Weak Convergence Methods for Approximation of Path-dependent Functionals

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Outline

Introduction and Examples
- Motivations
- Problem setup

Convergence of value function given approximating processes
- Intuition and counter examples
- Continuity under Skorohod topology
- Convergence results and discussions on its assumptions

Approximating underlying process

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Option pricing

We consider a single stock, with the unique equivalent local martingale measure (ELMM) $\mathbb{P}$, under which the deflated stock price follows

$$dX(s) = \sigma(X(s))dW(s), \; X(t) = x \geq 0,$$

where W is a standard Brownian motion w.r.t. $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s : s \geq t\})$.

The price of an European option with maturity T and payoff $f(X(T))$ is

$$V(x,t) = \mathbb{E}_{x,t}[f(X(T))].$$

There are two numerical methods for the computation of V: PDE numerics; Monte-Carlo methods.

Suppose volatility σ and payoff f satisfy standing assumptions

(A1) σ is locally Holder continuous with exponent $\frac{1}{2}$ satisfying $\sigma(x) > 0$ for all $x \in \mathbb{R}^+$, $\sigma(0) = 0$.

(A2) $f : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ is a continuous payoff function with growth condition $|f(x)| \le K(1 + |x|^\gamma)$ for some $\gamma \in [0, 1]$.

Option price V solves PDE $BS(Q,f)$ in $C^{2,1}(Q) \cap C(\overline{Q})$,

$$BS(Q,f) : \begin{cases} u_t + \frac{1}{2}\sigma^2(x)u_{xx} = 0 & \text{on } Q := \mathbb{R}^+ \times (0,T) \\ u(x,t) = f(x) & \text{on } \partial^* Q := [0,\infty) \times \{T\} \cup \{0\} \times (0,T). \end{cases}$$

One can use PDE numerics to compute option price, ex Finite Difference Methods (FDM), Finite Element Methods (FEM)

Monte Carlo method is a class of computational algorithms that relies on some repeated random sampling to evaluate its deterministic value using its probabilistic fact.
Among of many, we show Euler-Maruyama (EM) approximation as an example below. Consider EM with step size Δ, each simulation X^Δ is the piecewise constant interpolation of $\{X_n^\Delta : n \geq 0\}$, i.e.

$$X^\Delta(s) = X^\Delta_{[s/\Delta]}, \quad \forall s > 0.$$

where

$$X^\Delta_{n+1} = X^\Delta_n + \sigma(X^\Delta_n)(W(n\Delta + \Delta) - W(n\Delta)), \quad X^\Delta_0 = x.$$

Let $X^\Delta(\cdot)$ be The approximated value function simply is the average,

$$V_\Delta(x, 0) := \mathbb{E}_{x,0}[f(X^\Delta(T))] \to V(x, 0). \quad \square$$

Two examples on path-dependent value

1. Barrier option

A class of option prices are path-dependent, including look-back option, rebate/barrier option, Asian option, Bermuda option etc. For ex.,

- (Barrier option) We consider up-and-in barrier call with strike $1/2$ at maturity $T = 1$, which price formula is given by

$$V = e^{-r}\mathbb{E}[(X(1) - \frac{1}{2})^{+} I_{[0,1)}(\tau)]$$

where $\tau = \inf\{s > 0 : X(s) \notin (-\infty, 1)\} \wedge 1$.

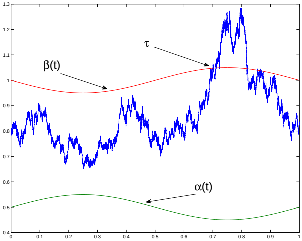

Figure: No PDE representation. Thus, Monte-Carlo for the computation.

Two examples on path-dependent value

2. Discrete-monitoring-barrier option

Most works assume continuous monitoring of the barrier by default like the previous example. However, in practice most barrier options traded in markets are monitored at discrete time.

- (Discrete-monitoring-barrier option) The price formula of up-and-in barrier call with strike $1/2$, monitoring at discrete time instants $\mathcal{T} = \{1/m, 2/m, \ldots, m-1/m, 1\}$ is

$$V = e^{-r}\mathbb{E}\left[\left(X(1) - \frac{1}{2}\right)^{+} I_{[0,1)}(\hat{\tau})\right].$$

  where $\hat{\tau} = \inf\{s \in \mathcal{T} : X(s) \notin (-\infty, 1)\} \wedge 1$.

Unlike their continuous-time counterparts, there is essentially no closed form solution available, and even numerical pricing is more difficult; see [BGK99] Broadie, Glasserman, Kou, Connecting discrete and continuous path-dependent options. *Finance Stoch.*, 3:55–82, 1999.

[Kou03] S. G. Kou. On pricing of discrete barrier options. *Statistica Sinica*, 13:955–964, 2003.

The path-dependent value function

General framework on 1-D

Denote $C[0,1] = C([0,1],\mathbb{R})$ and $D[0,1] = D([0,1],\mathbb{R})$.
Given a filtered probability space $(\Omega, \mathcal{F} = \mathcal{F}_1, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t : t \in [0,1]\})$, consider an $\mathbb{F}$-adapted continuous process $X : [0,1] \times \Omega \mapsto C[0,1]$ of

$$dX(t) = b(X(t),t)dt + \sigma(X(t),t)dW(t);\; X(0) = x.$$

We are interested in the computation of the objective functional V defined by, for some given function $F : D[0,1] \mapsto \mathbb{R}$

$$V = \mathbb{E}[F(X)].$$

Overview of computational methods

Why for MC?

To evaluate the path-dependent objective functions, a major difficulty arises from the lack of Markovian property. This gives added difficulty to the conventional numerical-PDE-based methods including the finite difference or finite element methods.

Thus, one naturally turns to approximation methods using Monte Carlo method. This includes Euler-Maruyama approximation and Markov chain approximation, others.

The general idea of the Monte Carlo method

To compute $V = \mathbb{E}[F(X)]$

We need to do

Step 1. For each sample point $\omega \in \Omega$, we simulate the underlying process $X(\cdot, \omega) \in C[0,1]$ by a certain discretization method with a small parameter h (maybe a step size), denoted by $X^h(\cdot, \omega) \in D[0,1]$.

Step 2. Then, one can approximate V by computing the average

$$V^h = \mathbb{E}[F(X^h)].$$

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Three useful operators used in the payoff structures

Hitting time $\mathcal{T}$, Projection Π, and Running Maximum $\mathcal{M}$

Let $\Gamma = (\alpha, \beta)$ be an open interval of $\mathbb{R}$.
Let $\mathcal{T} : D[0,1] \mapsto [0,1]$ be the first hitting time

$$\mathcal{T}(x) = \inf\{t > 0 : x(t) \notin \Gamma\} \wedge 1, \quad \forall x \in D[0,1],$$

Projection operator $\Pi : D[0,1] \times [0,1]^m \mapsto \mathbb{R}^m$ is defined by

$$\Pi(x, \nu) = (x(\nu_1), \ldots, x(\nu_m))' \quad \forall x \in D[0,1], \nu \in [0,1]^m.$$

Let $\mathcal{M} : D[0,1] \mapsto D[0,1]$ be the maximum process operator of

$$\mathcal{M}(x)(t) = \left(\sup_{0 \le s \le t} x(t) \right)', \quad \forall x \in D[0,1].$$

Value function of our interests

The value function is designed to cover many path-dependent applications

We are interested in the computation of $V = \mathbb{E}[F(X)]$ with F given in the form of

$$F(x) = g\Big(\Pi(x, \mathcal{T}(x)), \Pi(\mathcal{M}(x), \mathcal{T}(x))\}, \Pi(x, \nu), \mathcal{T}(x)\Big)$$

where $g(\cdot)$ is a given measurable real function, and $\nu = \{\nu_1, \ldots, \nu_m\}$ is a given finite dimensional vector.

This value function covers many path-dependent applications, including look-back option, rebate/barrier option, Asian option, Bermuda option, etc.

ex. Recall payoff of Discrete-Monitoring-barrier option:

$F(X) = (X(1) - \frac{1}{2})^+ I_{[0,1]}(\hat{\tau}) = (\Pi(x, 1) - \frac{1}{2})^+ I_{[1,\infty)}(\|\Pi(x, \nu)\|_\infty)$.

Goal. What is the sufficient condition for the convergence $V^h \to V$?

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Intuition

Approximation of points in $\mathbb{R}^1$

Let $X, X^h \in \mathbb{R}$, and $F : \mathbb{R} \mapsto \mathbb{R}$.
To ensure convergence of $V^h = F(X^h)$ to $V = F(X)$, i.e. $\lim_{h\to 0} V^h = V$, the usual requirements are as follows:

(H1) $X^h \to X$ as $h \to 0$.

(H2) F is continuous.

Q. How shall one generalize the above fact to the random world for $V = \mathbb{E}[F(X)]$? Note that, X is random curve, and $F : D[0,1] \times \Omega \to \mathbb{R}$.

Possible generalization of the usual requirement for MC

To compute $V = \mathbb{E}[F(X)]$ via $V^h = \mathbb{E}[F(X^h)]$, due to V is invariant under the same distribution, the usual requirements for the Monte Carlo method are intuitively given as follows:

(H1) X^h converges weakly (in distribution) to X as $h \to 0$, denoted by $X^h \Rightarrow X$.

(H2) F is continuous on its domain $D[0,1]$ (with some topology).

A couple of natural questions are as follows.

- Are (H1)-(H2) sufficient to guarantee the desired convergence $\lim_{h\to 0} V^h = V$?
- Can (H2) be possibly weakened to some discontinuous function (like indicator function) so that the barrier option pricing can be included?

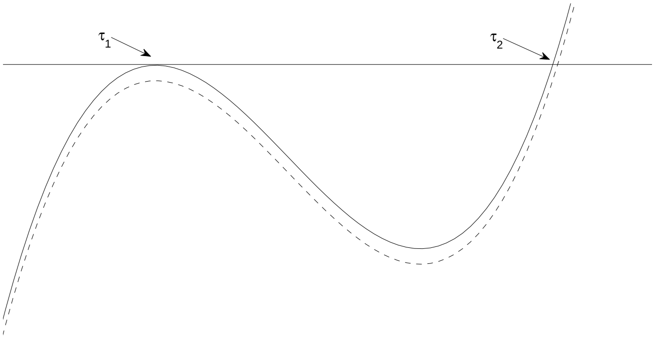
Counter example (TP): Tangency problem

(H1)-(H2) are NOT sufficient to guarantee the desired convergence $\lim_{h\to 0} V^h = V$

Let $\{X(s) : 0 \le s \le 1\}$ be a deterministic process given
The exit time of X is $\tau = \inf\{s > 0 : X(s) \notin (-\infty, 1)\} \wedge 1 = \frac{1}{2}$.
Define a family of processes parameterized by h with $X^h(s) = X(s) - h$.
Although X^h converges to X in L^∞ as $h \to 0^+$, we note that

$$\tau^h = \inf\{s > 0 : X^h(s) \notin (\alpha(s), \beta(s))\} \wedge 1 = 1$$

not converging to $\tau = 1/2$.



Counter example (3B): Bessel process-1

(H1)-(H2) are NOT sufficient to guarantee the desired convergence $\lim_{h\to 0} V^h = V$

Let $X^{-1}(t)$ be a Bessel process of order 3 with initial $X(0) = 1$, i.e.,

$$X^{-1}(t) = 1 + W(t) + \int_0^t \frac{ds}{X^{-1}(s)},$$

which is a well known strict local martingale with $\mathbb{E}[X(1)] < 1$.
A stock price which follows a strict local martingale is termed as *bubble*.

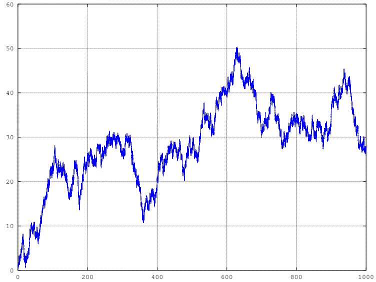


Figure: A sample path of Bessel process of order 3

Counter example (3B): Bessel process-2

(H1)-(H2) are NOT sufficient to guarantee the desired convergence $\lim_{h\to 0} V^h = V$

We denote interpolation of Euler approximation of X with step size h by X^h. X^h is a true martingale. Thus we have $\lim_{h\to 0} \mathbb{E}[X^h(1)] = 1 > \mathbb{E}[X(1)]$.
[Q.] In the above traditional Euler method fails to be convergent. Indeed, numerical PDE method also fails here. Then, how can one find an apporximation of $\mathbb{E}[X(1)]$?
[FK10] Daniel Fernholz and Ioannis Karatzas. On optimal arbitrage. *Ann. Appl. Probab.*, 20(4):11791204, 2010.

Next move?

What are we missing in (H1)-(H2)?

The above examples show that (H1)-(H2) may not be sufficient for Monte Carlo simulation to be convergent to the right value. This suggests the following question:

(Q1) Given $X^h \Rightarrow X$, what are sufficient conditions to ensure the convergence $\lim_{h\to 0} V^h = V$? Is it possible to weaken the continuity of g to cover the barrier option?

To complete the approximation of the value, one should also consider the following question in addition to (Q1).

(Q2) Given X, how does one construct an approximating sequence of processes X^h such that $X^h \Rightarrow X$?

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Skorohod topology

Define a uniform topology on $D[0,1]$ by

$$\|x-y\| = \sup_{t\in[0,1]} |x(t)-y(t)|, \ \forall x,y \in D[0,1].$$

Also, $D[0,1]$ can be equipped with the Skorohod topology with the metric

$$\|x-y\|_s = \inf_{\lambda\in\Lambda}\{\|\lambda-\mathbb{I}\|, \|x\circ\lambda - y\|\}, \ \forall x,y \in D[0,1],$$

where $x\circ y$ denotes the composite function of x and y, and Λ is the collection of all continuous increasing functions λ on $[0,1]$ with $\lambda(0)=0$ and $\lambda(1)=1$, and $\mathbb{I}\in\Lambda$ is the identity mapping.

Proposition

$x_n \to x$ in Skorohod topology if and only if there exists $\lambda_n \in \Lambda$ such that

$$\|\lambda_n - \mathbb{I}\| \to 0, \quad \|x_n - x\circ\lambda_n\| \to 0, \ as\ n\to\infty.$$

Continuity of operators-1

Lemma

$\mathcal{M}$ is continuous on $D[0,1]$ in Skorohod topology.

Lemma

Π is continuous at $(x,\nu) \in D[0,1] \times [0,1]^m$ whenever x is continuous at each ν_i of $i = 1, 2, \ldots, m$ in Skorohod topology.

See proofs in

[SYZ13] Qingshuo Song, George Yin, Qing Zhang, Weak Convergence Methods for Approximation of Path-dependent Functionals, SIAM J. Control Optim., 51(5): 4189-4210, 2013.

Continuity of operators-2

The continuity of $\mathcal{T}$

Let us partition the space $C[0,1]$ as follows:

$$
\begin{aligned}
C_1 = \{x \in C[0,1] : \quad & \mathcal{T}(x) < 1,\ x(\mathcal{T}(x)) = \beta, \\
& \inf\{t > \mathcal{T}(x) : x(t) > \beta\} = \mathcal{T}(x)\},
\end{aligned}
$$

$$
\begin{aligned}
C_2 = \{x \in C[0,1] : \quad & \mathcal{T}(x) < 1,\ x(\mathcal{T}(x)) = \alpha, \\
& \inf\{t > \mathcal{T}(x) : x(t) < \alpha\} = \mathcal{T}(x)\},
\end{aligned}
$$

and

$$C_3 = \{x \in C[0,1] : \mathcal{T}(x) = 1\}, \text{ and } C_4 = C[0,1] \setminus (\cup_{i=1}^{3} C_i).$$

Lemma

The $\mathcal{T}$ is continuous at each $x \in \cup_{i=1}^{3} C_i$.

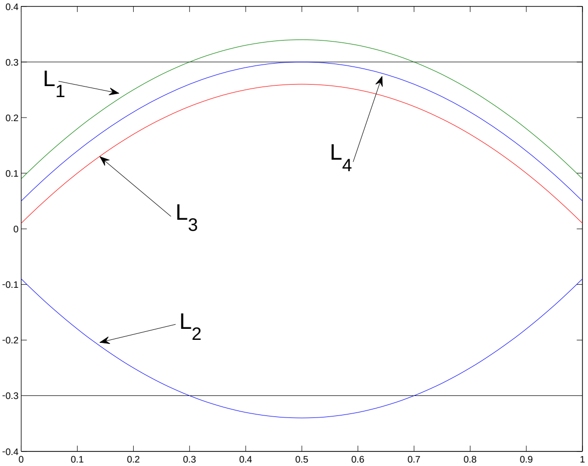
Proof.

See [SYZ13]. □

Continuity of operators-3

Illustration of the continuity of $\mathcal{T}$

$C[0,1] = \cup_{i=1}^{4} C_i$ and $C_i \cap C_j = \emptyset$ for $i \neq j$. As for the illustration, one can see that the four curves depicted in the Figure belong to four different subsets separately, that is, $L_i \in C_i$ for $i = 1, 2, 3, 4$.



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Convergence results

We make the following assumptions.

(A1) $X^h \Rightarrow X$.

(A2) $\mathbb{P}\{X \in \cup_{i=1}^{3} C_i\} = 1$.

(A3) g is an almost surely continuous function.

(A4) One of the following conditions holds:

1. g is a bounded function;
2. g is a function with linear growth and $\{X^h(t) : h > 0, t \in [0,1]\}$ is uniformly integrable.

Theorem (see [SYZ13])

Assume (A1)-(A4). Then, $\lim_{h \to 0} V^h = V$.

Proof.

Use continuous mapping theorem. □

Discussion on (A2)

Reason behind ex. (TP)

(A2) $\mathbb{P}\{X \in \cup_{i=1}^{3} C_i\} = 1.$

In fact, (A2) is a requirement on the regularity of the boundary $\partial\Gamma$ with respect to the process X, and it is referred to as τ'-regularity for simplicity. Note that since X in ex.(TP) violates τ'-regularity (A2), by observing

$$\inf\{t > \tau : X(t) \notin \Gamma\} \wedge 1 = 1 > 1/2 = \tau,$$

it yields the convergence to the wrong value. In other words, (A2) is crucial for the investigation of the convergence.

Discussion on (A3)

Allows discontinuous payoff functions

(A3) g is an almost surely continuous function.

is the requirement on the function g.
First of all, it allows discontinuity of g, but it cannot be too bad in the sense of (A3). However, it is already enough to include barrier option pricing.
Recall its payoff

$$g(x) = e^{-r}(x(1) - \tfrac{1}{2})^{+} I_{[0,1)}(\mathcal{T}(x)) = e^{-r}(x(1) - \tfrac{1}{2})^{+} I_{[0,1)}(\mathcal{T}'(x))$$

where $\mathcal{T}'(x) = \inf\{t > 0 : x(t) \notin \Gamma\}$. In fact, g is discontinuous only at $\{\mathcal{T}'(x) = 1\}$.
Suppose the stock price X follows a geometric Brownian motion, then the probability measure $\mathbb{P}$ satisfies $\mathbb{P}\{\mathcal{T}'(X) = 1\} = \mathbb{P}\{\mathcal{M}(X)(1) = 1\} = 0$, and g is continuous almost surely.

Discussion on (A4)

Explanation on ex. (3B)

[Fact] Let $Y^h \Rightarrow Y$. Then $\mathbb{E}[Y^h] \to \mathbb{E}[Y]$ if $\{Y^h\}$ is uniformly integrable (UI).
[Q.] Can we replace *if* with *if and only if*?
Note that (A1-A3) ensures $F(X^h) \Rightarrow F(X)$. To have $\mathbb{E}[F(X^h)] \to \mathbb{E}[F(X)]$, we need the growth condition of g, i.e.

(A4) One of the following conditions holds:

1. g is a bounded function;
2. g is a function with linear growth and $\{X^h(t) : h > 0, t \in [0,1]\}$ is uniformly integrable.

to further ensure UI of $\{F(X^h)\}$.
In particular, if g is of linearly growth function of the underlying price like in the call type option, then one must verify the uniform integrability. We have already seen that, in Example (3B) of Bessel process, the approximation converges to a wrong value by violating uniform integrability.

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Stochastic differential equation

Unique solvability

Let $Y = \{Y(t) : t \in [0,1]\}$ be the unique solution of

$$dY(t) = b(Y(t),t)dt + \sigma(Y(t),t)dW(t); \;\; Y(0) = y,$$

where $b : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$, W is a standard $\mathbb{R}^{d_1}$ BM, and $\sigma : \mathbb{R}^{d+1} \mapsto \mathbb{R}^{d \times d_1}$.
Unique solvability holds under the standing assumption

(A5) b and σ are Lipschitz in y and Hölder-1/2 continuous in t.

MC approximation of SDE

Let $t_0^h = 0 \le t_1^h \le \cdots \le t_N^h = 1$ be a sequence of increasing predictable (i.e., t_i^h is $\mathcal{F}_{i-1}^h$-measurable.) random times with respect to a discrete filtration $\{\mathcal{F}_i^h : i = 0, 1, \ldots\}$, and $\{Y_i^h : i = 1, 2, \ldots N\}$ be a sequence of $\{\mathcal{F}_i^h\}$-adapted Markov chain in $\mathbb{R}^d$ with transition probability

$$\mathbb{P}\{Y_{i+1}^h \in dy | Y_i^h = x, t_i^h = t\} = p^h(t, x, y).$$

We use $Y^h = \{Y^h(t) : t \in [0, 1]\}$ to denote piecewise constant interpolation

$$Y^h(t) = \sum_{i=0}^{n-1} Y_i^h I_{\{t_i^h \le t < t_{i+1}^h\}}. \tag{1}$$

Convergence of MC approximation

The interpolation of the Markov chain process Y^h is said to be *locally consistent*, if

(LC1) $\mathbb{E}_n^h[\Delta Y_n^h] = \mathbb{E}_n^h[\Delta t_n^h] \cdot (b(Y_n^h, t_n^h) + O(h))$,

(LC2) $\mathrm{cov}(\Delta Y_n^h | \mathcal{F}_n^h) = \mathbb{E}_n^h[\Delta t_n^h] \cdot ((\sigma\sigma')(Y_n^h, t_n^h) + O(h))$, where $O(h)$ is either a d-dimensional vector or $d \times d$ dimensional matrix that is $\mathcal{F}_n^h$-measurable with each element being $O(h)$.

To proceed, we also require quasi-uniform step size.

(QU) The step size $\{\Delta t_i^h\}$ satisfies $\frac{h}{K} \leq \inf_i \Delta t_i^h \leq \sup_n \Delta t_n^h \leq Kh$.

Theorem (see [SYZ13])

Locally consistent MC approximation with quasi-uniform step size is weakly convergent to solution of SDE, i.e. $Y^h \Rightarrow Y$ *as* $h \to 0$.

Convergence of MC approximation

The above construction of Markov chain approximation is based on the local consistency, which is slightly different from the local consistency given by [Theorem 10.4.1] of the book
[KD01] H.J. Kushner and P. Dupuis. Numerical Methods for Stochastic Control Problems in Continuous Time, Springer, 2001.
As a result, the convergence result of the Markov chain approximation is generalized in the following sense. σ and b may be unbounded but have linear growth. Therefore, the geometric Brownian motion is covered by weak convergence result of Theorem 3.1 as an important application. In fact, locally consistent MC approximation is flexible for its various choices, since it covers Euler approximation, Binomial approximation, and so on.

Can We Expect the Strong Convergence for MC?

Classical result

We consider a special case of Euler approximation.
Suppose $\hat{Y}^h$ is a continuous interpolation of Euler approximation

$$\hat{Y}^h(t) = \hat{Y}^h_{nh} + b(nh, \hat{Y}^h_{nh})(t - nh) + \sigma(nh, \hat{Y}^h_{nh})(W(t) - W(nh)), \quad \text{for } t \in [nh, nh + h).$$

Then a classical result on a strong convergence shows that:

$$\mathbb{E}\Big[\sup_{0 \le t \le T} |Y(t) - \hat{Y}^h(t)|\Big] \le Kh^{1/2}.$$

Q. Can We Expect the Strong Convergence in the similar fashion for locally consistent MC?

Can We Expect the Strong Convergence for MC?

Counter example: When $b = 0$ and $\sigma = 1$.

Locally consistent MC allows constant interpolation of Markov chain. However, the above inequality on strong convergence fails for the constant interpolation.

Consider EM of W_t on $[0, 1]$ by equal step size $h = 1/N$. Then, center

$$\mathbb{E}\Big[\sup_{0\le t\le 1}|W(t)-W([Nt]/N)|\Big] = \mathbb{E}\Big[\sup_{1\le n\le N}\sup_{(n-1)/N\le t<n/N}|W(t)-W(\frac{n-1}{N})|\Big].$$

Note that, $\bar{W}(t) = \sqrt{N}W(t/N)$ is a standard Brownian motion w.r.t. a time-scaled filtration. So one can reduce the above equality as

$$\mathbb{E}\Big[\sup_{0\le t\le 1}|W(t) - W([Nt]/N)|\Big] = \frac{1}{\sqrt{N}}\mathbb{E}\Big[\sup_{1\le n\le N}\Lambda_n\Big],$$

where $\{\Lambda_n\}$ are i.i.d. random variables defined by

$$\Lambda_n = \sup_{n-1\le t<n}|\bar{W}(t) - \bar{W}(n-1)|.$$

Since, Λ_n's are unbounded iid random variables, $\mathbb{E}\Big[\sup_{1\le n\le N}\Lambda_n\Big]$ goes to infinity with as $N \to \infty$. This shows that

$$\mathbb{E}\Big[\sup_{0\le t\le 1}|W(t) - W([Nt]/N)|\Big] > O(N^{-1/2}).$$

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This work has been devoted to analyzing approximation to path-dependent functionals. Using the methods of weak convergence, we have provided a unified approach for proving the convergence of numerical approximation of path-dependent functionals for a wide range of applications.

As byproduct, approximation of value in the presence of bubble can be answered, see

[SY13] Qingshuo Song, Pengfei Yang, Approximating Functional of Local Martingale Under the Lack of Uniqueness of Black-Scholes PDE, Quantitative Finance, (2013)

Further development can be done in multi-dimensional problem.