Quasi-Regular Dirichlet Forms on Riemannian Loop Spaces

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Outline

Introduction

Quasi-regularity

Let $(E, \mathcal{F}, \mu)$ be a σ-finite measure space, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ a D-F on $L^2(\mu)$. If

- There exists an $\mathcal{E}$-nest $(F_n)_{n \geq 1}$ consisting of compact sets
- There exist $u_n \in \mathcal{D}(\mathcal{E})$, having $\mathcal{E}$-continuous μ-versions $\tilde{u}_n$ and $\tilde{u}_n$ separates points of $E \backslash N$, where N is a $\mathcal{E}$-exceptional set of E.
- There exists an $\mathcal{E}_1$-dense subset of $\mathcal{D}(\mathcal{E})$ whose elements have $\mathcal{E}$-continuous μ-versions.

  Then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called **quasi-regular**.

Importance: Existence and Uniqueness of associated diffusion Process.

Framework

Let M be a d-dimensional **complete** and **stochastically complete** connected Riemannian manifold, and X be the Brownian motion on M starting from a fixed point $o \in M$.

Path space

$$W_o(M) := \{\gamma \in C([0,1];M) | \gamma(0) = o\}$$

Loop space

$$L_o(M) := \{\gamma \in C([0,1];M) | \gamma(0) = \gamma(1) = o\}.$$

Horizontal lift

Let U_t be the horizontal lift of X, that is,

$$\mathrm{d}U_t = \sum_{i=1}^{d} H_i(U_t) \circ \mathrm{d}B_t^i, \quad t \geq 0,$$

where U_0 is an othornormal basis of $T_o M$, $B_t^1, \cdots, B_t^d$ are independent one dimensional Brownian motions, $\{H_i\}_{i=1}^d$ is the standard othornormal basis of horizontal vector fields.

Cameron-Martin space

$$\mathbb{H} := \Big\{ h \in C([0,1];\mathbb{R}^d) \Big| h \text{ is absolutely continuous,} \quad h_0 = 0, \| h \|_{\mathbb{H}} = \int_0^1 |h_s'|^2 \mathrm{d}s < \infty \Big\}$$

$$\mathbb{H}_0 := \{h \in \mathbb{H} | h(1) = 0\}.$$

Cylindrical function

$\mathcal{F}C_b$ =cylindrical functions on $W_o(M)$,

$F \in \mathcal{F}C_b \Longleftrightarrow \exists\ 0 \leqslant t_1 < \cdots < t_n < 1$ and $f \in C_c^1(M^n)$

$$F(\gamma) = f(\gamma_{t_1}, \cdots, \gamma_{t_n}), \quad \gamma \in W_o(M).$$

Let μ_o be the distribution of X, and ν_o be Brownian bridge measure on $L_o(M)$. Then for any $F \in \mathcal{F}C_b$ with $F(\gamma) := f(\gamma_{t_1}, \cdots, \gamma_{t_n})$,

$$\nu_o\left[f\left(\gamma(t_1), ...\gamma(t_n)\right)\right] := \mu_o\left[f\left(\gamma(t_1), \cdots \gamma(t_n)\right)\Big|\gamma(1) = o\right].$$

Equivalently,

$$\begin{aligned}&\nu_o\left[f\left(\gamma(t_1), ...\gamma(t_n)\right)\right]\\&= \frac{1}{p(1, o, o)} \int_{M^n} f\left(x_1, \cdots, x_n\right) p\left(t_1, o, x_1\right)\\&\quad \times p\left(t_2 - t_1, x_1, x_2\right) \cdots p\left(1 - t_n, x_n, o\right) dx_1 \cdots dx_n.\end{aligned}$$

Gradient operator

For any $F \in \mathcal{F}C_b$ with $F(\gamma) := f(\gamma_{t_1}, \cdots, \gamma_{t_n})$ and any $h \in \mathbb{H}$, let

$$D_h F(\gamma) := \sum_{i=1}^{n} \langle \nabla_i f(\gamma), U_{t_i} h_{t_i} \rangle,$$

where ∇_i is the (distributional) gradient operator in the i-th component. By Riesz Representation theorem, $\exists\ DF(\gamma) \in \mathbb{H}, D^0 F(\gamma) \in \mathbb{H}_0, \gamma \in W_o(M)$ s.t.

$$D_h F(\gamma) = \langle DF(\gamma), h \rangle_{\mathbb{H}}, D_{\tilde{h}} F(\gamma) = \langle D^0 F(\gamma), \tilde{h} \rangle_{\mathbb{H}_0}, \quad h \in \mathbb{H}, \tilde{h} \in \mathbb{H}_0.$$

In particular, for the above function F,

$$(DF(\gamma))_s = \sum_{i=1}^{n} (s \wedge t_i) U_{t_i}(\gamma)^{-1} \nabla_i f(\gamma),$$

$$(D^0 F(\gamma))_s = \sum_{i=1}^{n} (s \wedge t_i - s t_i) U_{t_i}(\gamma)^{-1} \nabla_i f(\gamma), \nu_o - a.s, \ s \in [0, 1].$$

Define:

$$\mathcal{E}(F,G) := \int_{W_o(M)} \langle DF, DG\rangle_{\mathbb{H}} \mathrm{d}\mu_o,$$
$$\mathcal{E}_0(F,G) := \int_{L_o(M)} \langle D^0F, D^0G\rangle_{\mathbb{H}_0} \mathrm{d}\nu_o, \quad F,G \in \mathcal{F}C_b.$$

In [Chen-Wu(JFA2014)], $(\mathcal{E}, \mathcal{F}C_b)$ is closable in $L^2(\mu_o)$ and its closure is a quasi-regular Dirichlet form if M is only complete and stochastic complete. Our aim is now to show that $(\mathcal{E}_0, \mathcal{F}C_b)$ is closable in $L^2(\nu_o)$ and its closure is a quasi-regular Dirichlet under the same condition.

Difficulty $\leftrightarrow$ closability$\leftrightarrow$ Integration by parts formula

Integration by parts formula(Path space)

Let $\mathbf{Ric}_{U_t} : \mathbb{R}^d \to \mathbb{R}^d$. Assume that

$$\mathbb{E}\int_0^1 \|\mathbf{Ric}_{U_t}\|^2 \mathrm{d}t < \infty.$$

We have

$$\int_{W_o(M)} FD_h G \mathrm{d}\mu_o = \int_{W_o(M)} GD_h^* F \mathrm{d}\mu_o, \quad F, G \in \mathcal{F}C_b,$$

where

$$D_h^* = -D_h + \int_0^1 \left\langle \dot{h}_t + \frac{1}{2}\mathbf{Ric}_{U_t} h_t, \mathrm{d}B_t \right\rangle,$$

where B_t is an $\mathbb{R}^d$-valued standard Brownian motion relative to the filtered probability spaces $(W_o(M), \{\mathcal{F}\}_{0\le s\le 1}, \mathcal{F}, \mu_o)$.

Integration by parts formula(Loop space)

Assume that M is compact connected Riemannian manifold or hyperbolic space. If

$$\mathbb{E}_{\nu_o} \int_0^1 |\dot{h}_s|^{2+\delta} ds < \infty$$

for some constant $\delta > 0$. We have

$$\int_{L_o(M)} F D_h G \mathrm{d}\nu_o = \int_{L_o(M)} G D_h^* F \mathrm{d}\nu_o, \quad F, G \in \mathcal{F}C_b,$$

where

$$D_h^* = -D_h + \int_0^1 \left\langle \dot{h}_t + \frac{1}{2}\mathbf{Ric}_{U_t} h_t - Hess_{Ut} \log p_{1-t}(\cdot, o) h_t, \mathrm{d}\beta_t \right\rangle,$$

where β_t is a is an $\mathbb{R}^d$-valued standard Brownian motion relative to the filtered probability spaces $(L_o(M), \{\mathcal{C}\}_{0 \le s \le 1}, \mathcal{C}, \nu_o)$.

Quasi-regular D-F on loop spaces

known results(Path)

- Driver-Rökner(92): **Compactness** $+$(**IPF**) $\Rightarrow$ $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ Quasi-reguar local D-F.
- Elworthy-Ma(97, compact): **Finsler Manifold**$+$(**closability**) $\Rightarrow$ Quasi-reguar D-F(damped).
- Chen-Wu(14): **complete and stochastic complete**$\Rightarrow$ $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ Quasi-reguar D-F.

known results(Loop)

- Driver-Rökner(92): **Compactness** $+$(**IPF**) $\Rightarrow (\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ Quasi-reguar local D-F.
- Hsu(97, M=compact): **Compactness** $+$(**IPF**) $\Rightarrow (\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ Quasi-reguar local D-F.
- Aida(00): **Hyperbolic**$+$(**IPF**) $\Rightarrow (\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ Quasi-reguar D-F.

Main Result

Theorem A

Assume that M is a complete and stochastic complete connected Riemannian manifold. Then the quadratic form $(\mathcal{E}_0, \mathcal{F}C_b)$ is closable on $L^2(\nu_o)$, and its closure $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ is a quasi-regular Dirichlet form on $L^2(\nu_o)$.

Remark

Since loop space has Finsler structure, we only show that $(\mathcal{E}_0, \mathcal{F}C_b)$ is closable. And this will be obtained by the integration by parts formula. By observing the integration by parts formula on path space and loop space(compact) respectively, for path space, we may take a method of approximation by change of metric, but for loop space, we can not take this way, because this will depend on the heat kernel.

Sketch of Proof of Theorem A

We only show that $(\mathcal{E}, \mathcal{F}C_b)$ is closable.
Similar to the case on path space, we will first prove the local integration by parts formula on loop space, and obtain the local closability, then the proof derive from the approximation process.
(a) Constructing a sequence of random $l_n(t)$ such that

$$\mathbb{E}_{\nu_o}\left[\int_0^1 |l_n'(t)|^2 dt\right] \leq \infty, \quad n \geq 1$$

and

$$l_n|_{B_{n-1}} = 1, \quad l_n|_{B_n^c} = 0.$$

(b) (**Local IPF on path space**)For any $h \in \mathbb{H}$, let $h_m(s) := h(s) l_m(s)$, then for every $m \geq 1$ and $F \in \mathcal{F}C_b$, we have

$$\mathbb{E}_{\mu_o}\left[D_{h_m} F(\gamma)\right] = \mathbb{E}_{\mu_o}\left[F(\gamma) \int_0^1 \left(h_m'(t,\gamma) + \frac{1}{2}\mathrm{Ric}_{U_t(\gamma)} h_m(t,\gamma)\right) d\omega_t\right]$$

(c) (**Local IPF on loop space**)For any $h \in \mathbb{H}_0$ with $h(t) = 0, \forall\ t \in [t_0, 1]$ for some $0 < t_0 < 1$, let $h_m(s) := h(s) l_m(s)$, then for every $m \geq 1$ and $F \in \mathcal{F}C_b$, we have

$$\mathbb{E}_{\nu_o}\left[D_{h_m} F(\gamma)\right] = \mathbb{E}_{\nu_o}\left[F(\gamma) \int_0^{t_0} \left(h_m'(t,\gamma) + \frac{1}{2}\mathrm{Ric}_{U_t(\gamma)} h_m(t,\gamma)\right) d\omega_t\right]$$

(d) Finally, by the approximation procedure, it is not difficulty to show that $(\mathcal{E}_0, \mathcal{F}C_b)$ is closable.

General quasi-regular D-F

Let d_M be the Riemannian distance on M, and define

$$\rho(\gamma) := \sup_{t\in[0,1]} d_M(\gamma(t), o).$$

and

$$\mathcal{F}C_{b,loc} := \left\{ Fl(\rho) : F \in \mathcal{F}C_b, \quad l \in C_0^\infty(\mathbb{R}) \right\}$$

be the collection of "local" bounded Lipschitz continuous cylinder functions.

Let $\mathbf{B} : L(L_o(M) \times \mathbb{H}_0) \to \mathbb{H}_0$ be a measurable operator.

- (**A**1) For a.s. $\gamma \in L_o(M), \mathbf{B}(\gamma) : \mathbb{H}_0 \to \mathbb{H}_0$ densely defined self-adjoint with the domain $\mathcal{D}(\mathbf{B}(\gamma))$
- (**A**2) $\forall \ F \in \mathcal{F}C_{b,loc}, DF(\gamma) \in \mathcal{D}(\mathbf{B}(\gamma)^{1/2})$ a.e. and

$$\int_{L_o(M)} |\mathbf{B}(\gamma)^{1/2}(DF(\gamma))|^2_{\mathbb{H}_0} d\nu_o < \infty,$$

- (**A**3) For every $R > 0, \exists \varepsilon(R) > 0$ s.t.

$$\mathbf{B}(\gamma) \geqslant \varepsilon(R)\mathbf{Id}, \quad \gamma \subset B_R.$$

Define:

$$\mathcal{E}_{\mathbf{B}}(F,G)=\int_{L_o(M)}\langle \mathbf{B}^{1/2}D^0F,\mathbf{B}^{1/2}D^0G\rangle_{\mathbb{H}_0}\mathrm{d}\nu_o, F,G\in\mathcal{F}C_{b,loc}.$$

Theorem B

Assume that M is a complete and stochastic complete connected Riemannian manifold and $\mathbf{B}$ satisfies with $(\mathbf{A}1),(\mathbf{A}2)$ and $(\mathbf{A}3)$. Then the quadratic form $(\mathcal{E}_{\mathbf{B}},\mathcal{F}C_{b,loc})$ is closable on $L^2(\nu_o)$, and its closure $(\mathcal{E}_{\mathbf{B}},\mathcal{D}(\mathcal{E}_{\mathbf{B}}))$ is a quasi-regular Dirichlet form on $L^2(\nu_o)$.

Main references

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The End

Thank you for your attention!