(\sigma(t, X_t))^{-1} b(t, X_t)\right|^2 dt + \langle (\sigma(t, X_t))^{-1} b(t, X_t), dB_t \rangle .
$$

Now viewing $v(t, X_t)$ as the composition of the deterministic $C^{1,2}$-function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ with the continuous semi-martingale X_t, we can apply Itô's formula to $v(t, X_t)$ and further with the help of our original SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

we have the following derivation

$$\begin{aligned} dv(t, X_t) &= \left\{ \frac{\partial}{\partial t} v(t, X_t) + \frac{1}{2}[Tr(\sigma\sigma^*)\nabla^2 v](t, X_t) \right. \\ &\quad \left. + \langle b, \nabla v\rangle(t, X_t) \right\} dt + \langle (\sigma^* \nabla v)(t, X_t), dB_t \rangle \end{aligned}$$

since

$$\langle \nabla v(t, X_t), \sigma(t, X_t)dB_t \rangle = \langle \sigma^*(t, X_t)\nabla v(t, X_t), dB_t \rangle \, .$$

Now comparing this with the previously obtained

$$dv(t, X_t) = \frac{1}{2}\left|(\sigma(t, X_t))^{-1}b(t, X_t)\right|^2 dt + \langle(\sigma(t, X_t))^{-1}b(t, X_t), dB_t\rangle$$

and using the uniqueness of Doob-Meyer's decomposition of continuous semi-martingale, we conclude that the coefficients of dt and dB_t must coincide, respectively, namely

$$(\sigma^{-1}b)(t, X_t) = (\sigma^* \nabla v)(t, X_t)$$

$$\frac{1}{2}|(\sigma^{-1}b)(t, X_t)|^2 = \frac{\partial}{\partial t}v(t, X_t) + \frac{1}{2}[Tr(\sigma\sigma^* \nabla^2 v)](t, X_t) + \langle b, \nabla v\rangle(t, X_t)$$

holds for all $t > 0$.

Since our SDE is non-degenerate, the support of $X_t, t \in [0, \infty)$ is the whole space $\mathbb{R}^d$. Hence, the following two equalities

$$(\sigma^{-1} b)(t, x) = (\sigma^* \nabla) v(t, x)$$

$$\frac{1}{2} |(\sigma^{-1} b)(t, x)|^2 = \frac{\partial}{\partial t} v(t, x) + \langle b, \nabla v \rangle (t, x) + \frac{1}{2} [Tr(\sigma \sigma^* \nabla v)](t, x)$$

hold on $[0, \infty) \times \mathbb{R}^d$. From these equalities we derive

$$b(t, x) = (\sigma \sigma^* \nabla v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d$$

and v satisfies the Burgers-KPZ type equation

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2} \left\{ \left[Tr(\sigma \sigma^* \nabla^2 v) \right] (t, x) + |\sigma^* \nabla v|^2 (t, x) \right\} .$$

Sufficiency Assume that there exists a $C^{1,2}$ scalar function $v : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ solving the Burgers-KPZ equation. Specify the drift b of the original SDE via

$$b(t, x) = (\sigma\sigma^* \nabla v)(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^d .$$

We then have

$$\begin{aligned} dv(t, X_t) &= \Big[-\frac{1}{2}|\sigma^* \nabla v|^2(t, X_t) + \langle b, \nabla v\rangle(t, X_t)\Big] dt \\ &\quad +\langle(\sigma^* \nabla v)(t, X_t), dB_t\rangle \\ &= \frac{1}{2}|\sigma^{-1} b|^2(t, X_t)dt + \langle(\sigma^{-1} b)(t, X_t), dB_t\rangle . \end{aligned}$$

The above clearly implies

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \frac{1}{2}\int_0^t \left|(\sigma(s, X_s))^{-1} b(s, X_s)\right|^2 ds \\ &\quad + \int_0^t \langle (\sigma(s, X_s))^{-1} b(s, X_s), dB_s \rangle \end{aligned}$$

by taking stochastic integration. This completes the proof.

For the simplest case that $d = 1$, we have more consequences from the characterisation theorem. In this case we have

$$\gamma(t,x) = -\frac{b(t,x)}{\sigma(t,x)}$$

since $\sigma(t,x) \neq 0$. Set

$$u(t,x) := \frac{b(t,x)}{\sigma^2(t,x)} = -\frac{\gamma(t,x)}{\sigma(t,x)}, \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

With the assumption on γ for the Girsanov theorem, we can rephrase our previous theorem in a slightly more concise manner

Theorem 1 in one dimension case

Let $v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be C^1 with respect to the first variable and C^2 with respect to the second variable. Then

$$v(t, X_t) = v(0, X_0) - \int_0^t \frac{b(s, X_s)}{\sigma(s, X_s)} dB_s - \frac{1}{2} \int_0^t \left| \frac{b(s, X_s)}{\sigma(s, X_s)} \right|^2 ds$$

iff $u(t, x) := \frac{\partial}{\partial x} v(t, x)$ satisfies the following nonlinear PDE

$$\frac{\partial}{\partial t} u = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} u - \sigma \left(\frac{\partial}{\partial x} \sigma + \sigma u \right) \frac{\partial}{\partial x} u - \sigma u^2 \frac{\partial}{\partial x} \sigma \, .$$

Theorem 2

Let $v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be C^1 with respect to the first variable and C^2 with respect to the second variable. Then

$$v(t, X_t) = v(0, X_0) - \int_0^t \frac{b(s, X_s)}{\sigma(s, X_s)} dB_s - \frac{1}{2} \int_0^t \left|\frac{b(s, X_s)}{\sigma(s, X_s)}\right|^2 ds$$

iff there exists a C^1-function $\Phi : \mathbb{R} \to \mathbb{R}$ such that for $u := \frac{\partial}{\partial x} v$

$$b(t, x) = \Phi(u(t, x)), \quad (t, x) \in [0, \infty) \times \mathbb{R}$$

and u satisfies the following (time-reversed) generalized Burgers equation

$$\frac{\partial}{\partial t} u(t, x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi_1(u(t, x)) - \frac{1}{2} \frac{\partial}{\partial x} \Psi_2(u(t, x))$$

Theorem 2 (cont'd)

where

$$\Psi_1(r) := \int \frac{\Phi(r)}{r} dr, \quad \Psi_2(r) := r\Phi(r), \quad r \in \mathbb{R}.$$

The above generalized Burgers equation covers much more classes of specific nonlinear PDEs. Here we give three examples to explicate this point.

Example 1 Give a constant $\sigma > 0$. Let $b(t,x) = \sigma^2 u(t,x)$ and $\sigma(t,x) \equiv \sigma$, our SDE then becomes

$$dX_t = \sigma^2 u(t, X_t)dt + \sigma dB_t.$$

The C^1-function Φ is simply given by $\Phi(r) = \sigma^2 r$ and the corresponding PDE is a classical Burgers equation (time-reversed)

$$\frac{\partial}{\partial t}u(t,x) = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}u(t,x) - \sigma^2 u(t,x)\frac{\partial}{\partial x}u(t,x).$$

The next example shows that our generalized Burgers equation can be a porous media type PDE.

Example 2 We fix $m \in \mathbb{N}$. Let $b(t,x) = m[u(t,x)]^m$ and $\sigma(t,x) = \sqrt{m}[u(t,x)]^{\frac{m-1}{2}}$, our SDE then becomes

$$dX_t = m[u(t,X_t)]^m dt + \sqrt{m}[u(t,X_t)]^{\frac{m-1}{2}} dB_t.$$

The C^1-function Φ is then given by $\Phi(r) = mr^m$ and the corresponding PDE is a porous media type nonlinear PDE

$$\frac{\partial}{\partial t}u(x,t) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}u^m(t,x) - m\frac{\partial}{\partial x}u^{m+1}(t,x).$$

The third example is to show that in the time-homogeneous case in the sense that b and σ are functions of the variable $x \in \mathbb{R}$ only, the corresponding PDE then determines a harmonic function.

Example 3 Let $b(t, x) = b(x)$ and $\sigma(t, x) = \sigma(x)$, our original SDE then reads as

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

and the corresponding PDE is a second order elliptic equation for harmonic functions

$$\frac{\partial^2}{\partial x^2}\Psi_1(u(x)) + \frac{\partial}{\partial x}\Psi_2(u(x)) = 0$$

where

$$\Psi_1(r) = \int \frac{\Phi(r)}{r} dr, \quad \Psi_2(r) = r\Phi(r), \quad r \in \mathbb{R}.$$

Here we'd like to extend our theorem to the case of SDEs on a general connected complete differential manifold. To this end, we need a proper framework to start with. Let us start with the following observation. In the situation of the SDEs on $\mathbb{R}^d$, if $X = (X_t)_{t\in[0,\infty)}$ solve

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

then, via martingale problem, the diffusion process X is associated with the Markov generator

$$L_t f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b^i(t,x)\frac{\partial f(x)}{\partial x_i}, \quad f \in C^2(\mathbb{R}^d)$$

with $a(t,x) := \sigma(t,x)\sigma^*(t,x)$.So let

$$g_t = (g_t^{ij}(\cdot)) := (\sigma\sigma^*)^{-1}(t,\cdot).$$

Then we have a time-dependent metric on $\mathbb{R}^d$ defined as follow

$$\langle x, y \rangle_{g_t} := \sum_{i,j=1}^{d} g_t^{ij} x_i y_j = \langle g_t x, y \rangle, \quad x, y \in \mathbb{R}^d.$$

Let ∇_{g_t} and Δ_{g_t} be the associated gradient and Laplacian, respectively. Then the generator for X can be reformulated as follows (cf. e.g. the classic books by D. Elworthy or by N. Ikeda and S. Watanabe)

$$L_t f = \frac{1}{2} \Delta_{g_t} f + \langle \tilde{b}(t, \cdot), \nabla_{g_t} f \rangle_{g_t}$$

for some smooth function $\tilde{b} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$. From this point of view, we intend to extend our theorem to a general connected complete differential manifold.

Let M be a d-dimensional connected complete differential manifold with a family of Riemannian metrics $\{g_t\}_{t\in[0,\infty)}$, which is smooth in $t \in [0,\infty)$. Clearly (M, g_t) is a Riemannian manifold for each $t \in [0,\infty)$. Let $\{b(t,\cdot)\}_{t\in[0,\infty)}$ be a family of smooth vector fields on M which is smooth in t as well. Let ∇_{g_t} and Δ_{g_t} denote the gradient and Laplacian operators induced by the metric g_t, respectively. Then the diffusion process X on M generated by the operator

$$L_t := \frac{1}{2}\Delta_{g_t} + b(t,\cdot)$$

can be constructed by solving the following SDE on M

$$dX_t = b(t, X_t)dt + \Phi_t \circ dB_t$$

where $\{B_t\}_{t\in[0,\infty)}$ is the d-dimensional Brownian motion, $\circ d$ stands for the Stratonovich differential, and Φ_t is the horizontal lift of X_t onto the frame bundle $O_t(M)$ of the Riemannian manifold (M, g_t), namely, Φ_t solves the following equation

$$d\Phi_t = H_{t,\Phi_t} \circ dX_t - \frac{1}{2}\Big\{\sum_{i,j=1}^{d}(\partial_t g_t)(\Phi_t e_i, \Phi_t e_j)V_{ij}(\Phi_t)\Big\}d\,t,$$

where $H_{t,\cdot} : T(M) \to O_t(M)$ is the horizontal lift w.r.t. the metric g_t, $\{e_i\}_{1\le i\le d}$ is the canonical basis on $\mathbb{R}^d$ and $\{V_{ij}\}_{1\le i,j\le d}$ is the canonical basis of vertical vector fields. Here $T(M)$ denotes the tangent bundle of M (cf. M. Arnaudon, K.A. Coulibaly and A. Thalmaier, *C. R. Acad. Sci. Paris Ser. I* **346** (2008)). The next result is an extension of our characterisation theorem to M.

Theorem 3

Let $v : [0, \infty) \times M \to \mathbb{R}$ be $C^{1,2}$. Then

$$v(t, X_t) = v(0, X_0) + \frac{1}{2}\int_0^t |b(s, X_s)|_{g_t}^2 ds + \int_0^t \langle (\Phi_s^{-1} b(s, X_s), dB_s\rangle_{g_t}$$

holds if and only if

$$b(t, x) = (\nabla_{g_t} v)(t, x), \quad (t, x) \in [0, \infty) \times M$$

and the following time-reversed Burgers-KPZ type equation

$$\frac{\partial}{\partial t} v(t, x) = -\frac{1}{2}\left[(\Delta_{g_t} v)(t, x) + |\nabla_{g_t} v|_{g_t}^2 (t, x)\right]$$

hold, where $|z|_{g_t}^2 := \langle z, z\rangle_{g_t}$ for any vector z on M.

Recently, an interesting study by colleagues in Swansea

G. Alhamzi, E.J. Beggs, A.D. Neate: From homotopy to Itô calculus and Hodge theory, arXiv.1307.3119

derives a similar link by pure algebraic approach, which is more close to quantum probability calculations.

Path-independent phenomenon also appeared in Calculus of Variation and Stochastic Deformation of Classical Mechanics [cf. J.-C. Zambrini, The research program of Stochastic Deformation (with a view toward Geometric Mechanics), arXiv.1212.4186]. In

A.B. Cruzeiro, J.-L. Wu and J.-C. Zambrini: On stochastically complete integrability of stochastic dynamical systems, working paper.

we link the complete integrability (via Ito-Dynkin formula) to the path-independence of the action functionals and we then characterise the integrability by certain Hamilton-Jacobi-Bellman equation.

Degenerate case

. Joint with Bo Wu, we recently consider the following SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq 0$$

where
$b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$,
$\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m$, and
B_t is m-dimensional $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-Brownian motion.
Under the condition

$$b(t, x) + \sigma(t, x)\gamma(t, x) = 0$$

we recover the characterisation theorem on the support of the solution X_t. Furthermore, we discuss this on Riemannian manifolds.

This part based on

[1] M. Wang, J.-L. Wu: Necessary and sufficient conditions for path-independence of Girsanov transformation for infinite-dimensional stochastic evolution equations, *Frontiers of Mathematics in China* **9** (2014), Issue 3, 601-622.
[2] F.-Y. Wang, J.-L. Wu: On infinite-dimensional stochastic differential equations driving by Q-Wiener processes, in preparation.

Given a separable $(H, \langle\cdot,\cdot\rangle, \|\cdot\|_H)$ with $\{e_i\}_{i\geq 1}$ a complete orthonormal basis for H. Let $L(H)$ be the Banach space of all linear operators $T : H \to H$ endowed with the usual operator norm $\|T\| := \sup_{\|x\|=1} \|Tx\|_H$ and $L_{HS}(H)$ the Hilbert space of all Hilbert-Schmidt operators $T : H \to H$ endowed with the norm $\|T\|_{HS} := (\sum_{i=1}^{\infty} \|Te_i\|_H^2)^{\frac{1}{2}}$. For a given symmetric, nonnegative operator $Q \in L_{HS}$, let $\{\beta_i(t,\omega)\}_{i\geq 1}$ is a family of independent one-dim. Brownian motions. A *Q-Wiener process* $\{W_t\}_{t\geq 0}$ is formulated as

$$W_t := W_t(\omega := \sum_{i=1}^{\infty} \beta_i(t,\omega) e_i, \quad \omega \in \Omega,\ t \in [0,\infty)$$

with

$$\mathbb{E}(\langle W_t, x\rangle\langle W_s, y\rangle) = t \wedge s\langle x, Qy\rangle, \quad t, s \in [0,\infty), \quad x, y \in H$$

We are concerned with the initial value problem for a semi-linear stochastic differential equation on H

$$\begin{cases} dX_t = \{AX_t + b(t, X_t)\}dt + \sigma(t, X_t)dW_t, \quad t > 0 \\ X_0 = x \in H, \end{cases} \tag{1}$$

where $A : H \to H$ is an unbound, linear operator with its domain $\mathcal{D}(A) \subset H$, $b : [0, \infty) \times H \to H$ and $\sigma : [0, \infty) \times H \to L(H)$ are $C^{1,2}$, in Fréchet differentiation.
As is known, in order the stochastic differentiation term makes sense, σ must be $L_{HS}(H)$-valued! This then causes a problem as we require that σ must be invertible but Hilbert-Schmidt operators are NOT invertible! So we need to find an appropriate way to formulate our problem. For simplicity, we assume $Q = \mathit{Identity}$ (which was done with Miao Wang in [1]. Extension to general Q-Wiener process driven SDEs is discussed in [2] joint with Feng-Yu Wang).

Let $(A, \mathcal{D}(A))$ be a linear, unbounded, negative definite, self-adjoint operator on H generating a contraction C_0-semigroup $\{e^{tA}\}_{t\geq 0}$. Let $L_A(H)$ be defined as

$$L_A(H) := \{L : H \to H \mid e^{tA}L \in L_{HS}(H), \ \ \forall t > 0\}$$

endowed with the σ-algebra induced by the family

$$\{L \to \langle e^{tA}Lx, y\rangle_H \mid t > 0, x, y \in H\}$$

from $\mathcal{B}(\mathbb{R})$ so that $L_A(H)$ is a measurable space. Consider mild equation associated with (1)

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}b(s, X_s)ds + \int_0^t e^{(t-s)A}\sigma(s, X_s)dW_s\,, \quad t \geq 0.$$

So we require $\sigma : [0, \infty) \times H \to L_A(H)$ to make the stochastic integral well-defined. To ensure the existence of a unique solution, we put following

(H1) Assume that $-A$ has discrete spectrum with eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots$$

counting multiplicities such that $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty$.

(H2) There exist a constant $\epsilon \in (0,1)$ and an increasing function $L : [0,\infty) \to (0,\infty)$ such that $\forall t \geq 0, \ \ \forall x, y \in H$

$$\sup_{t\in[0,T]} \left\{ \|b(t,0)\|_H^2 + \int_0^t \|e^{(t-s)A}\sigma(s,0)\|_{HS}^2 s^{-\epsilon} ds \right\} < \infty$$

and

$$\|b(t,x) - b(t,y)\|_H + \|e^{tA}\left(\sigma(t,x) - \sigma(t,y)\right)\|_{HS} \leq L(t)\|x-y\|_H.$$

One can show that under the above conditions, there is a unique mild solution with

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|X_t\|_H^2\right)<\infty,\quad \forall T>0.$$

Remark: Under the assumption (H1), it clear that the space $L_A(H)$ allows to have invertible operators from H to H, such as the identity operator.

Next, we need Itô formula for real-valued functions of X_t. Here we notice that the diffusion coefficient σ in (1) is not Hilbert-Schmidt, thus the usual infinite-dimensional Itô formula can not apply. it seems so far there is no Itô formula for functions of solutions of infinite-dimensional semi-linear SDEs containing our SEEs (1) which are only solved with midl solutions. We could succeed Itô formula here by using Galerkin approximation.

For any $n \geq 1$, let $\pi_n : H \to H_n := \text{span}\{e_1, \cdots, e_n\}$ be the (orthogonal) projection operator, that is

$$\pi_n x := \sum_{i=1}^{n} \langle x, e_i \rangle_H e_i, \quad x \in H.$$

We note that the project operator π_n commutes with the semigroup $e^{tA}, t \geq 0$. Let $A_n := A\,|_{H_n}, b_n := \pi_n b$ and $\sigma_n := \pi_n \sigma$. We then consider the following (finite-dim.) stochastic differential equation in H_n

$$\begin{cases} dX_t^n = \{A_n X_t^n + b_n(t, X_t^n)\}dt + \sigma_n(t, X_t^n)dW_t, \\ X^n(0) = \pi_n x. \end{cases} \tag{2}$$

The assumption (H2) implies b_n and σ_n fulfill the usuall growth and Lipschitz conditions so that there exists a unique strong solution $X_t^n \in H_n, t \in [0, \infty)$ to (2). One can show

$$\lim_{n\to\infty} \mathbb{E}\|X_t^n - X_t\|_H^2 = 0, \quad t \geq 0. \tag{3}$$

By using Itô formula for X_t^n and the above limit, we have the following Itô formula for X_t

Proposition

Assume (H1), (H2), and let $v : [0, \infty) \times H \to \mathbb{R}$ be in $C_b^{1,2}([0, \infty) \times H)$ such that $[\nabla v(t, x)] \in Dom(A)$ for any $(t, x) \in [0, \infty) \times H$ and $\|A\nabla v(t, \cdot)\|_H$ is bounded locally and uniformly in $t \in [0, \infty)$. Then we have

$$\begin{aligned} v(t, X_t) &= v(0, X_0) + \int_0^t \langle \sigma^*(s, X_s)\nabla v(s, X_s), dW_s\rangle_H \\ &\quad + \int_0^t \Big[\frac{\partial}{\partial s} v(s, X_s) + \langle \nabla v(s, X_s), b(s, X_s)\rangle_H \\ &\qquad + \langle A\nabla v(s, X_s), X_s\rangle_H\Big] ds \\ &\quad + \frac{1}{2}\int_0^t Tr[(\sigma\sigma^*)(s, X_s)\nabla^2 v(s, X_s)] ds . \qquad (4) \end{aligned}$$

Next, we assume

(H3) $\sigma(t,x)$ is invertible for each $(t,x) \in [0,\infty) \times H$ and the two coefficients b, σ in Equation (1) fulfill

$$\mathbb{E}\left(\exp\left\{\frac{1}{2}\int_0^T \|\sigma^{-1}(t,X_t)b(t,X_t)\|_H^2 dt\right\}\right) < \infty, \quad \forall T > 0.$$

So under (H1), (H2) and (H3), the *Girsanov density*

$$\begin{aligned}\frac{d\tilde{P}_t}{dP}(\omega) \quad := \quad & \exp\Big\{ -\int_0^t \langle \sigma^{-1}(s,X_s(\omega))b(s,X_s(\omega)), dW_s(\omega)\rangle_H \\ & -\frac{1}{2}\int_0^t \|\sigma^{-1}(s,X_s(\omega))b(s,X_s(\omega))\|_H^2 ds\Big\}, \quad t \geq 0\end{aligned}$$

is a well-defined process for the SDE in (1).

The following main result gives a necessary and sufficient conditions, and hence a characterization of path-independence of the Girsanov density process for (infinite-dimensional) SDEs on separable Hilbert spaces.

Theorem 4

Assume (H1), (H2), (H3) and let $v : [0, \infty) \times H \to \mathbb{R}$ be in $C_b^{1,2}([0, \infty) \times H)$ such that $[\nabla v(t, x)] \in Dom(A)$ for any $(t, x) \in [0, \infty) \times H$ and $\|A\nabla v(t, \cdot)\|_H$ is bounded locally and uniformly in $t \in [0, \infty)$. Then the Girsanov density (5) for (1) fulfills the following path-indendenpent property

$$\frac{d\tilde{P}_t}{dP} = \exp\{v(0, X_0) - v(t, X_t)\}, \quad t \geq 0$$

if and only if v satisfies

Theorem 4 (cont'd)

$$\frac{\partial}{\partial t} v(t,x) = -\frac{1}{2}\left\{ Tr[(\sigma\sigma^*)\nabla^2 v](t,x) + \|\sigma^*\nabla v\|_H^2(t,x) \right\} - \langle x, A\nabla v(t,x)\rangle_H$$

and

$$b(t,x) = [(\sigma\sigma^*)\nabla v](t,x), \quad \forall (t,x) \in (0,\infty)\times H.$$

With Feng-Yu Wang, we study SDEs on Hilbert spaces. The key starting point is to extend the finite dimensional condition

$$b(t,x)+\sigma(t,x)\gamma(t,x)=0$$

to infinite dimensional so that we allow σ to be Hilbert-Schmidt operator valued.

Thank You!