On A Class of Skew Diffusion Processes

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Outline

A Survey

- Skew Brownian motion was first introduced in Ito & McKean (1965)
- One construction was by J. B. Walsh (1978)
- The SDE expression was established by J. M. Harrison & L. A. Shepp (1981)
- As a solution of generalized SDE with local time, see J. F. Le Gall (1984)
- Multi-skewed Brownian motion studied by J. M. Ramirez (2011)

Other References

- Singular Drift Diffusions
  M. Barlow (1988), Barlow-Pitman-Yor(1989), Bass & Chen (2002,2003,2005)
- Population Biology
  R. S. Cantrell & C. Cosner (1999)
- Physics
  M. Zhang (2000), A. Lejay (2003) and J. M. Ramirez (2011)
- Financial Mathematics
  M. Decamps & M. Goovaerts & W. Schoutens (2006)
- A Survey Paper
  A. Lejay(2006).

The Talk Topics

- Skew Brownian Motion
- Skew O-U Processes
- Skew Feller's branching Processes

Outline

A Discrete Approximation

Let $\{S_0, S_1, \cdots\}$ be a Markov chain on the integers with $S_0 = 0$ and transition probabilities

$$\mathbb{P}(S_{k+1} = S_k + 1 | S_0, \cdots, S_k) = \begin{cases} \alpha & \text{if } S_k = 0, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

$$\mathbb{P}(S_{k+1} = S_k - 1 | S_0, \cdots, S_k) = \begin{cases} 1 - \alpha & \text{if } S_k = 0, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

For fixed $t > 0$, $n^{-1/2} S_{[nt]}$ converges in finite dimensional distribution to α-*skew* Brownian motion as $n \uparrow \infty$ (also converges weakly). [*Harrison* & *Shepp* (1981)].

Ito & Mckean's Construction

Let $\{W_t, t \geq 0\}$ be a standard B.M. and $J_1, J_2, \cdots$ denote the excursion intervals of the reflected process $\{|W_t|, t \geq 0\}$.

For a given $\alpha \in (0,1)$, let $\{A_m^{(\alpha)} : m = 0, 1, \cdots\}$ be a sequence of i.i.d. Bernoulli r.v.s with $\mathbb{P}(A_m^{(\alpha)} = 1) = \alpha$.

Define an *α-skew* Brownian motion X_t^α started at 0 via

$$X_t^\alpha = \sum_{m=1}^{\infty} \mathbf{1}_{J_m}(t) A_m^{(\alpha)} |W_t|,$$

where $\mathbf{1}_A$ denotes the indicator function of the set A.

See also [*Appuhamillage & Sheldon* (2011)].

Ito & *Mckean's* Construction

For a α-*skew* Brownian motion X_t^α, as a diffusion processe, the scale $S(x)$ and speed $m(x)$ functions can be defined, respectively by

$$S(x) = \begin{cases} \frac{1}{1-\alpha}x & x \leq 0, \\ \frac{1}{\alpha}x & x \geq 0. \end{cases}$$

$$m(x) = \begin{cases} 2(1-\alpha)x & x \leq 0, \\ 2\alpha x & x \geq 0. \end{cases}$$

[*Ito* and *McKean* (1965)].

Walsh's Construction

Now let $r(x)$ be the inverse function of $S(x)$

$$r(x) = \begin{cases} (1-\alpha)x & x \leq 0, \\ \alpha x & x \geq 0, \end{cases}$$

and define

$$A_t = \int_0^t \frac{1}{\sigma(W_s)} \mathrm{d}W_s,$$

where $\sigma(y) = \frac{1}{\alpha}\mathbf{1}_{\{y \geq 0\}} + \frac{1}{1-\alpha}\mathbf{1}_{\{y<0\}}$.

Walsh's Construction

Next define a stopping time T_t, by

$$T_t = \inf\{s \geq 0, \langle A \rangle_s > t\},$$

for $0 \leq t < \infty$.
If set $X_t^\alpha = r(W_{T_t})$. Then X_t^α is actually a α-*skew* Brownian motion with parameter α. [*Walsh* (1978)]

Harrison & *Shepp's* SDE

Define

$$Y_t = \int_0^t \sigma(Y_s)\mathrm{d}W_s,$$

Then $Y_t = W_{T_t}$. Set $X_t^\alpha = r(Y_t)$. Apply a generalized *Ito* formula to $r(Y_t)$, we may have

$$\mathrm{d}r(Y_t) = \mathrm{d}W_t + \frac{2\alpha - 1}{2}\mathrm{d}\hat{L}_t^Y(0)$$

Note that, $\hat{L}_t^{X^\alpha}(0) = \frac{1}{2}\mathrm{d}\hat{L}_t^Y(0)$. Then X_t^α satisfy a SDE as follows.

$$\mathrm{d}X_t^\alpha = \mathrm{d}W_t + (2\alpha - 1)\mathrm{d}\hat{L}_t^{X^\alpha}(0).$$

[*Harrison* & *Shepp* (1978)].

Outline

Skew OU Processes

Define a Skew OU processes X_t^1 to be the unique solution of SDE:

$$\mathrm{d}X_t^1 = (\mu - aX_t^1)\mathrm{d}t + \sigma \mathrm{d}W_t^1 + \beta \mathrm{d}\hat{L}_t^{X^1}(0), \tag{1}$$

where $|\beta| \le 1, a > 0, \sigma > 0, \mu \in \mathbb{R}$. W_t^1 is a standard B.M. and $\hat{L}_t^{X^1}(0)$ is the symmetric local time of X^1 at point 0. We certainly have to check the existence and particularly the uniqueness under the condition $|\beta| \le 1$. [*Song* – *Wang* – *Wang* (2014)]

A Transform

Define

$$G_1(x) = \begin{cases} \frac{(1+\beta)x}{2}, & x < 0, \\ \frac{(1-\beta)x}{2}, & x \geq 0, \end{cases}$$

and set

$$\theta_1 = \frac{1+\beta}{2a}\mu, \quad \theta_2 = \frac{1-\beta}{2a}\mu,$$

A Transform

and

$$\sigma_1 = \frac{1+\beta}{2}\sigma, \qquad \sigma_2 = \frac{1-\beta}{2}\sigma.$$

Then $Y_t^1 = G_1(X_t^1)$ has the following form:

$$\mathrm{d}Y_t^1 = \begin{cases} a\left(\theta_1 - Y_t^1\right)\mathrm{d}t + \sigma_1 \mathrm{d}W_t^1, & \textit{if} \quad Y_t^1 < 0, \\ a\left(\theta_2 - Y_t^1\right)\mathrm{d}t + \sigma_2 \mathrm{d}W_t^1, & \textit{if} \quad Y_t^1 \geq 0. \end{cases} \tag{2}$$

Transition density of Skew OU

Proposition

The transition density $p_1(t; x_0, x)$ of Skew OU process X_t^1 with initial state x_0 can be given by

$$p_1(t; x_0, x) = G_1'(x) m_1(G_1(x)) \sum_{n=1}^{+\infty} e^{-\lambda_n t} \varphi_n(G_1(x_0)) \varphi_n(G_1(x)), \quad x \neq 0,$$

where m_1 is the speed function of process Y_t^1, and

Transition density

Proposition

the eigenvalues $0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ *as* $n \uparrow \infty$ *are the discrete zeros of the Wronskian equation*

$$\omega(\lambda) = e^{\varrho} 2^{1-v} v \sqrt{a} \left[\frac{H_v(-\frac{\alpha_1}{\sqrt{2}}) H_{v-1}(\frac{\alpha_2}{\sqrt{2}})}{\sigma_2} + \frac{H_{v-1}(-\frac{\alpha_1}{\sqrt{2}}) H_v(\frac{\alpha_2}{\sqrt{2}})}{\sigma_1} \right],$$

where $\alpha_1 = -\frac{\sqrt{2a}}{\sigma_1}\theta_1, \quad \alpha_2 = -\frac{\sqrt{2a}}{\sigma_2}\theta_2, \quad v = \frac{\lambda}{a}, \quad \varrho = -\frac{\mu^2}{a\sigma^2},$

Transition density

Proposition

and the normalized eigenfunctions

$$\varphi_n(x) = \begin{cases} \sqrt{\dfrac{\eta(0,\lambda_n)}{\omega'(\lambda_n)\xi(0,\lambda_n)}}\xi(x,\lambda_n), & x < 0, \\ sign(\xi(0,\lambda_n)\eta(0,\lambda_n))\sqrt{\dfrac{\xi(0,\lambda_n)}{\omega'(\lambda_n)\eta(0,\lambda_n)}}\eta(x,\lambda_n), & x \geq 0, \end{cases}$$

with $\xi(x,\lambda) = e^{z_1^2/4}D_v(-z_1)$, $\eta(x,\lambda) = e^{z_2^2/4}D_v(z_2)$, *for* $z_1 = \frac{\sqrt{2a}}{\sigma_1}(x-\theta_1)$, $z_2 = \frac{\sqrt{2a}}{\sigma_2}(x-\theta_2)$. $D_v(z)$ *and* $H_v(z)$ *are parabolic cylinder and Hermite function, respectively .*

Hitting time of Skew OU

For fixed $x_0 < x$, define the first hitting time up $\mathcal{T}^1_{x_0\uparrow x}$ of X^1_t by

$$\mathcal{T}^1_{x_0\uparrow x} = \inf\{t \geq 0; X^1_t = x\},$$

and respectively, the first hitting time up $\hat{T}^1_{y_0\uparrow y}$ of Y^1_t by

$$\hat{T}^1_{y_0\uparrow y} = \inf\{t \geq 0; Y^1_t = y\}.$$

Due to the relations between X^1_t and Y^1_t, we have

$$\mathbb{P}(\mathcal{T}^1_{x_0\uparrow x} \leq t) = \mathbb{P}(\hat{T}^1_{G_1(x_0)\uparrow G_1(x)} \leq t).$$

Hitting time of Skew OU

Proposition

If $G_1(x) > 0$, *the spectral expansion of* $\mathbb{P}_{x_0}(\mathcal{T}_{x_0\uparrow x} \leq t)$ *has a form as:*

$$\mathbb{P}_{x_0}(\mathcal{T}^1_{x_0\uparrow x} \leq t) = 1 - \sum_{n=1}^{\infty} c_{n,x} e^{-\lambda_{n,x} t} \varphi_{n,x}(G_1(x_0)), \tag{3}$$

where $0 \leq \lambda_{1,x} < \lambda_{2,x} < \cdots$ *are eigenvalues, and* $\varphi_{n,x}(x)$ *are eigenfunctions. Moreover,*

Hitting time of Skew OU

Proposition

(1) The eigenvalues $0 \le \lambda_{1,x} < \lambda_{2,x} < \cdots$ *are the zeros of the Wronskian equation*

$$\omega(\lambda) = \exp\left(\varrho + \frac{\alpha_1^2 + \alpha_2^2 - \beta_2^2}{4}\right) 2^{-\frac{v}{2}-2} \Bigg[\frac{\sqrt{2}\sigma_2}{\sigma_1} v D_{v-1}(-\alpha_1) \left(E_v^{(1)}(\beta_2)E_v^{(0)}(\alpha_2) - E_v^{(0)}(\beta_2)E_v^{(1)}(\alpha_2)\right) - D_v(-\alpha_1)\left(vE_v^{(1)}(\beta_2)E_{v-1}^{(1)}(\alpha_2) + 2E_v^{(0)}(\beta_2)E_{v-1}^{(0)}(\alpha_2)\right)\Bigg],$$

(to be continued)

Hitting time of Skew OU

Proposition

(2) The eigenfunctions $\varphi_{n,x}(x)$ satisfy:

$$\varphi_{n,x}(x) = \begin{cases} \sqrt{\dfrac{\eta(0,\lambda_{n,x})}{\omega'(\lambda_{n,x})\xi(0,\lambda_{n,x})}}\,\xi(x,\lambda_{n,x}), & x < 0, \\ C_1\sqrt{\dfrac{\xi(0,\lambda_{n,x})}{\omega'(\lambda_{n,x})\eta(0,\lambda_{n,x})}}\,\eta(x,\lambda_{n,x}), & 0 \le x < G_1(x), \end{cases}$$

where $C_1 = sign(\xi(0,\lambda_{n,x})\eta(0,\lambda_{n,x}))$, *(to be continued)*

Hitting Time

Proposition

$$\beta_1 = \frac{\sqrt{2a}}{\sigma_1}(G_1(x) - \theta_1), \;\; \beta_2 = \frac{\sqrt{2a}}{\sigma_2}(G_1(x) - \theta_2),$$

$$\xi(x,\lambda) = e^{\frac{z_1^2}{4}} D_v(-z_1),$$

$$\eta(x,\lambda) = 2^{-\frac{v}{2}-2} e^{\frac{z_2^2-\beta_2^2}{4}} \frac{\sigma_2}{\sqrt{a}} \left(E_v^{(1)}(\beta_2) E_v^{(0)}(z_2) - E_v^{(0)}(\beta_2) E_v^{(1)}(z_2) \right),$$

and the coefficients $c_{n,x} = -\frac{\varphi'_{n,x}(G_1(x))}{\lambda_{n,x}\varsigma_1(G_1(x))}$, $D_v(z)$, $E_v^{(0)}(z)$, $E_v^{(1)}(z)$ *are the parabolic cylinder functions.*

Hitting Time

Proposition

If $G_1(x) \leq 0$, the spectral expansion of $\mathbb{P}_{x_0}(\mathcal{T}^1_{x_0\uparrow x} \leq t)$ also takes the same form as the case of $G_1(x) > 0$, but the Wronskian $\omega(\lambda)$ should be changed to be:

$$\omega(\lambda) = -\frac{e^{\frac{\beta_1^2}{4}} D_v(-\beta_1)}{\varsigma_1(G_1(x))}.$$

Hitting Time

Proof

To compute the first hitting time up, we mainly use the spectral expansion and the properties of special functions.
[Song – Wang – Wang (2014)].

Remark

On the other hand, similarly we are able to compute the distribution of first hitting time down of X_t^1 as to compute the first hitting time up.
[Song – Wang – Wang (2014)].

Laplace Transform

Proposition

The Laplace transforms of $\mathcal{T}^1_{x_0\uparrow x} = \inf\{t \geq 0;\ X^1_t = x\}$ *and* $\mathcal{T}^1_{x_0\downarrow x} = \inf\{t \geq 0;\ X^1_t = x\}$ *are given by*

$$\mathbb{E}_{x_0}\left(e^{-\vartheta \mathcal{T}^1_{x_0\uparrow x}}\right) = \frac{I_\vartheta(G_1(x_0))}{I_\vartheta(G_1(x))},$$

$$\mathbb{E}_{x_0}\left(e^{-\vartheta \mathcal{T}^1_{x_0\downarrow x}}\right) = \frac{D_\vartheta(G_1(x_0))}{D_\vartheta(G_1(x))},$$

(to be continued)

Laplace Transform

Proposition

where the increasing function $I_\vartheta(x)$ satisfies:

$$I_\vartheta(x) = \begin{cases} H_{-\chi}\left(-\frac{z_1}{\sqrt{2}}\right), & x \leq 0, \\ c_1 H_{-\chi}\left(-\frac{z_2}{\sqrt{2}}\right) + c_2 H_{-\chi}\left(\frac{z_2}{\sqrt{2}}\right), & x > 0, \end{cases} \tag{4}$$

(to be continued)

Laplace Transform

Proposition

and the decreasing function $D_\vartheta(x)$ satisfies:

$$D_\vartheta(x) = \begin{cases} c_3 H_{-\chi}\left(-\frac{z_1}{\sqrt{2}}\right) + c_4 H_{-\chi}\left(\frac{z_1}{\sqrt{2}}\right), & y \le 0, \\ H_{-\chi}\left(\frac{z_2}{\sqrt{2}}\right), & y > 0, \end{cases} \tag{5}$$

where $\chi = \vartheta/a$, $z_1 = \frac{\sqrt{2a}}{\sigma_1}(x-\theta_1)$, $z_2 = \frac{\sqrt{2a}}{\sigma_2}(x-\theta_2)$, *$H_v(z)$ is Hermite function, (to be continued)*

Laplace transform

Proposition

and the coefficients $c_i, i = 1, 2, 3, 4$ satisfy

$$c_1 = \frac{\sigma_1 H_{-\chi}\left(-\frac{\alpha_1}{\sqrt{2}}\right) H_{-\chi-1}\left(\frac{\alpha_2}{\sqrt{2}}\right) + \sigma_2 H_{-\chi}\left(\frac{\alpha_2}{\sqrt{2}}\right) H_{-\chi-1}\left(-\frac{\alpha_1}{\sqrt{2}}\right)}{\sigma_1 H_{-\chi}\left(\frac{\alpha_2}{\sqrt{2}}\right) H_{-\chi-1}\left(-\frac{\alpha_2}{\sqrt{2}}\right) + \sigma_1 H_{-\chi}\left(-\frac{\alpha_2}{\sqrt{2}}\right) H_{-\chi-1}\left(\frac{\alpha_2}{\sqrt{2}}\right)},$$

$$c_2 = \frac{\sigma_1 H_{-\chi}\left(-\frac{\alpha_1}{\sqrt{2}}\right) H_{-\chi-1}\left(-\frac{\alpha_2}{\sqrt{2}}\right) - \sigma_2 H_{-\chi}\left(-\frac{\alpha_2}{\sqrt{2}}\right) H_{-\chi-1}\left(-\frac{\alpha_1}{\sqrt{2}}\right)}{\sigma_1 H_{-\chi}\left(\frac{\alpha_2}{\sqrt{2}}\right) H_{-\chi-1}\left(-\frac{\alpha_2}{\sqrt{2}}\right) + \sigma_1 H_{-\chi}\left(-\frac{\alpha_2}{\sqrt{2}}\right) H_{-\chi-1}\left(\frac{\alpha_2}{\sqrt{2}}\right)},$$

(to be continued)

Laplace transform

Proposition

$$c_3 = \frac{\sigma_2 H_{-\chi}\left(\frac{\alpha_2}{\sqrt{2}}\right) H_{-\chi-1}\left(\frac{\alpha_1}{\sqrt{2}}\right) - \sigma_1 H_{-\chi}\left(\frac{\alpha_1}{\sqrt{2}}\right) H_{-\chi-1}\left(\frac{\alpha_2}{\sqrt{2}}\right)}{\sigma_2 H_{-\chi}\left(-\frac{\alpha_1}{\sqrt{2}}\right) H_{-\chi-1}\left(\frac{\alpha_1}{\sqrt{2}}\right) + \sigma_2 H_{-\chi}\left(\frac{\alpha_1}{\sqrt{2}}\right) H_{-\chi-1}\left(-\frac{\alpha_1}{\sqrt{2}}\right)},$$

$$c_4 = \frac{\sigma_2 H_{-\chi}\left(\frac{\alpha_2}{\sqrt{2}}\right) H_{-\chi-1}\left(-\frac{\alpha_1}{\sqrt{2}}\right) + \sigma_1 H_{-\chi}\left(-\frac{\alpha_1}{\sqrt{2}}\right) H_{-\chi-1}\left(\frac{\alpha_2}{\sqrt{2}}\right)}{\sigma_2 H_{-\chi}\left(-\frac{\alpha_1}{\sqrt{2}}\right) H_{-\chi-1}\left(\frac{\alpha_1}{\sqrt{2}}\right) + \sigma_2 H_{-\chi}\left(\frac{\alpha_1}{\sqrt{2}}\right) H_{-\chi-1}\left(-\frac{\alpha_1}{\sqrt{2}}\right)}.$$

Outline

Definition

Skew Feller's branching processes X_t^2 is the unique solution of the following SDE:

$$\mathrm{d}X_t^2 = \tilde{\mu}(\tilde{a} - X_t^2)\mathrm{d}t + \tilde{\sigma}\sqrt{X_t^2}\mathrm{d}W_t^2 + \tilde{\beta}\mathrm{d}\hat{L}_t^{X^2}(b), \tag{6}$$

where $|\tilde{\beta}| \leq 1, b > 0, \tilde{a} > 0, \tilde{\mu}, \tilde{\sigma} > 0$, W_t^2 is a standard B.M. and $\hat{L}_t^{X^2}(b)$ is the symmetric local time of X^2 at point b. [Trutnau (2010, 2011)].

A Transform

Define

$$G_2(x) = \begin{cases} \frac{1+\tilde{\beta}}{2}(x-b)+b, & 0 < x < b, \\ \frac{1-\tilde{\beta}}{2}(x-b)+b, & x \geq b, \end{cases}$$

and

$$\tilde{\theta}_1 = \frac{(1-\tilde{\beta})\tilde{a}}{2} + \frac{(1+\tilde{\beta})b}{2}, \quad \tilde{\sigma}_1 = \sigma\sqrt{\frac{1-\tilde{\beta}}{2}}, \tilde{l}_1 = \frac{(1+\tilde{\beta})b}{2}$$

$$\tilde{\theta}_2 = \frac{(1+\tilde{\beta})\tilde{a}}{2} + \frac{(1-\tilde{\beta})b}{2}, \quad \tilde{\sigma}_2 = \sigma\sqrt{\frac{1+\tilde{\beta}}{2}}, \tilde{l}_2 = \frac{(1-\tilde{\beta})b}{2}.$$

A Transform

The transformed process $Y_t^2 = G_2(X_t^2)$ has the following form:

$$\mathrm{d}Y_t^2 = \begin{cases} \tilde{\mu}\left(\tilde{\theta}_2 - Y_t^2\right)\mathrm{d}t + \tilde{\sigma}_2\sqrt{Y_t^2 - \tilde{l}_2}\,\mathrm{d}W_t^2, & \tilde{l}_2 < Y_t^2 < b, \\ \tilde{\mu}\left(\tilde{\theta}_1 - Y_t^2\right)\mathrm{d}t + \tilde{\sigma}_1\sqrt{Y_t^2 - \tilde{l}_1}\,\mathrm{d}W_t^2, & Y_t^2 \geq b. \end{cases} \tag{7}$$

Transition density

Proposition

The transition density $p_2(t; x_0, x)$ of Skew Feller's branching process X_t^2 with initial state x_0 can be given by

$$p_2(t; x_0, x) = G_2'(x) m_2(G_2(x)) \sum_{n=1}^{+\infty} e^{-\lambda_n t} \varphi_n(G_2(x_0)) \varphi_n(G_2(x)), \quad x \neq b$$

where m_2 is the speed function of process Y_t^2. (to be continued)

Transition density

Proposition

and the eigenvalues $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ *as* $n \uparrow \infty$ *are the simple discrete zeros of the Wronskian equation*

$$\omega(\lambda) = -\frac{1}{\varsigma_2(b)}\Big[\frac{2\tilde{\mu}\alpha}{\tilde{\sigma}_1^2}F(\alpha,\pi,\Lambda_2)U(\alpha+1,\pi+1,\Lambda_1) + \frac{2\tilde{\mu}\alpha}{\tilde{\sigma}_2^2\pi}U(\alpha,\pi,\Lambda_1)F(\alpha+1,\pi+1,\Lambda_2)\Big],$$

where $\pi = \frac{2\tilde{\mu}\tilde{a}}{\tilde{\sigma}^2}$, $\alpha = -\frac{\lambda}{\tilde{\mu}}$, $\Lambda_1 = \frac{2\tilde{\mu}(b-\tilde{l}_1)}{\tilde{\sigma}_1^2}$, $\Lambda_2 = \frac{2\tilde{\mu}(b-\tilde{l}_2)}{\tilde{\sigma}_2^2}$, ς_2 *is the scale function of* Y_t^2, *(to be continued)*

Transition density

Proposition

and the normalized eigenfunctions follow

$$\varphi_n(x) = \begin{cases} \sqrt{\frac{\psi(b,\lambda_n)}{\omega'(\lambda_n)\phi(b,\lambda_n)}}\phi(x,\lambda_n) & \tilde{l}_2 < x < b, \\ sign(\phi(b,\lambda_n)\psi(b,\lambda_n))\sqrt{\frac{\phi(b,\lambda_n)}{\omega'(\lambda_n)\psi(b,\lambda_n)}}\psi(x,\lambda_n) & x \geq b, \end{cases}$$

where $\psi(x,\lambda) = U(\alpha,\pi,\tilde{z}_1)$, $\phi(x,\lambda) = F(\alpha,\pi,\tilde{z}_2)$ *and* $\tilde{z}_1 = \frac{2\tilde{\mu}(x-\tilde{l}_1)}{\tilde{\sigma}_1^2}$, $\tilde{z}_2 = \frac{2\tilde{\mu}(x-\tilde{l}_2)}{\tilde{\sigma}_2^2}$. $U(\alpha,\pi,z)$ *and* $F(\alpha,\pi,z)$ *are the Tricomi and Kummer function.*

Hitting time

For fixed $x_0 < x$, define the first hitting time up $\mathcal{T}^2_{x_0\uparrow x}$ of X^2_t by

$$\mathcal{T}^2_{x_0\uparrow x} = \inf\{t \geq 0; X^2_t = x\},$$

and the first hitting time up $\hat{T}^2_{y_0\uparrow y}$ of Y^2_t by

$$\hat{T}^2_{y_0\uparrow y} = \inf\{t \geq 0; Y^2_t = y\}.$$

Due to the relationship of X^2_t and Y^2_t, we have

$$\mathbb{P}(\mathcal{T}^2_{x_0\uparrow x} \leq t) = \mathbb{P}(\hat{T}^2_{G_2(x_0)\uparrow G_2(x)} \leq t).$$

Hitting time

Proposition

If $G_2(x) > b$, the spectral expansion of the distribution $\mathbb{P}_{x_0}(\mathcal{T}^2_{x_0\uparrow x} \leq t)$ takes the form:

$$\mathbb{P}_{x_0}(\mathcal{T}^2_{x_0\uparrow x} \leq t) = 1 - \sum_{n=1}^{\infty} c_{n,x} e^{-\lambda_{n,x} t} \varphi_{n,x}(G_2(x_0)), \tag{8}$$

where

Hitting time

Proposition

(1) the eigenvalues $0 \le \lambda_{1,x} < \lambda_{2,x} < \cdots$ *are the zero points of the Wronskian equation*

$$\omega(\lambda) = -\frac{1}{\varsigma_2(b)\Gamma_1}\{F(\alpha,\pi,\Lambda_2)[F(\alpha+1,\pi+1,\Lambda_1)U(\alpha,\pi,\nu_1) + \pi F(\alpha,\pi,\nu_1)U(\alpha+1,\pi+1,\Lambda_1)] + \frac{\tilde{\sigma}_1^2}{\tilde{\sigma}_2^2}F(\alpha+1,\pi+1,\Lambda_2) [F(\alpha,\pi,\nu_1)U(\alpha,\pi,\Lambda_1) - F(\alpha,\pi,\Lambda_1)U(\alpha,\pi,\nu_1)]\},$$

and

Hitting time

Proposition

$\Gamma_1 = F(\alpha+1, \pi+1, \nu_1)U(\alpha, \pi, \nu_1) + \beta F(\alpha, \pi, \nu_1)U(\alpha+1, \pi+1, \nu_1)$,
with

$$\nu_1 = \frac{2\tilde{\mu}(G_2(x) - \tilde{l}_1)}{\tilde{\sigma}_1^2}, \quad \nu_2 = \frac{2\tilde{\mu}(G_2(x) - \tilde{l}_2)}{\tilde{\sigma}_2^2}.$$

(to be continued)

Hitting time

Proposition

(2) the eigenfunctions $\varphi_{n,x}(x)$ satisfy:

$$\varphi_{n,x}(x) = \begin{cases} \sqrt{\frac{\psi(b,\lambda_{n,x})}{\omega'(\lambda_{n,x})\phi(b,\lambda_{n,x})}}\phi(x,\lambda_{n,x}) & \tilde{l}_2 < y < b, \\ C_2\sqrt{\frac{\phi(b,\lambda_{n,x})}{\omega'(\lambda_{n,x})\psi(b,\lambda_{n,x})}}\psi(x,\lambda_{n,x}) & b \leq x \leq G_2(x), \end{cases}$$

where

$$C_2 = sign(\psi(b,\lambda_{n,x})\phi(b,\lambda_{n,x})), \quad \phi(x,\lambda) = F(\alpha,\pi,\tilde{z}_2),$$

$$\psi(x,\lambda) = \frac{\tilde{\sigma}_1^2}{2\tilde{\mu}\alpha}\frac{\pi}{\Gamma_1}[F(\alpha,\pi,\nu_1)U(\alpha,\pi,\tilde{z}_1) - F(\alpha,\pi,\tilde{z}_1)U(\alpha,\pi,\nu_1)],$$

(to be continued)

Hitting time

Proposition

and the coefficients $c_{n,x}$ satisfy

$$c_{n,x} = -\frac{\varphi'_{n,x}(G_2(x))}{\lambda_{n,x}\varsigma_2(G_2(x))}, \quad \varphi'_{n,x}(G_2(x)) = \frac{\mathrm{d}\varphi_{n,x}(x)}{dx}\Big|_{x\uparrow G_2(x)}$$

(to be continued)

Hitting time

Proposition

If $G_2(x) \leq b$, the spectral expansion of $\mathbb{P}_{x_0}(\mathcal{T}^2_{x_0 \uparrow x} \leq t)$ also takes the same form as the case of $G_2(x) > b$, but the Wronskian $\omega(\lambda)$ equation should be changed to:

$$\omega(\lambda) = -\frac{F(\alpha, \pi, \nu_2)}{(G_2(x) - \tilde{l}_2)^{-\pi} e^{\nu_2}}.$$

Hitting time

Proof

To compute the first hitting time up, we mainly use the spectral expansion and the properties of special functions. [Song – Xu – Wang (2014)].

Remark

On the other hand, similarly to compute the distribution of first hitting time down fo X_t^2 as to compute the first hitting time up. [Song – Xu – Wang (2014)].

Laplace transform

Proposition

The Laplace transforms of $\mathcal{T}^2_{x_0\uparrow x} = \inf\{t \geq 0;\ X^2_t = x\}$ *and* $\mathcal{T}^2_{x_0\downarrow x} = \inf\{t \geq 0;\ X^2_t = x\}$ *are given by*

$$E_{x_0}\left(e^{-\vartheta \mathcal{T}^2_{x_0\uparrow x}}\right) = \frac{\tilde{I}_\vartheta(G_2(x_0))}{\tilde{I}_\vartheta(G_2(x))},$$

$$E_{x_0}\left(e^{-\vartheta \mathcal{T}^2_{x_0\downarrow x}}\right) = \frac{\tilde{D}_\vartheta(G_2(x_0))}{\tilde{D}_\vartheta(G_2(x))},$$

where

Laplace transform

Proposition

(1) the increasing function $\tilde{l}_\vartheta(x)$ is determined by

$$\tilde{l}_\vartheta(x) = \begin{cases} F(\frac{\vartheta}{\tilde{\mu}}, \pi, \tilde{z}_2) & x \leq b, \\ \tilde{c}_1 F(\frac{\vartheta}{\tilde{\mu}}, \pi, \tilde{z}_1) + \tilde{c}_2 U(\frac{\vartheta}{\tilde{\mu}}, \pi, \tilde{z}_1) & x > b, \end{cases} \quad (9)$$

and

Laplace transform

Proposition

the decreasing function $\tilde{D}_\vartheta(x)$ is defined by

$$\tilde{D}_\vartheta(x) = \begin{cases} \tilde{c}_3 F(\frac{\vartheta}{\tilde{\mu}}, \pi, \tilde{z}_2) + \tilde{c}_4 U(\frac{\vartheta}{\tilde{\mu}}, \pi, \tilde{z}_2) & x < b, \\ U(\frac{\vartheta}{\tilde{\mu}}, \pi, \tilde{z}_1) & x \geq b. \end{cases}$$

(to be continued)

Laplace transform

Proposition

(2) Set

$$\Xi_1 = \tilde{\sigma}_2^2 F(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_1) U(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_1)$$
$$+ \tilde{\sigma}_2^2 \pi F(\frac{\vartheta}{k}, \pi, \Lambda_1) U(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_1),$$
$$\Xi_2 = \tilde{\sigma}_1^2 F(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_2) U(\frac{\vartheta}{k}, \pi, \Lambda_2)$$
$$+ \tilde{\sigma}_1^2 \pi F(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_2) U(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_2),$$

(to be continued)

Laplace transform

Proposition

and the coefficients $\tilde{c}_i, i = 1, 2, 3, 4$, *are determined, respectively by*

$$\tilde{c}_1 = \frac{1}{\Xi_1}[\tilde{\sigma}_1^2 F(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_2) U(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_1)$$
$$+ \tilde{\sigma}_2^2 \pi F(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_2) U(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_1)],$$
$$\tilde{c}_2 = \frac{1}{\Xi_1}[\tilde{\sigma}_2^2 F(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_1) F(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_2)$$
$$- \tilde{\sigma}_1^2 F(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_1) F(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_2)],$$

Laplace transform

Proposition

$$\tilde{c}_3 = \frac{1}{\Xi_2}[\tilde{\sigma}_1^2 \pi U(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_1) U(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_2)$$
$$- \tilde{\sigma}_2^2 \pi U(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_1) U(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_2)],$$
$$\tilde{c}_4 = \frac{1}{\Xi_2}[\tilde{\sigma}_1^2 F(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_2) U(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_1)$$
$$+ \tilde{\sigma}_2^2 \pi F(\frac{\vartheta}{\tilde{\mu}}, \pi, \Lambda_2) U(\frac{\vartheta}{\tilde{\mu}} + 1, \pi + 1, \Lambda_1)].$$

The End

Thank You!