L^p-Wasserstein Distance for Stochastic Differential Equations

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Outline

1. **Setting**
   - Wasserstein distance
   - Uniformly dissipative condition

2. **Diffusion processes**

3. **SDEs with Lévy jumps**

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Wasserstein distance

Given two probability measures μ and ν on $\mathbb{R}^d$, and $p \in [1, \infty)$,

- $$W_p(\mu, \nu) = \inf_{\Pi \in \mathcal{P}(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, \Pi(dx, dy) \right)^{1/p},$$

  where $\mathcal{P}(\mu, \nu)$ is the collection of measures on $\mathbb{R}^d \times \mathbb{R}^d$ having μ and ν as marginals.

- Let $\xi \sim \mu$ and $\eta \sim \nu$. Then

  $$W_p(\mu, \nu) \leqslant \left(\mathbb{E}|\xi - \eta|^p \right)^{1/p}.$$

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   - Wasserstein distance
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Uniformly dissipative condition

- $$dX_t = b(X_t)\,dt + dB_t,$$

  where

  $$\langle b(x) - b(y), x - y\rangle \leqslant -K|x-y|^2 \quad \text{for all } x, y \in \mathbb{R}^d$$

  with a constant $K > 0$.

- For any $p \geqslant 1$, $t > 0$ and two probability measures μ and ν on $\mathbb{R}^d$,

  $$W_p(\mu P_t, \nu P_t) \leqslant e^{-Kt}\, W_p(\mu, \nu).$$

Proof by coupling of marching soldiers

- $$\begin{aligned} dX_t &= b(X_t)\,dt + dB_t, \quad X_0 = x \\ dY_t &= b(Y_t)\,dt + dB_t, \quad Y_0 = y \end{aligned}$$

- $$d(X_t - Y_t) = (b(X_t) - b(Y_t))\,dt$$

- $$\langle b(x) - b(y), x - y\rangle \leqslant -K|x-y|^2 \quad \underline{\text{for all } x, y \in \mathbb{R}^d}$$
$$\Downarrow$$
$$d|X_t - Y_t|^p \leqslant -pK|X_t - Y_t|^p\,dt, \quad t > 0.$$

Ergodicity: uniformly dissipative condition

- $$\langle b(x) - b(y), x - y\rangle \leqslant -K|x-y|^2 \quad \underline{\text{for all } x, y \in \mathbb{R}^d}$$

- $$\langle b(x), x\rangle \leqslant -K|x|^2 + \langle b(0), x\rangle \leqslant -K|x|^2 + |b(0)||x| \quad \text{for all } x \in \mathbb{R}^d.$$

- $$\langle b(x) - b(y), x - y\rangle \leqslant -K|x-y|^2 \quad \underline{\text{for all } |x-y| > L}$$

  with a constant $L > 0$ large enough.

Question

- For any $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leqslant \begin{cases} K_1 |x-y|^2, & |x-y| \leqslant L; \\ -K_2 |x-y|^2, & |x-y| > L \end{cases}$$

  holds with some positive constants K_1, K_2 and $L > 0$.

- For any $p \geqslant 1$, $t > 0$ and two probability measures μ and ν on $\mathbb{R}^d$,

$$W_p(\mu P_t, \nu P_t) \leqslant e^{-Kt} W_p(\mu, \nu). \quad (???)$$

- (Eberle, 2011 and 2014) There exist constants C and $\lambda > 0$ such that for $t > 0$, and any two probability measures μ and ν on $\mathbb{R}^d$,

$$W_1(\mu P_t, \nu P_t) \leqslant C e^{-\lambda t} W_1(\mu, \nu).$$

Question

- (Eberle, 2011 and 2014)

$$W_1(\mu P_t, \nu P_t) \leqslant Ce^{-\lambda t} W_1(\mu, \nu). \quad (p > 1???)$$

$$W_p(\mu P_t, \nu P_t) \leqslant Ce^{-\lambda t} W_p(\mu, \nu). \quad (???)$$

- Idea: Coupling by reflection.
- (Cattiaux and Guillin, 2014) "The coupling by reflection cannot furnish some information on W_2". (???)

Diffusion processes

$$dX_t = dB_t + b(X_t)\,dt.$$

Theorem (Luo and W., 2014)

Suppose that for any $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y\rangle \leqslant \begin{cases} K_1|x-y|^2, & |x-y| \leqslant L; \\ -K_2|x-y|^{\theta}, & |x-y| > L \end{cases}$$

holds with some positive constants K_1, K_2, $L > 0$ *and* $\theta \geqslant 2$. *Then there is a constant* $\lambda > 0$ *such that for all* $p \in [1, \infty)$, $t > 0$ *and* $x, y \in \mathbb{R}^d$

(1)

$$W_p(\delta_x P_t, \delta_y P_t) \leqslant C(p,\theta) e^{-\lambda t/p} |x-y|^{1/p}, \quad |x-y| \leqslant 1.$$

Diffusion processes

Theorem (Luo and W., 2014)

Suppose that for any $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leqslant \begin{cases} K_1|x-y|^2, & |x-y| \leqslant L; \\ -K_2|x-y|^{\theta}, & |x-y| > L \end{cases}$$

holds with some positive constants K_1, K_2, $L > 0$ *and* $\theta \geqslant 2$. *Then there is a constant* $\lambda > 0$ *such that for all* $p \in [1, \infty)$, $t > 0$ *and* $x, y \in \mathbb{R}^d$

(2) *for* $|x-y| \geqslant 1$,

$$W_p(\delta_x P_t, \delta_y P_t) \leqslant C(p, \theta) e^{-\lambda t/p} \begin{cases} |x-y|, & \theta = 2; \\ |x-y| 1_{\{0<t\leqslant 1\}} + 1_{\{t>1\}}, & \theta > 2. \end{cases}$$

Proof: Coupling by reflection

- Lindvall and Rogers (1986); Chen and Li (1989)

$$dY_t = (I - 2e_t e_t^*)\, dB_t + b(Y_t)\, dt, \quad t < T,$$

where

$$e_t = \frac{\sigma^{-1}(X_t - Y_t)}{|\sigma^{-1}(X_t - Y_t)|}$$

and

$$T = \inf\{t > 0 : X_t = Y_t\}.$$

- The different process $(Z_t)_{t\geqslant 0} = (X_t - Y_t)_{t\geqslant 0}$ satisfies

$$dZ_t = \frac{2Z_t}{|\sigma^{-1}Z_t|}\, dW_t + (b(X_t) - b(Y_t))\, dt, \quad t < T.$$

Proof: Coupling metric functions

- Chen and Wang (1997)

$$\psi(r) = \frac{C}{(1+r)^{2\theta-2}}\left[\exp\left(\frac{\varepsilon}{\theta}(r^\theta \vee r)\right) - 1\right],$$

where $\varepsilon \in (0, K_2)$ and $C := C(K_1, K_2, L) > 0$.

- … … … … … …

SDEs with additive Lévy noises

- $$dX_t = b(X_t)\,dt + dZ_t,$$
  where $(Z_t)_{t\geqslant 0}$ is a d-dimensional Lévy process.
- $(Z_t)_{t\geqslant 0}$ is a symmetric α-stable process with Lévy measure
  $$\frac{C_{d,\alpha}}{|z|^{d+\alpha}}\,dz.$$

SDEs with symmetric stable jumps

Theorem (W., 2014+)

Let $(Z_t)_{t\geqslant 0}$ be a symmetric α-stable process on $\mathbb{R}^d$ with $\alpha \in (1,2)$. Suppose that for any $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leqslant \begin{cases} K_1 |x-y|^2, & |x-y| \leqslant L; \\ -K_2 |x-y|^{\theta}, & |x-y| > L \end{cases}$$

holds with some positive constants K_1, K_2, $L > 0$ and $\theta \geqslant 2$. Then there is a constant $\lambda > 0$ such that for all $p \in [1, \infty)$, $t > 0$ and $x, y \in \mathbb{R}^d$

(1)

$$W_p(\delta_x P_t, \delta_y P_t) \leqslant C(p,\theta) e^{-\lambda t/p} |x-y|^{1/p}, \quad |x-y| \leqslant 1.$$

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(2) *for* $|x - y| \geqslant 1$,

$$W_p(\delta_x P_t, \delta_y P_t) \leqslant C(p, \theta) e^{-\lambda t/p} \begin{cases} |x - y|, & \theta = 2; \\ |x - y| \mathbb{1}_{\{0 < t \leqslant 1\}} + \mathbb{1}_{\{t > 1\}}, & \theta > 2. \end{cases}$$

SDEs with symmetric stable jumps

Theorem (W., 2014+)

Let $(Z_t)_{t\geqslant 0}$ be a symmetric α-stable process on $\mathbb{R}^d$ with $\alpha \in (0,1]$. Suppose that for any $x, y \in \mathbb{R}^d$,

$$\langle b(x)-b(y), x-y\rangle \leqslant \begin{cases} K_1|x-y|^2, & |x-y|\leqslant L; \\ -K_2|x-y|^{\theta}, & |x-y|>L \end{cases}$$

holds with some positive constants K_1, K_2, $L>0$ and $\theta \geqslant 2$.
If furthermore

$$\frac{\alpha C_{d,\alpha}\omega_d 3^{\alpha-1}}{d} > K_1 L^{\alpha},$$

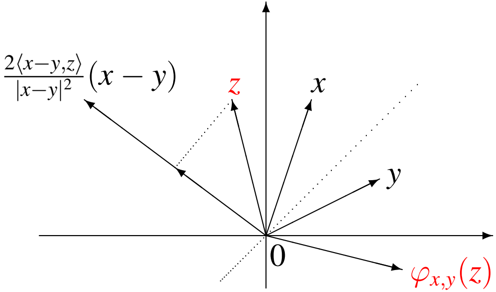
then ... ...

- Bogdan and Jakubowski (2007); Chen, Kim and Song (2012) (Dirichlet) heat kernel estimates for fractional Laplacian with gradient perturbation.
- Wang and W. (2014) Dimensional free Harnack inequality.

Coupling by reflection for symmetric stable jumps

- For any x, y and $z \in \mathbb{R}^d$, set

$$\varphi_{x,y}(z) := \begin{cases} z - \frac{2\langle x-y,z\rangle}{|x-y|^2}(x-y), & x \neq y; \\ -z, & x = y. \end{cases}$$



- $|\varphi_{x,y}(z)| = |z|$ and $(z + \varphi_{x,y}(z)) \perp (x-y)$.

Coupling techniques: coupling operator

- $$\begin{aligned}&\widetilde{L}f(x,y)\\&=\frac{1}{2}\Big[\int_{\{|z|\leqslant a|x-y|\}}\Big(f(x+z,y+\varphi_{x,y}(z))-f(x,y)-\langle\nabla_x f(x,y),z\rangle 1_{\{|z|\leqslant 1\}}\\&\qquad-\langle\nabla_y f(x,y),\varphi_{x,y}(z)\rangle 1_{\{|z|\leqslant 1\}}\Big)\frac{C_{d,\alpha}}{|z|^{d+\alpha}}\,dz\\&\quad+\int_{\{|z|\leqslant a|x-y|\}}\Big(f(x+\varphi_{x,y}(z),y+z)-f(x,y)-\langle\nabla_y f(x,y),z\rangle 1_{\{|z|\leqslant 1\}}\\&\qquad-\langle\nabla_x f(x,y),\varphi_{x,y}(z)\rangle 1_{\{|z|\leqslant 1\}}\Big)\frac{C_{d,\alpha}}{|z|^{d+\alpha}}\,dz\Big]\\&\quad+\ldots\ldots\end{aligned}$$
- $a\in(0,1/2)$. (Why?) Let $x,y,z\in\mathbb{R}$. Coupling operator

$$(x,y)\to(x+z,y-z)$$

$$\text{“}|(x+z)-(y-z)|\leqslant|x-y|\text{”}\Leftarrow\text{“}|z|\leqslant|x-y|/2\text{”}.$$

Coupling by reflection for Lévy jump processes

- Wang (2011); Schilling and W. (2012) ... ...
- W. (2014)

$$dX_t = b(X_t)\,dt + dZ_t.$$

- Luo and W. (2014) Continuity of semigroups for stable-like processes with symbol $p(x,\xi) = |\xi|^{\alpha(x)}$.

Coupling metric functions

- $a = 1/c_1$.
- For $\alpha \in (1,2)$,

$$\psi(r) = \begin{cases} 1 - e^{-c_1 r}, & r \leqslant 2L; \\ Ae^{c_2(r-2L)} + B(r-2L)^2 + \left(1 - e^{-2c_1 L} - A\right), & r \geqslant 2L, \end{cases}$$

where

$$A = \frac{c_1}{c_2} e^{-2Lc_1}, \quad B = -\frac{(c_1+c_2)c_1}{2} e^{-2Lc_1}, \quad c_2 = 20c_1$$

and c_1 is a constant determined by later.

Coupling metric functions

- $a = 1/4$.
- For $\alpha \in (0, 1]$,

$$\psi(r)=\begin{cases} r - cr^{1+\alpha} & r \leqslant 2L; \\ Ae^{c_0(r-2L)} + B(r-2L)^2 + \Big(2L - c(2L)^{1+\alpha} - A\Big), & r \geqslant 2L, \end{cases}$$

where

$$c = \frac{1}{2^{1+\alpha}(1+\alpha)L^\alpha}, \quad A = \frac{1}{2c_0}, \quad B = -\frac{1}{2}\left[\frac{\alpha}{4L} + \frac{c_0}{2}\right],$$

and c_0 is a constant determined by later.

Example

Example

- Let $(Z_t)_{t\geqslant 0}$ be a symmetric α-stable process in $\mathbb{R}^d$ with $\alpha \in (0,2]$.
- $b(x) = \nabla V(x)$ with $V(x) = -|x|^{2\beta}$ and $\beta > 1$.

- There exists a constant $\lambda := \lambda(\alpha,\beta) > 0$ such that for all $p \geqslant 1$, $x, y \in \mathbb{R}^d$ and $t > 0$,

$$W_p(\delta_x P_t, \delta_y P_t) \leqslant C(\alpha,\beta,p)e^{-\lambda t/p}\left[\frac{|x-y|^{1/p} \vee |x-y|}{1+|x-y|\mathbb{1}_{[1,\infty)}(t)}\right].$$

- $$\langle b(x)-b(y), x-y\rangle \leqslant -\beta 2^{4-3\beta}\underline{|x-y|^{2\beta}}.$$

Thank you for your attention!