Birth and death process with bounded jumps in random environment

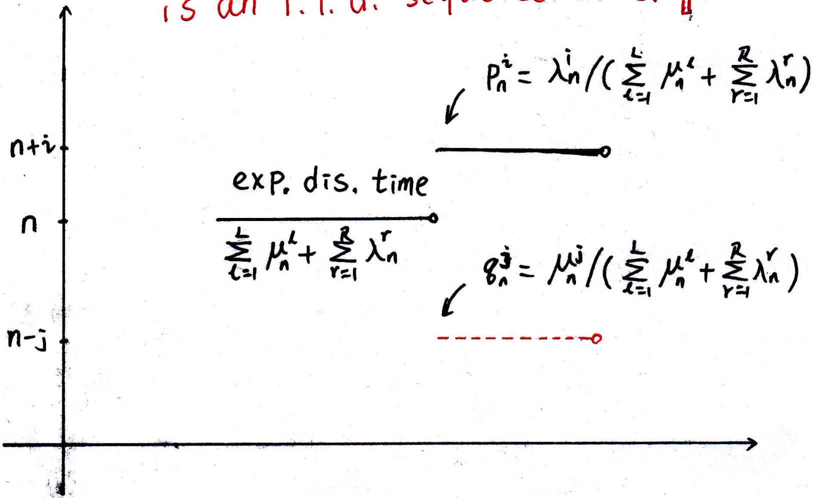
Huaming Wang

Department of Mathematics, Anhui Normal University

The model

$\omega = (\omega_n)_{n\in\mathbb{Z}} = (\mu_n^L, \cdots, \mu_n^1, \lambda_n^1, \cdots, \lambda_n^R)_{n\in\mathbb{Z}}$

is an i.i.d. sequence under $\mathbb{P}$



Space of Random Environment

$L, \ R \geq 1$ are two integers(**jump size**).

Ω : collection of $\omega = (\omega_i)_{i \in \mathbb{Z}} = (\mu_i^L, ..., \mu_i^1, \lambda_i^1, ..., \lambda_i^R)_{i \in \mathbb{Z}}$, $\mu_i^l, \lambda_i^r \geq 0$, $i \in \mathbb{Z}$, $l = 1, .., L$, $r = 1, ..., R$.

$\mathcal{F}$: Borel σ-algebra on Ω.

θ : shift operator on Ω defined by $(\theta\omega)_i = \omega_{i+1}$.

$\mathbb{P}$: a probability measure on $(\Omega, \mathcal{F})$ which is assumed to be i.i.d. or sometimes stationary and ergodic.

Random environment ω is a random element of Ω chosen according to $\mathbb{P}$.

(L,R) BDPRE

Given a realization of ω,
let $\{N_t\}_{t\geq 0}$ be a continuous time Markov chain,
which waits at a state n an exponentially distributed time with parameter $\sum_{l=1}^{L}\mu_n^l+\sum_{r=1}^{R}\lambda_n^r$ and then
jumps to $n-i$ with probability $\mu_n^i/(\sum_{l=1}^{L}\mu_n^l+\sum_{r=1}^{R}\lambda_n^r)$, $i=1,...,L$
or to $n+j$ with probability $\lambda_n^j/(\sum_{l=1}^{L}\mu_n^l+\sum_{r=1}^{R}\lambda_n^r)$, $j=1,...,R$.
$\{N_t\}_{t\geq 0}$ is called a *birth and death process with bounded jumps in random environment* ((L,R) BDPRE in short).

P_ω : **quenched probability;**

P : **annealed probability.**

Background

$\{N_t\}$: continuous time analogue of *random walk with bounded jumps in random environment.*

Key [K84]

Letchikov [L89]

Brémont [B02, B09]

Hong and Zhang [HZ10]

Hong and Wang [HW13, HW14] etc.

Ritter [R80] ($L = R = 1$), recurrence criteria, LLN.

Embedded process and Skeleton process

$\tau_n, n \geq 0$: discontinuities of $\{N_t\}$.

Embedded process and Skeleton process

$\tau_n, n \geq 0$: discontinuities of $\{N_t\}$.

$\chi_n := N_{\tau_n}$.

Embedded process and Skeleton process

$\tau_n, n \geq 0$: discontinuities of $\{N_t\}$.

$\chi_n := N_{\tau_n}$.

$\{\chi_n\}$ is called the embedded process of $\{N_t\}$.

Embedded process and Skeleton process

$\tau_n, n \geq 0$: discontinuities of $\{N_t\}$.

$\chi_n := N_{\tau_n}$.

$\{\chi_n\}$ is called the embedded process of $\{N_t\}$.

Fix $h > 0$, and let $X_n := N_{nh}$.

Embedded process and Skeleton process

$\tau_n, n \geq 0$: discontinuities of $\{N_t\}$.

$\chi_n := N_{\tau_n}$.

$\{\chi_n\}$ is called the embedded process of $\{N_t\}$.

Fix $h > 0$, and let $X_n := N_{nh}$.

$\{X_n\}$ is called the h-skeleton process of $\{N_t\}$.

Embedded process and Skeleton process

$\tau_n, n \geq 0$: discontinuities of $\{N_t\}$.

$\chi_n := N_{\tau_n}$.

$\{\chi_n\}$ is called the embedded process of $\{N_t\}$.

Fix $h > 0$, and let $X_n := N_{nh}$.

$\{X_n\}$ is called the h-skeleton process of $\{N_t\}$.

Recurrence criteria of embedded process

$$\Leftrightarrow$$

Recurrence criteria of $\{N_t\}$.

Condition C

(C1) $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ forms a stationary and ergodic system.

Condition C

(C1) $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ forms a stationary and ergodic system.

(C2) the measure $\mathbb{P}$ is uniformly elliptic, that is,

$$\mathbb{P}\Big(\varepsilon < \mu_0^l, \lambda_0^r < M, 1 \leq l \leq L, 1 \leq r \leq R\Big) = 1$$

for some small $\varepsilon > 0$ and large $M > 0$.

Define $T_1 = \inf[t > 0 : N_t > 0]$.

Define $T_1 = \inf[t > 0 : N_t > 0]$.

Theorem 1 (LLN for (L,R) BDPRE)

Suppose that conditions (C1) and (C2) are satisfied and $\gamma_R \geq 0$. Then

(a) $ET_1 < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = v_{\mathbb{P}} > 0$, P-a.s.;

(b) $ET_1 = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$, P-a.s..

Define $T_1 = \inf[t > 0 : N_t > 0]$.

Theorem 1 (LLN for (L,R) BDPRE)

Suppose that conditions (C1) and (C2) are satisfied and $\gamma_R \geq 0$.
Then
(a) $ET_1 < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = v_{\mathbb{P}} > 0$, P-a.s.;
(b) $ET_1 = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$, P-a.s..

$v_{\mathbb{P}} =$

$$\frac{\mathbb{E}\Big(\sum_{r=1}^{R} \sum_{k\leq 0} E_{\theta^{-k}\omega}\big(\sum_{j=1}^{U_k} \xi_{kj} | N_{T_1} = r\big)\big(\sum_{l=1}^{L} (-l)\mu_0^l + \sum_{r=1}^{R} r\lambda_0^r\big)\Big)}{\sum_{r=1}^{R} E(T_1 | N_{T_1} = r)}$$

Define $T_1 = \inf[t > 0 : N_t > 0]$.

Theorem 1 (LLN for (L,R) BDPRE)

Suppose that conditions (C1) and (C2) are satisfied and $\gamma_R \geq 0$. Then
(a) $ET_1 < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = v_{\mathbb{P}} > 0$, P-a.s.;
(b) $ET_1 = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$, P-a.s..

$v_{\mathbb{P}} =$

$$\frac{\mathbb{E}\Big(\sum_{r=1}^{R} \sum_{k\leq 0} E_{\theta^{-k}\omega}\big(\sum_{j=1}^{U_k} \xi_{kj} | N_{T_1} = r\big)\big(\sum_{l=1}^{L} (-l)\mu_0^l + \sum_{r=1}^{R} r\lambda_0^r\big)\Big)}{\sum_{r=1}^{R} E(T_1 | N_{T_1} = r)}$$

$U_k := \#\{n : N_{\tau_n} = k, \tau_n < T_1\}$,

Define $T_1 = \inf[t > 0 : N_t > 0]$.

Theorem 1 (LLN for (L,R) BDPRE)

Suppose that conditions (C1) and (C2) are satisfied and $\gamma_R \geq 0$. Then

(a) $ET_1 < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = v_{\mathbb{P}} > 0$, P-a.s.;

(b) $ET_1 = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$, P-a.s..

$v_{\mathbb{P}} =$

$$\frac{\mathbb{E}\Big(\sum_{r=1}^{R} \sum_{k\leq 0} E_{\theta^{-k}\omega}\big(\sum_{j=1}^{U_k} \xi_{kj} | N_{T_1} = r\big)\big(\sum_{l=1}^{L}(-l)\mu_0^l + \sum_{r=1}^{R} r\lambda_0^r\big)\Big)}{\sum_{r=1}^{R} E(T_1 | N_{T_1} = r)}$$

$U_k := \#\{n : N_{\tau_n} = k, \tau_n < T_1\}$,

$P_\omega(\xi_{kj} > t) = e^{-(\sum_{l=1}^{L} \mu_k^l + \sum_{r=1}^{R} \lambda_k^r)t}, \ t \geq 0.$

Define $T_1 = \inf[t > 0 : N_t > 0]$.

Theorem 1 (LLN for (L,R) BDPRE)

Suppose that conditions (C1) and (C2) are satisfied and $\gamma_R \geq 0$. Then

(a) $ET_1 < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = v_{\mathbb{P}} > 0$, P-a.s.;

(b) $ET_1 = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$, P-a.s..

$v_{\mathbb{P}} =$

$$\frac{\mathbb{E}\Big(\sum_{r=1}^{R} \sum_{k\leq 0} E_{\theta^{-k}\omega}\big(\sum_{j=1}^{U_k} \xi_{kj} | N_{T_1} = r\big)\big(\sum_{l=1}^{L}(-l)\mu_0^l + \sum_{r=1}^{R} r\lambda_0^r\big)\Big)}{\sum_{r=1}^{R} E(T_1 | N_{T_1} = r)}$$

$U_k := \#\{n : N_{\tau_n} = k, \tau_n < T_1\}$,

$P_\omega(\xi_{kj} > t) = e^{-(\sum_{l=1}^{L} \mu_k^l + \sum_{r=1}^{R} \lambda_k^r)t}, \ t \geq 0.$

Theorem 2 (LLN for (2,2) BDPRE)

Let $\pi(\omega)$ and $D(\omega)$ be certain functions of ω. Suppose $L = R = 2$ and $\gamma_R \geq 0$. Then $\mathbb{P}$-a.s.,

(a) $\mathbb{E}(\pi(\omega)) < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = \frac{\mathbb{E}\big(\pi(\omega)(2\lambda_0^2+\lambda_0^1-\mu_0^1-2\mu_0^2)\big)}{\mathbb{E}(D(\omega))}$.

(b) $\mathbb{E}(\pi(\omega)) = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$.

Theorem 2 (LLN for (2,2) BDPRE)

Let $\pi(\omega)$ and $D(\omega)$ be certain functions of ω. Suppose $L = R = 2$ and $\gamma_R \geq 0$. Then $\mathbb{P}$-a.s.,

(a) $\mathbb{E}(\pi(\omega)) < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = \frac{\mathbb{E}\left(\pi(\omega)(2\lambda_0^2+\lambda_0^1-\mu_0^1-2\mu_0^2)\right)}{\mathbb{E}(D(\omega))}$.

(b) $\mathbb{E}(\pi(\omega)) = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$.

Idea: branching structure for (2,2) random walk.

Theorem 2 (LLN for (2,2) BDPRE)

Let $\pi(\omega)$ and $D(\omega)$ be certain functions of ω. Suppose $L = R = 2$ and $\gamma_R \geq 0$. Then $\mathbb{P}$-a.s.,

(a) $\mathbb{E}(\pi(\omega)) < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = \frac{\mathbb{E}\big(\pi(\omega)(2\lambda_0^2+\lambda_0^1-\mu_0^1-2\mu_0^2)\big)}{\mathbb{E}(D(\omega))}$.

(b) $\mathbb{E}(\pi(\omega)) = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$.

Idea: branching structure for (2,2) random walk.

[HW14] Hong W. M., and Wang H. M., Intrinsic branching structure within random walk on Z, Teor. Veroyatnost. i Primenen., Vol.58(4), 730 - 751, 2013 (English version will appear in Theory of Probability and Its Applications, 2014)

LLN for (1,1) BDPRE

$$T_n := \inf[t \geq 0 : N_t = n];$$

LLN for (1,1) BDPRE

$T_n := \inf[t \geq 0 : N_t = n]$;

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{ET_1}$.

LLN for (1,1) BDPRE

$T_n := \inf[t \geq 0 : N_t = n];$

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{ET_1}.$

$\exists\ n$ s.t. $T_n < t < T_{n+1};$

LLN for (1,1) BDPRE

$T_n := \inf[t \geq 0 : N_t = n];$

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{ET_1}.$

$\exists\ n$ s.t. $T_n < t < T_{n+1};$

$n - \nu_n < N_t < n + 1;$

LLN for (1,1) BDPRE

$T_n := \inf[t \geq 0 : N_t = n];$

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{ET_1}.$

$\exists\ n$ s.t. $T_n < t < T_{n+1};$

$n - \nu_n < N_t < n + 1;$

$\frac{n-\nu_n}{T_{n+1}} < \frac{N_t}{t} < \frac{n+1}{T_n}.$

LLN for (1,1) BDPRE

$T_n := \inf[t \geq 0 : N_t = n]$;

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{ET_1}$.

$\exists\ n$ s.t. $T_n < t < T_{n+1}$;

$n - \nu_n < N_t < n + 1$;

$\frac{n-\nu_n}{T_{n+1}} < \frac{N_t}{t} < \frac{n+1}{T_n}$.

Key point: $\{T_n - T_{n-1}\}$ is a stationary and mixing sequence under P.

LLN for (1,1) BDPRE

$T_n := \inf[t \geq 0 : N_t = n]$;

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{ET_1}$.

$\exists\ n$ s.t. $T_n < t < T_{n+1}$;

$n - \nu_n < N_t < n + 1$;

$\frac{n-\nu_n}{T_{n+1}} < \frac{N_t}{t} < \frac{n+1}{T_n}$.

Key point: $\{T_n - T_{n-1}\}$ is a stationary and mixing sequence under P.

For (L,1) BDPRE transient to the right, the above approach still works.

LLN for (1,1) BDPRE

$T_n := \inf[t \geq 0 : N_t = n]$;

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{ET_1}$.

$\exists\, n$ s.t. $T_n < t < T_{n+1}$;

$n - \nu_n < N_t < n + 1$;

$\frac{n-\nu_n}{T_{n+1}} < \frac{N_t}{t} < \frac{n+1}{T_n}$.

Key point: $\{T_n - T_{n-1}\}$ is a stationary and mixing sequence under P.

For (L,1) BDPRE transient to the right, the above approach still works.

But for general (L,R) BDPRE, it does not work.

(L,R) BDPRE-difficulties of LLN

$$T_0 = 0,\ T_n = \inf[t > T_{n-1} : N_t > N_{T_{n-1}}], n \geq 1;$$

(L,R) BDPRE-difficulties of LLN

$T_0 = 0$, $T_n = \inf[t > T_{n-1} : N_t > N_{T_{n-1}}], n \geq 1;$

$N_{T_n} = ???$

(L,R) BDPRE-difficulties of LLN

$T_0 = 0,\ T_n = \inf[t > T_{n-1} : N_t > N_{T_{n-1}}], n \geq 1;$

$N_{T_n} =???$

$n \leq N_{T_n} \leq nR$

(L,R) BDPRE-difficulties of LLN

$T_0 = 0,\ T_n = \inf[t > T_{n-1} : N_t > N_{T_{n-1}}], n \geq 1;$

$N_{T_n} =???$

$n \leq N_{T_n} \leq nR$

$\exists\ n \text{ s.t. } T_n \leq t < T_{n+1};$

(L,R) BDPRE-difficulties of LLN

$T_0 = 0,\ T_n = \inf[t > T_{n-1} : N_t > N_{T_{n-1}}], n \geq 1;$

$N_{T_n} = ???$

$n \leq N_{T_n} \leq nR$

$\exists\ n$ s.t. $T_n \leq t < T_{n+1};$

$n - \nu_n \leq N_t < (n+1)R;$

(L,R) BDPRE-difficulties of LLN

$T_0 = 0,\ T_n = \inf[t > T_{n-1} : N_t > N_{T_{n-1}}], n \geq 1;$

$N_{T_n} = ???$

$n \leq N_{T_n} \leq nR$

$\exists\ n$ s.t. $T_n \leq t < T_{n+1};$

$n - \nu_n \leq N_t < (n+1)R;$

$\frac{n-\nu_n}{T_{n+1}} \leq \frac{N_t}{t} < \frac{(n+1)R}{T_n}.$

(L,R) BDPRE-difficulties of LLN

$T_0 = 0,\ T_n = \inf[t > T_{n-1} : N_t > N_{T_{n-1}}], n \geq 1;$

$N_{T_n} = ???$

$n \leq N_{T_n} \leq nR$

$\exists\ n$ s.t. $T_n \leq t < T_{n+1};$

$n - \nu_n \leq N_t < (n+1)R;$

$\frac{n-\nu_n}{T_{n+1}} \leq \frac{N_t}{t} < \frac{(n+1)R}{T_n}.$

$\{T_n - T_{n-1}\}_{n\geq 1}$ is neither stationary nor ergodic.

(L,R) BDPRE-difficulties of LLN

$T_0 = 0,\ T_n = \inf[t > T_{n-1} : N_t > N_{T_{n-1}}], n \geq 1;$

$N_{T_n} = ???$

$n \leq N_{T_n} \leq nR$

$\exists\ n$ s.t. $T_n \leq t < T_{n+1};$

$n - \nu_n \leq N_t < (n+1)R;$

$\frac{n-\nu_n}{T_{n+1}} \leq \frac{N_t}{t} < \frac{(n+1)R}{T_n}.$

$\{T_n - T_{n-1}\}_{n\geq 1}$ is neither stationary nor ergodic.

Idea: consider the skeleton process $\{N_{nh}\}_{n\geq 0}$.

$nh \leq t < (n+1)h;$

$$nh \le t < (n+1)h;$$

$$N_{nh} - J_t L \le N_t < N_{nh} + J_t R,$$

$nh \leq t < (n+1)h;$

$N_{nh} - J_t L \leq N_t < N_{nh} + J_t R,$

where $J_t := \#\{$ jumps of $\{N_t\}$ in [t,t+h)$\}$;

$nh \leq t < (n+1)h;$

$N_{nh} - J_t L \leq N_t < N_{nh} + J_t R,$

where $J_t :=$ #{ jumps of $\{N_t\}$ in [t,t+h)};

$\frac{N_{nh} - J_t L}{(n+1)h} \leq \frac{N_t}{t} < \frac{N_{nh} + J_t R}{nh};$

$nh \leq t < (n+1)h;$

$N_{nh} - J_t L \leq N_t < N_{nh} + J_t R,$

where $J_t := \#\{$ jumps of $\{N_t\}$ in [t,t+h)$\}$;

$\frac{N_{nh} - J_t L}{(n+1)h} \leq \frac{N_t}{t} < \frac{N_{nh} + J_t R}{nh};$

If $\lim_{n\to\infty} \frac{N_{nh}}{n} = v_{\mathbb{P}}^h$ then

$nh \leq t < (n+1)h;$

$N_{nh} - J_t L \leq N_t < N_{nh} + J_t R,$

where $J_t := \#\{$ jumps of $\{N_t\}$ in [t,t+h)$\};$

$\frac{N_{nh} - J_t L}{(n+1)h} \leq \frac{N_t}{t} < \frac{N_{nh} + J_t R}{nh};$

If $\lim_{n\to\infty} \frac{N_{nh}}{n} = v_{\mathbb{P}}^h$ then

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{N_{nh}}{nh} = \frac{v_{\mathbb{P}}^h}{h}.$

$nh \le t < (n+1)h;$

$N_{nh} - J_t L \le N_t < N_{nh} + J_t R,$

where $J_t :=$ #{ jumps of $\{N_t\}$ in [t,t+h)};

$\frac{N_{nh} - J_t L}{(n+1)h} \le \frac{N_t}{t} < \frac{N_{nh} + J_t R}{nh};$

If $\lim_{n\to\infty} \frac{N_{nh}}{n} = v_{\mathbb{P}}^h$ then

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{N_{nh}}{nh} = \frac{v_{\mathbb{P}}^h}{h}.$

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{h\to 0} \frac{v_{\mathbb{P}}^h}{h} =: v_{\mathbb{P}}.$

$nh \le t < (n+1)h;$

$N_{nh} - J_t L \le N_t < N_{nh} + J_t R,$

where $J_t :=$ #{ jumps of $\{N_t\}$ in [t,t+h)};

$\frac{N_{nh} - J_t L}{(n+1)h} \le \frac{N_t}{t} < \frac{N_{nh} + J_t R}{nh};$

If $\lim_{n\to\infty} \frac{N_{nh}}{n} = v_{\mathbb{P}}^h$ then

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{n\to\infty} \frac{N_{nh}}{nh} = \frac{v_{\mathbb{P}}^h}{h}.$

$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{h\to 0} \frac{v_{\mathbb{P}}^h}{h} =: v_{\mathbb{P}}.$

Idea to prove LLN of skeleton process $\{N_{nh}\}$

- Approach: "the environment viewed from particle".
- Difficulty: $\{N_{nh}\}_{n\ge 0}$ is a discrete time random walk in random environment with unbounded jumps.

Define $T_1^h = \inf[n : X_n > 0]$.

Define $T_1^h = \inf[n : X_n > 0]$.

$p_\omega(h, i, j) := P_\omega(N_h = i + j | N_0 = i)$.

Define $T_1^h = \inf[n : X_n > 0]$.

$p_\omega(h,i,j) := P_\omega(N_h = i + j | N_0 = i)$.

Theorem 3 (LLN of the skeleton process)

Suppose that (C1) and (C2) hold. Then P-a.s., $\{X_n\}$ is transient to the right, recurrent or transient to the left according as $\gamma_R \geq 0$, $\gamma_R = 0$ or $\gamma_R \leq 0$. Moreover, if $\gamma_R \geq 0$, then
$E(T_1^h) = \infty \Rightarrow P$-a.s., $\lim_{n\to\infty} \frac{X_n}{n} = 0$;
$E(T_1^h) < \infty \Rightarrow P$-a.s., $\lim_{n\to\infty} \frac{X_n}{n} = v_{\mathbb{P}}^h > 0$,
where $v_{\mathbb{P}}^h = \frac{\mathbb{E}\left(\sum_{i=1}^{\infty} \sum_{k\leq 0} E_{\theta^{-k}\omega}\left(U_k^h | X_{T_1^h = i}\right) \sum_{j\in\mathbb{Z}} j p_\omega(h,0,j)\right)}{\sum_{i=1}^{\infty} E\left(T_1^h | X_{T_1^h} = i\right)}$, with
$U_k^h = \#\{0 \leq n < T_1^h : X_n = k\}$.

Define $T_1^h = \inf[n : X_n > 0]$.

$p_\omega(h, i, j) := P_\omega(N_h = i + j | N_0 = i)$.

Theorem 3 (LLN of the skeleton process)

Suppose that (C1) and (C2) hold. Then P-a.s., $\{X_n\}$ is transient to the right, recurrent or transient to the left according as $\gamma_R \geq 0$, $\gamma_R = 0$ or $\gamma_R \leq 0$. Moreover, if $\gamma_R \geq 0$, then
$E(T_1^h) = \infty \Rightarrow P$-a.s., $\lim_{n\to\infty} \frac{X_n}{n} = 0$;
$E(T_1^h) < \infty \Rightarrow P$-a.s., $\lim_{n\to\infty} \frac{X_n}{n} = v_{\mathbb{P}}^h > 0$,
where $v_{\mathbb{P}}^h = \frac{\mathbb{E}\left(\sum_{i=1}^{\infty} \sum_{k\leq 0} E_{\theta^{-k}\omega}\left(U_k^h | X_{T_1^h = i}\right) \sum_{j\in\mathbb{Z}} j p_\omega(h,0,j)\right)}{\sum_{i=1}^{\infty} E\left(T_1^h | X_{T_1^h} = i\right)}$, with
$U_k^h = \#\{0 \leq n < T_1^h : X_n = k\}$.

$E(T_1^h) < \infty \Leftrightarrow E(T_1) < \infty$.

Sketch of the proof of Theorem 3

Define $\overline{\omega}(n) = \theta^{X_n}\omega$.

Sketch of the proof of Theorem 3

Define $\overline{\omega}(n) = \theta^{X_n}\omega$.

Define the local drift $d(x, \omega) = E_\omega^x(X_1 - X_0)$ and set

$$M_n = X_n - X_0 - \sum_{k=0}^{n-1} d(X_n, \omega)$$
$$= X_n - X_0 - \sum_{k=0}^{n-1} d(0, \overline{\omega}(k))$$

Sketch of the proof of Theorem 3

Define $\overline{\omega}(n) = \theta^{X_n}\omega$.

Define the local drift $d(x,\omega) = E_\omega^x(X_1 - X_0)$ and set

$$M_n = X_n - X_0 - \sum_{k=0}^{n-1} d(X_n, \omega)$$
$$= X_n - X_0 - \sum_{k=0}^{n-1} d(0, \overline{\omega}(k))$$

Lemma 1

Under P_ω, $\{M_n\}$ is a martingale and P-a.s., $\lim_{n\to\infty} \frac{M_n}{n} = 0$.

Sketch of the proof of Theorem 3

Define $\overline{\omega}(n) = \theta^{X_n}\omega$.

Define the local drift $d(x,\omega) = E_\omega^x(X_1 - X_0)$ and set

$$M_n = X_n - X_0 - \sum_{k=0}^{n-1} d(X_n, \omega)$$

$$= X_n - X_0 - \sum_{k=0}^{n-1} d(0, \overline{\omega}(k))$$

Lemma 1

Under P_ω, $\{M_n\}$ is a martingale and P-a.s., $\lim_{n\to\infty} \frac{M_n}{n} = 0$.

Lesigne and Volný (SPA, 2001):

$E(e^{|M_n - M_{n-1}|}) < \infty \Rightarrow P(|M_n| > \sqrt{n}\lambda) \leq e^{-c_4\lambda^{\frac{2}{3}}}$

$\Rightarrow P$-a.s., $\lim_{n\to\infty} \frac{M_n}{n} = 0$.

Lemma 2

Suppose that Condition (C2) is satisfied. Then for $\mathbb{P}$-a.a. ω,

$$p_\omega(h, i, j) < e^{c_0 h} e^{-c_1 |j|},$$

for some constant $c_0, c_1 > 0$.

Lemma 2

Suppose that Condition (C2) is satisfied. Then for $\mathbb{P}$-a.a. ω,

$$p_\omega(h,i,j) < e^{c_0 h} e^{-c_1|j|},$$

for some constant $c_0, c_1 > 0$.

By Lemma 2,

$$E_\omega(e^{|X_n - X_{n-1}|}) < c_3$$

Lemma 2

Suppose that Condition (C2) is satisfied. Then for $\mathbb{P}$-a.a. ω,

$$p_\omega(h,i,j) < e^{c_0 h} e^{-c_1|j|},$$

for some constant $c_0, c_1 > 0$.

By Lemma 2,

$$E_\omega(e^{|X_n - X_{n-1}|}) < c_3$$

which implies

$$E(e^{|M_n - M_{n-1}|}) < \infty.$$

Lemma 2

Suppose that Condition (C2) is satisfied. Then for $\mathbb{P}$-a.a. ω,

$$p_\omega(h, i, j) < e^{c_0 h} e^{-c_1 |j|},$$

for some constant $c_0, c_1 > 0$.

By Lemma 2,

$$E_\omega(e^{|X_n - X_{n-1}|}) < c_3$$

which implies

$$E(e^{|M_n - M_{n-1}|}) < \infty.$$

What left to prove is only

$$P\text{-a.s.}, \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} d(0, \overline{\omega}(k)) = v_{\mathbb{P}}^h.$$

Define

$$K^h(\omega, d\omega') = \sum_{j\in\mathbb{Z}} p_\omega(h, 0, j)\delta_{\omega'=\theta^j\omega}.$$

Define

$$K^h(\omega, d\omega') = \sum_{j \in \mathbb{Z}} p_\omega(h, 0, j)\delta_{\omega' = \theta^j \omega}.$$

Lemma 3

Under either P or P_ω, $\{\overline{\omega}(n)\}_{n \geq 0}$ is a Markov chain with transition kernel $K^h(\omega, \omega')$.

Define

$$K^h(\omega, d\omega') = \sum_{j\in\mathbb{Z}} p_\omega(h,0,j)\delta_{\omega'=\theta^j\omega}.$$

Lemma 3

Under either P or P_ω, $\{\overline{\omega}(n)\}_{n\geq 0}$ is a Markov chain with transition kernel $K^h(\omega,\omega')$.

Whenever $E(T_1^h) < \infty$, define the measures

$$Q^h(d\omega) := E\left(\sum_{i\geq 1} \frac{\mathbf{1}_{X_{T_1^h}=i}}{P_\omega(X_{T_1^h}=i)} \sum_{k=0}^{T_1^h-1} 1_{\overline{\omega}(k)\in d\omega}\right), \ \overline{Q}^h(d\omega) = \frac{Q^h(d\omega)}{E(T_1^h)}.$$

Lemma 4

Suppose that conditions (C1), (C2) hold and $E(T_1^h) < \infty$. Then Q^h is invariant under the kernel K^h, that is

$$Q^h(B) = \iint \mathbf{1}_{\omega' \in B} K^h(\omega, d\omega') Q^h(d\omega).$$

Moreover, $Q^h \sim \mathbb{P}$ and

$$\frac{dQ^h}{d\mathbb{P}} = \sum_{k \le 0} \sum_{i \ge 1} E_{\theta^{-k}\omega}(U_k^h | X_{T_1^h} = i) =: \pi^h(\omega),$$

where $U_k^h = \#\{n \le T_1^h : X_n = k\}$.

Lemma 4

Suppose that conditions (C1), (C2) hold and $E(T_1^h) < \infty$. Then Q^h is invariant under the kernel K^h, that is

$$Q^h(B) = \iint \mathbf{1}_{\omega' \in B} K^h(\omega, d\omega') Q^h(d\omega).$$

Moreover, $Q^h \sim \mathbb{P}$ and

$$\frac{dQ^h}{d\mathbb{P}} = \sum_{k \le 0} \sum_{i \ge 1} E_{\theta^{-k}\omega}(U_k^h | X_{T_1^h} = i) =: \pi^h(\omega),$$

where $U_k^h = \#\{n \le T_1^h : X_n = k\}$.

Lemma 5

Under the conditions of Lemma 4, $\{\overline{\omega}(n)\}$ is stationary and ergodic under the probability measure $\overline{Q}^h \times P_\omega$.

Using Birkhoff's ergodic theorem, we have that for $\overline{Q}^h$-a.a. or $\mathbb{P}$-a.a. ω, P_ω-a.s.,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} d(X_k,\omega) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} d(0,\overline{\omega}(k)) = \int d(0,\omega)d\overline{Q}^h.$$

Using Birkhoff's ergodic theorem, we have that for $\overline{Q}^h$-a.a. or $\mathbb{P}$-a.a. ω, P_ω-a.s.,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} d(X_k,\omega) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} d(0,\overline{\omega}(k)) = \int d(0,\omega)d\overline{Q}^h.$$

Then P-a.s.,

$$\lim_{n\to\infty}\frac{X_n}{n} = \int d(0,\omega)d\overline{Q}^h =: v_{\mathbb{P}}^h.$$

Using Birkhoff's ergodic theorem, we have that for $\overline{Q}^h$-a.a. or $\mathbb{P}$-a.a. ω, P_ω-a.s.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} d(X_k, \omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} d(0, \overline{\omega}(k)) = \int d(0,\omega) d\overline{Q}^h.$$

Then P-a.s.,

$$\lim_{n\to\infty} \frac{X_n}{n} = \int d(0,\omega) d\overline{Q}^h =: v_{\mathbb{P}}^h.$$

$$\begin{aligned} v_{\mathbb{P}}^h &= \int d(0,\omega) d\overline{Q}^h \\ &= \frac{\mathbb{E}\left(\sum_{i=1}^{\infty} \sum_{k\leq 0} E_{\theta^{-k}\omega}(U_k^h | X_{T_1^h = i}) \sum_{j\in\mathbb{Z}} j p_\omega(h,0,j) \right)}{\sum_{i=1}^{\infty} E(T_1^h | X_{T_1^h} = i)}. \end{aligned}$$

Recurrence criteria

Given ω, define for $i \in \mathbb{Z}$,

$$b_i(k) = \begin{cases} \frac{\sum_{j=R-k+1}^{R} \lambda_i^j}{\mu_i^L} \text{ if } 1 \leq k \leq R, \\ -\frac{\sum_{j=k-R}^{L} \mu_i^j}{\mu_i^L} \text{ if } R+1 \leq k \leq R+L-1, \end{cases}$$

Recurrence criteria

Given ω, define for $i \in \mathbb{Z}$,

$$b_i(k) = \begin{cases} \frac{\sum_{j=R-k+1}^{R} \lambda_i^j}{\mu_i^L} \text{ if } 1 \le k \le R, \\ -\frac{\sum_{j=k-R}^{L} \mu_i^j}{\mu_i^L} \text{ if } R+1 \le k \le R+L-1, \end{cases}$$

and let

$$A_i = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ b_i(1) & b_i(2) & \cdots & b_i(L+R-1) \end{pmatrix}$$

be an $(L+R-1) \times (L+R-1)$ matrix.

Recurrence criteria

Given ω, define for $i \in \mathbb{Z}$,

$$b_i(k) = \begin{cases} \frac{\sum_{j=R-k+1}^{R} \lambda_i^j}{\mu_i^L} \text{ if } 1 \leq k \leq R, \\ -\frac{\sum_{j=k-R}^{L} \mu_i^j}{\mu_i^L} \text{ if } R+1 \leq k \leq R+L-1, \end{cases}$$

and let

$$A_i = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ b_i(1) & b_i(2) & \cdots & b_i(L+R-1) \end{pmatrix}$$

be an $(L+R-1) \times (L+R-1)$ matrix.

The Lyapunov exponents (Oseledec's multiplicative ergodic theorem) of the sequence $\{A_i\}_{i \in \mathbb{Z}}$ are

$$-\infty < \gamma_1 \leq \gamma_2 \leq ... \leq \gamma_{R+L-1} < \infty.$$

Proposition 1 (Recurrence criteria)

Suppose that (C1) and (C2) are satisfied. Let $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{R+L-1}$ be the Lyapunov exponents of the sequence $\{A_i\}_{i\in\mathbb{Z}}$ under $\mathbb{P}$. Then

(1) $\gamma_R > 0 \Rightarrow P(\lim_{t\to\infty} N_t = \infty) = 1$;
(2) $\gamma_R = 0 \Rightarrow P\left(-\infty = \underline{\lim}_{t\to\infty} N_t < \overline{\lim}_{t\to\infty} N_t = \infty\right) = 1$;
(3) $\gamma_R < 0 \Rightarrow P(\lim_{t\to\infty} N_t = -\infty) = 1$.

Proposition 1 (Recurrence criteria)

Suppose that (C1) and (C2) are satisfied. Let $\gamma_1 \le \gamma_2 \le ... \le \gamma_{R+L-1}$ be the Lyapunov exponents of the sequence $\{A_i\}_{i\in\mathbb{Z}}$ under $\mathbb{P}$. Then
(1) $\gamma_R > 0 \Rightarrow P(\lim_{t\to\infty} N_t = \infty) = 1$;
(2) $\gamma_R = 0 \Rightarrow P\left(-\infty = \underline{\lim}_{t\to\infty} N_t < \overline{\lim}_{t\to\infty} N_t = \infty\right) = 1$;
(3) $\gamma_R < 0 \Rightarrow P(\lim_{t\to\infty} N_t = -\infty) = 1$.

Proof. Since the recurrence criteria for $\{N_t\}$ is the same as its embedded process $\{\chi_n\}$, Proposition 1 is just a corollary of Theorem A in Letchikov [L89]. □

Proposition 1 (Recurrence criteria)

Suppose that (C1) and (C2) are satisfied. Let $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_{R+L-1}$ be the Lyapunov exponents of the sequence $\{A_i\}_{i\in\mathbb{Z}}$ under $\mathbb{P}$. Then

(1) $\gamma_R > 0 \Rightarrow P(\lim_{t\to\infty} N_t = \infty) = 1$;

(2) $\gamma_R = 0 \Rightarrow P\left(-\infty = \underline{\lim}_{t\to\infty} N_t < \overline{\lim}_{t\to\infty} N_t = \infty\right) = 1$;

(3) $\gamma_R < 0 \Rightarrow P(\lim_{t\to\infty} N_t = -\infty) = 1$.

Proof. Since the recurrence criteria for $\{N_t\}$ is the same as its embedded process $\{\chi_n\}$, Proposition 1 is just a corollary of Theorem A in Letchikov [L89]. □

[L89] Letchikov, A. V., Localization of one-dimensional random walks in random environments, Sov. Sci. Rev. C. Math. Phys., Vol. 8, pp 173-220, 1989

(L,1) BDPRE

Letting $R = 1$, consider (L,1) BDPRE.

(L,1) BDPRE

Letting $R = 1$, consider (L,1) BDPRE.

Condition B

(B1) $\mathbb{P}\left(\lambda_0 + \sum_{l=1}^{L} \mu_0^l > 0\right) = 1$;

(L,1) BDPRE

Letting $R = 1$, consider (L,1) BDPRE.

Condition B

(B1) $\mathbb{P}\left(\lambda_0 + \sum_{l=1}^{L} \mu_0^l > 0\right) = 1;$

(B2) $\mathbb{P}\Big(\sum_{n=1}^{\infty} \frac{1}{\lambda_n + \sum_{n=1}^{L} \mu_n^l} = \infty\Big) = 1,$

and $\mathbb{P}\Big(\sum_{n=-\infty}^{0} \frac{1}{\max_{1\leq k\leq L}\{\lambda_{nL-k}+\sum_{l=1}^{L} \mu_{nL-k}^l\}} = \infty\Big) = 1.$

(L,1) BDPRE

Letting $R = 1$, consider (L,1) BDPRE.

Condition B

(B1) $\mathbb{P}\left(\lambda_0 + \sum_{l=1}^{L} \mu_0^l > 0\right) = 1$;

(B2) $\mathbb{P}\Big(\sum_{n=1}^{\infty} \frac{1}{\lambda_n + \sum_{n=1}^{L} \mu_n^l} = \infty\Big) = 1,$

and $\mathbb{P}\Big(\sum_{n=-\infty}^{0} \frac{1}{\max_{1\leq k\leq L}\{\lambda_{nL-k}+\sum_{l=1}^{L} \mu_{nL-k}^l\}} = \infty\Big) = 1.$

(B3) $\mathbb{E}\ln \frac{\lambda_0}{\lambda_0+\sum_{l=1}^{L} \mu_0^l} > -\infty, \mathbb{E}\ln \frac{\mu_0^L}{\lambda_0+\sum_{l=1}^{L} \mu_0^l} > -\infty.$

(B4) $\mathbb{P}$ is i.i.d..

(L,1) BDPRE

Letting $R = 1$, consider (L,1) BDPRE.

Condition B

(B1) $\mathbb{P}\left(\lambda_0 + \sum_{l=1}^{L} \mu_0^l > 0\right) = 1$;

(B2) $\mathbb{P}\Big(\sum_{n=1}^{\infty} \frac{1}{\lambda_n + \sum_{n=1}^{L} \mu_n^l} = \infty\Big) = 1,$

and $\mathbb{P}\Big(\sum_{n=-\infty}^{0} \frac{1}{\max_{1\le k\le L}\{\lambda_{nL-k}+\sum_{l=1}^{L} \mu_{nL-k}^l\}} = \infty\Big) = 1.$

(B3) $\mathbb{E} \ln \frac{\lambda_0}{\lambda_0+\sum_{l=1}^{L} \mu_0^l} > -\infty, \mathbb{E} \ln \frac{\mu_0^L}{\lambda_0+\sum_{l=1}^{L} \mu_0^l} > -\infty.$

(B4) $\mathbb{P}$ is i.i.d..

(B1) and (B2) ensure the existence.

Under (B3) and (B4), we could use the multiplicative ergodic theorem for the following random matrices.

(L,1) BDPRE

Letting $R = 1$, consider (L,1) BDPRE.

Condition B

(B1) $\mathbb{P}\left(\lambda_0 + \sum_{l=1}^{L} \mu_0^l > 0\right) = 1;$

(B2) $\mathbb{P}\Big(\sum_{n=1}^{\infty} \frac{1}{\lambda_n + \sum_{n=1}^{L} \mu_n^l} = \infty\Big) = 1,$

and $\mathbb{P}\Big(\sum_{n=-\infty}^{0} \frac{1}{\max_{1\leq k\leq L}\{\lambda_{nL-k}+\sum_{l=1}^{L} \mu_{nL-k}^l\}} = \infty\Big) = 1.$

(B3) $\mathbb{E}\ln \frac{\lambda_0}{\lambda_0+\sum_{l=1}^{L} \mu_0^l} > -\infty, \mathbb{E}\ln \frac{\mu_0^L}{\lambda_0+\sum_{l=1}^{L} \mu_0^l} > -\infty.$

(B4) $\mathbb{P}$ is i.i.d..

(B1) and (B2) ensure the existence.

Under (B3) and (B4), we could use the multiplicative ergodic theorem for the following random matrices.

Introduce matrices

$$M_i = \begin{pmatrix} \frac{\mu_i^1}{\lambda_i} & ... & \frac{\mu_i^{L-1}}{\lambda_i} & \frac{\mu_i^L}{\lambda_i} \\ 1 + \frac{\mu_i^1}{\lambda_i} & ... & \frac{\mu_i^{L-1}}{\lambda_i} & \frac{\mu_i^L}{\lambda_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\mu_i^1}{\lambda_i} & ... & 1 + \frac{\mu_i^{L-1}}{\lambda_i} & \frac{\mu_i^L}{\lambda_i} \end{pmatrix}, \ i \in \mathbb{Z}.$$

For $i \in \mathbb{Z}$, let $a_i(k) = \frac{\sum_{l=k}^{L} \mu_i^l}{\lambda_i}$, $k = 1, ..., L$, $b_i(1) = \frac{\lambda_i}{\mu_i^L}$ and $b_i(k) = \frac{\sum_{l=k-1}^{L} \mu_i^l}{\mu_i^L}$, $k = 2, ..., L$.

Introduce matrices

$$M_i = \begin{pmatrix} \frac{\mu_i^1}{\lambda_i} & ... & \frac{\mu_i^{L-1}}{\lambda_i} & \frac{\mu_i^L}{\lambda_i} \\ 1 + \frac{\mu_i^1}{\lambda_i} & ... & \frac{\mu_i^{L-1}}{\lambda_i} & \frac{\mu_i^L}{\lambda_i} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\mu_i^1}{\lambda_i} & ... & 1 + \frac{\mu_i^{L-1}}{\lambda_i} & \frac{\mu_i^L}{\lambda_i} \end{pmatrix}, \ i \in \mathbb{Z}.$$

For $i \in \mathbb{Z}$, let $a_i(k) = \frac{\sum_{l=k}^{L} \mu_i^l}{\lambda_i}$, $k = 1, ..., L$, $b_i(1) = \frac{\lambda_i}{\mu_i^L}$ and $b_i(k) = \frac{\sum_{l=k-1}^{L} \mu_i^l}{\mu_i^L}$, $k = 2, ..., L$. Introduce matrices

$$B_i = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ b_i(1) & -b_i(2) & \cdots & -b_i(L) \end{pmatrix}, B_i^{-1} = \begin{pmatrix} a_i(1) & \cdots & a_i(L-1) & a_i(L) \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Theorem 4 (Recurrence criteria)

Suppose that (B1-B4) are all satisfied. Let $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_L$ be the Lyapunov exponents of the sequence $(M_i)_{i\in\mathbb{Z}}$. Then

$\gamma_L < 0 \Rightarrow P(\lim_{t\to\infty} N_t = \infty) = 1$;

$\gamma_L = 0 \Rightarrow P(-\infty = \underline{\lim}_{t\to\infty} N_t < \overline{\lim}_{t\to\infty} N_t = \infty) = 1$;

$\gamma_L > 0 \Rightarrow P(\lim_{t\to\infty} N_t = -\infty) = 1$.

Theorem 4 (Recurrence criteria)

Suppose that (B1-B4) are all satisfied. Let $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_L$ be the Lyapunov exponents of the sequence $(M_i)_{i\in\mathbb{Z}}$. Then
$\gamma_L < 0 \Rightarrow P(\lim_{t\to\infty} N_t = \infty) = 1$;
$\gamma_L = 0 \Rightarrow P(-\infty = \underline{\lim}_{t\to\infty} N_t < \overline{\lim}_{t\to\infty} N_t = \infty) = 1$;
$\gamma_L > 0 \Rightarrow P(\lim_{t\to\infty} N_t = -\infty) = 1$.

Define $T_n = \inf[t \geq 0 : N_t = n]$.

Theorem 5 (LLN)

Suppose that (B1-B4) are all satisfied and $\gamma_L \leq 0$. Then
(a) $\mathbb{E}T_1 < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = (ET_1)^{-1}$, P-a.s.;
(b) $\mathbb{E}T_1 = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$, P-a.s..

Theorem 4 (Recurrence criteria)

Suppose that (B1-B4) are all satisfied. Let $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_L$ be the Lyapunov exponents of the sequence $(M_i)_{i\in\mathbb{Z}}$. Then
$\gamma_L < 0 \Rightarrow P(\lim_{t\to\infty} N_t = \infty) = 1$;
$\gamma_L = 0 \Rightarrow P(-\infty = \underline{\lim}_{t\to\infty} N_t < \overline{\lim}_{t\to\infty} N_t = \infty) = 1$;
$\gamma_L > 0 \Rightarrow P(\lim_{t\to\infty} N_t = -\infty) = 1$.

Define $T_n = \inf[t \geq 0 : N_t = n]$.

Theorem 5 (LLN)

Suppose that (B1-B4) are all satisfied and $\gamma_L \leq 0$. Then
(a) $\mathbb{E}T_1 < \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = (ET_1)^{-1}$, P-a.s.;
(b) $\mathbb{E}T_1 = \infty \Rightarrow \lim_{t\to\infty} \frac{N_t}{t} = 0$, P-a.s..

Idea: $\{T_n - T_{n-1}\}_{n\geq 1}$ is a mixing sequence under P. An application of ergodic theorem yields that P-a.s., $\lim_{n\to\infty} T_n/n = E(T_1)$.

Theorem 6 (Decomposition of T_1)

Suppose that conditions (B1-B4) are all satisfied and $\gamma_L \leq 0$. Then $P(T_1 < \infty) = 1$ and

$$T_1 \stackrel{\mathscr{D}}{=} \xi_{0,1} + \sum_{i \leq -1} \sum_{k=1}^{U_{i,1}} \xi_{i,k} + \sum_{i \leq -1} \sum_{k=1}^{U_{i,1}+\ldots+U_{i,L}} \tilde{\xi}_{i+1,k},$$

$(U_{i,1}, \ldots, U_{i,L})_{i \leq 0}$ forms an L-type branching process in random environment ([HW13]) and

$$P_\omega(\tilde{\xi}_{i,k} \geq t) = P_\omega(\xi_{i,k} \geq t) = e^{-\left(\lambda_i + \sum_{i=1}^{L} \mu_i^l\right)t}, \ t \geq 0.$$

Moreover,

$$E_\omega T_1 = \sum_{i=-\infty}^{0} \frac{1}{\lambda_i} \mathbf{e}_1 M_0 M_{-1} \cdots M_{i+1} \mathbf{1}.$$

Theorem 6 (Decomposition of T_1)

Suppose that conditions (B1-B4) are all satisfied and $\gamma_L \leq 0$. Then $P(T_1 < \infty) = 1$ and

$$T_1 \overset{\mathscr{D}}{=} \xi_{0,1} + \sum_{i \leq -1} \sum_{k=1}^{U_{i,1}} \xi_{i,k} + \sum_{i \leq -1} \sum_{k=1}^{U_{i,1}+\ldots+U_{i,L}} \tilde{\xi}_{i+1,k},$$

$(U_{i,1}, \ldots, U_{i,L})_{i \leq 0}$ forms an L-type branching process in random environment ([HW13]) and

$$P_\omega(\tilde{\xi}_{i,k} \geq t) = P_\omega(\xi_{i,k} \geq t) = e^{-\left(\lambda_i + \sum_{i=1}^{L} \mu_i^l\right)t}, \ t \geq 0.$$

Moreover,

$$E_\omega T_1 = \sum_{i=-\infty}^{0} \frac{1}{\lambda_i} \mathbf{e}_1 M_0 M_{-1} \cdots M_{i+1} \mathbf{1}.$$

[HW13] Hong, W. M. and Wang, H. M., Intrinsic branching structure within (L-1) random walk in random environment and its applications, Infin. Dimens. Anal. Quantum Probab. Relat. Top., Vol. 16, 1350006, 2013

(1,R) BDP in fixed environment

Consider (1,R) BDP in fixed environment on positive half lattice.

(1,R) BDP in fixed environment

Consider (1,R) BDP in fixed environment on positive half lattice.

Recall that $(\mu_i, \lambda_i^1, ..., \lambda_i^R)_{i \geq 0}$ is the environment for BDP $\{N_t\}$. In order to limit the walker on $\mathbb{Z}^+$, set $\mu_0 = 0$.

(1,R) BDP in fixed environment

Consider (1,R) BDP in fixed environment on positive half lattice.

Recall that $(\mu_i, \lambda_i^1, ..., \lambda_i^R)_{i\geq 0}$ is the environment for BDP $\{N_t\}$. In order to limit the walker on $\mathbb{Z}^+$, set $\mu_0 = 0$.

For $i \geq 1$, let $a_i^k = \frac{\sum_{l=k}^R \lambda_i^l}{\mu_i}$, $k = 1, ..., R$, and introduce matrices

$$M_i = \begin{pmatrix} a_i^1 & \cdots & a_i^{R-1} & a_i^R \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}. \quad (1)$$

(1,R) BDP in fixed environment

Consider (1,R) BDP in fixed environment on positive half lattice.

Recall that $(\mu_i, \lambda_i^1, ..., \lambda_i^R)_{i\geq 0}$ is the environment for BDP $\{N_t\}$. In order to limit the walker on $\mathbb{Z}^+$, set $\mu_0 = 0$.

For $i \geq 1$, let $a_i^k = \frac{\sum_{l=k}^R \lambda_i^l}{\mu_i}$, $k = 1, ..., R$, and introduce matrices

$$M_i = \begin{pmatrix} a_i^1 & \cdots & a_i^{R-1} & a_i^R \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{1}$$

Let

$$P_t(i,j) = P(N_t = j | N_0 = i).$$

Theorem 7

Suppose that there are small $\kappa > 0$ and large $K > 0$ such that for all $n \geq 0$, $\kappa < \mu_n + \sum_{r=1}^{R} \lambda_n^r < K$..

Theorem 7

Suppose that there are small $\kappa > 0$ and large $K > 0$ such that for all $n \geq 0$, $\kappa < \mu_n + \sum_{r=1}^{R} \lambda_n^r < K$.. Then
(a) If $\lim_{n\to\infty} \mathbf{e}_1 M_1 M_2 \cdots M_n \mathbf{e}_1^T = 0$, then $\{N_t\}$ is recurrent.
(b) If $\sum_{n=1}^{\infty} \frac{1}{\mu_i} \mathbf{e}_1 M_1 M_2 \cdots M_n \mathbf{e}_1^T < \infty$, then $\{N_t\}$ is positive recurrent

Theorem 7

Suppose that there are small $\kappa > 0$ and large $K > 0$ such that for all $n \geq 0$, $\kappa < \mu_n + \sum_{r=1}^{R} \lambda_n^r < K$.. Then
(a) If $\lim_{n\to\infty} \mathbf{e}_1 M_1 M_2 \cdots M_n \mathbf{e}_1^T = 0$, then $\{N_t\}$ is recurrent.
(b) If $\sum_{n=1}^{\infty} \frac{1}{\mu_i} \mathbf{e}_1 M_1 M_2 \cdots M_n \mathbf{e}_1^T < \infty$, then $\{N_t\}$ is positive recurrent and the limits

$$\psi_0 := \lim_{t\to\infty} P_t(i,0) = \frac{(\sum_{r=1}^{R} \lambda_0^r)^{-1}}{(\sum_{r=1}^{R} \lambda_0^r)^{-1} + \sum_{n=1}^{\infty} \frac{1}{\mu_n} \mathbf{e}_1 M_1 M_2 \cdots M_{n-1} \mathbf{e}_1^T}$$

$$\psi_k := \lim_{t\to\infty} P_t(i,k) = \frac{\frac{1}{\mu_n} \mathbf{e}_1 M_1 M_2 \cdots M_{k-1} \mathbf{e}_1^T}{(\sum_{r=1}^{R} \lambda_0^r)^{-1} + \sum_{n=1}^{\infty} \frac{1}{\mu_n} \mathbf{e}_1 M_1 M_2 \cdots M_{n-1} \mathbf{e}_1^T}$$

define a stationary distribution for $\{N_t\}$ in the sense that for all $t > 0$,

$$\psi_k = \sum_{n=0}^{\infty} \psi_n P_t(n,k), \ \ k \geq 0.$$

The existence of $\{N_t\}$

Given ω, let $Q = (q_{ij})$ be a matrix with

$$q_{ij} = \begin{cases} \lambda_i^r, & \text{if } j = i + r,\ \ r = 1, ..., R; \\ \mu_i^l, & \text{if } j = i - l,\ \ l = 1, ..., L; \\ -\big(\sum_{i=1}^{L} \mu_i^l + \sum_{r=1}^{R} \lambda_i^r\big), & \text{if } j = i; \\ 0, & \text{else.} \end{cases}$$

Then Q is obviously a conservative Q-matrix.

The existence of $\{N_t\}$

Given ω, let $Q = (q_{ij})$ be a matrix with

$$q_{ij} = \begin{cases} \lambda_i^r, & \text{if } j = i + r,\ r = 1, ..., R; \\ \mu_i^l, & \text{if } j = i - l,\ l = 1, ..., L; \\ -\big(\sum_{i=1}^{L} \mu_i^l + \sum_{r=1}^{R} \lambda_i^r \big), & \text{if } j = i; \\ 0, & \text{else.} \end{cases}$$

Then Q is obviously a conservative Q-matrix.

We have from classical argument that there exists at least one transition matrix $(\overline{p}_\omega(t, i, j))$ such that

$$\lim_{t \to 0} \frac{\overline{p}_\omega(t, i, j) - \delta_{ij}}{t} = q_{ij},\ \ i, j \in \mathbb{Z}. \tag{2}$$

The existence of $\{N_t\}$

Theorem 8 (The existence)

Suppose that $\mathbb{P}\big(\sum_{l=1}^{L} \mu_0^l + \sum_{r=1}^{R} \lambda_0^r > 0\big) = 1$ and

$$\mathbb{P}\Big(\sum_{n=1}^{\infty}\Big(\max_{1\le k\le R}\Big\{\sum_{r=1}^{R}\lambda_{nR-k}^r + \sum_{l=1}^{L}\mu_{nR-k}^l\Big\}\Big)^{-1} = \infty\Big) = 1,$$

$$\mathbb{P}\Big(\sum_{n=-\infty}^{0}\Big(\max_{1\le k\le L}\Big\{\sum_{r=1}^{R}\lambda_{nL-k}^r + \sum_{l=1}^{L}\mu_{nL-k}^l\Big\}\Big)^{-1} = \infty\Big) = 1.$$

Then for $\mathbb{P}$-a.a. ω, there is a unique transition matrix $(p_\omega(h,i,j))$ which satisfies (2), that is $\{N_t\}$ exists.

Idea of the proof

$\tau_n, n \geq 1$: discontinuities of N_t

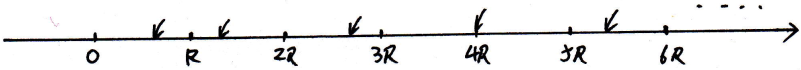
$X_n := N_{\tau_n}$: the embedded process

$q_i = -q_{ii}$?

$$P\Big(\sum_{n\geq 0} q_{X_n}^{-1} = \infty\Big) = 1$$

$$\Rightarrow P\big(\lim_{n\to\infty} \tau_n = \infty\big) = 1$$

$\Rightarrow \mathbb{P}$-a.s. $P_\omega(h,i,j)$ is unique



For example, if X_n is transient to the right, at least one point in each of $[kR, (k+1)R)$ would be visited.

Reference

[B02] Brémont, J., On some random walks on Z in random medium, Ann. Probab., Vol. 30(3), pp 1266-1312, 2002

[B09] Brémont, J., One-dimensional finite range random walk in random medium and invariant measure equation, Ann. Inst. H. PoincaréProbab. Statist., Vol. 45, pp 70-103, 2009

[C60] Chung, K. L., Markov chains with stationary transition probabilities, Springer-Verlag, 1960

[HZ10] Hong, W. M. and Zhang, L., Branching structure for the transient (1; R)-random walk in random environment and its applications, Infin. Dimens. Anal. Quantum Probab. Relat. Top., Vol. 13(4), pp 589-618, 2010

[K84] Key, E. S., Recurrence and transience criteria for random walk in a random environment, Ann. Probab., Vol. 12, pp 529-560, 1984

[K85] Kozlov, S. M., The method of averaging and walks in inhomogeneous environments, Russian Math. Surveys, Vol. 40 pp 73-145, 1985

[LV01] Lesigne, E. and Volný, D., Large deviations for martingales, Stochastic Process, Appl. Vol. 96, pp 143-159, 2001

[L89] Letchikov, A. V., Localization of one-dimensional random walks in random environments, Sov. Sci. Rev. C. Math. Phys., Vol. 8, pp 173-220, 1989

[R80] Ritter, G., A continuous-time analogue of random walk in a random environment, J. Appl. Probab. Vol. 17, pp 259-264, 1980

[W14a] Wang, H. M., Birth and death process with one-side bounded jumps in random environment, arXiv:1407.3385, 2014

[W14b] Birth and death process with bounded jumps in random environment, arXiv:1406.6222, 2014

[W14c] Stationary distribution for birth and death process with one-side bounded jumps, arXiv:1407.4007, 2014

[Z04] Zeitouni, O., Random walks in random environment, LNM 1837, J. Picard (Ed.), pp 189-312, Springer-Verlag Berlin Heidelberg, 2004

Thanks a lot

非常感谢

hmking@mail.ahnu.edu.cn